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پژوهشکده‌ی آمار

Higher Order Moments and Recurrence Relations of Order Statistics from the Exponentiated Gamma Distribution

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Abstract. Order statistics arising from exponentiated gamma (EG) distribution are considered. Closed form expressions for the single and double moments of order statistics are derived. Measures of skewness and kurtosis of the probability density function of the rth order statistic for different choices of r, n and θ are presented. Recurrence relations between single and double moments of rth order statistics are obtained. Single moment generating function (MGF) is derived in closed form. Also, we establish several recurrence relations between single MGF.

Keywords. Order statistics; recurrence relations; single moments; double moments; moment generating function; exponentiated gamma distribution.

1 Introduction

The subject of order statistics has been in process of development for many years and recently has become increasingly important. Articles relating to this area have appeared in numerous different publications. Many authors have studied order statistics; for example, David (1981), Balakrishnan and Cohen (1991), Arnold et al. (1992) and David and Nagaraja (2003). Moments of order statistics has been discussed by many authors; for example, Balakrishnan and Gupta (1998), Basu and Singh (1998) and Raqab (1998). Further, measures of skewness and kurtosis of the densities of order statistics from Burr type X distribution were computed by Raqab (1998). Moreover,

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recurrence relations for order statistics has dealt with many authors; for example, David (1995), Balakrishnan and Sultan (1998), Balakrishnan and Aggarwala (1998) and Balasubramanian and Beg (2004). Also, Raqab (2004) has established some recurrence relations for moment generating function (MGF) of order statistics from generalized exponential distribution.

For the gamma distribution with shape parameter $\alpha = m$ and scale parameter $\beta = 1$ i.e. $G(m, 1)$ explicit expressions for the moments of order statistics have been derived by Gupta (1960) for integer values of m, and by Krishnaiah and Rizvi (1967) for a general value of m; see also Breiter and Krishnaiah (1968). For integral values of m; Joshi (1979) has established recurrence relations satisfied by the single moments of order statistics. Young (1971) has also deduced a simple relation between moments of order statistics from the symmetrical inverse multinomial distribution and the order statistics of independent standardized gamma variable with integer parameter m.

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n from exponentiated gamma (EG) distribution with probability density function (p.d.f.)

$$f(x;\theta) = \theta x e^{-x}(1 - e^{-x} - xe^{-x})^{\theta-1}, \qquad x > 0, \quad \theta > 0 \tag{1}$$

and the cumulative distribution function (c.d.f.) is given by

$$F(x;\theta) = (1 - e^{-x} - xe^{-x})^{\theta}, \qquad x > 0, \quad \theta > 0 \tag{2}$$

Other statistical properties of this distribution are discussed by Shawky and Bakoban (2008c). Also, Bayesian estimations on the EG distribution studied by Shawky and Bakoban (2008a). Moreover, characterization from EG distribution based on record values deduced by Shawky and Bakoban (2008b). By putting $\theta = 1$ in (1), we have a gamma distribution $G(2, 1)$.

From (1) and (2), we observe that the characterization differential equation for the EG distribution is

$$(x+1)f(x) = \theta x\{F(x)\}^{1-L}[1 - \{F(x)\}^{L}], \qquad L = \frac{1}{\theta},\ x > 0, \tag{3}$$

L is a positive integer number.

In this paper, we derived exact expressions for the single and double moments of order statistics from EG distribution which is presented in section 2. We compute the measures of skewness and kurtosis of the distribution of the rth order statistic in a sample of size n for different choices of n, r

and θ. In section 3, recurrence relations for single and double moments of order statistics are derived. MGF for the single moments of order statistics from EG distribution is derived in section 4. Moreover, we establish some recurrence relations for MGF of order statistics from EG distribution in section 4.

2 Moments of Order Statistics

Let $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$ be the order statistics from the EG distribution given in (1). Then the single and double moments of order statistics are given as follows:

2.1 Single Moments

The p.d.f. of the rth order statistic $X_{r:n}, r = 1, 2, \ldots, n$ given by

$$f_{r:n}(x) = C_{r:n}\{F(x;\theta)\}^{r-1}\{1 - F(x;\theta)\}^{n-r} f(x;\theta), \qquad x > 0, \qquad (4)$$

(see David, 1981) where $f(\cdot)$ and $F(\cdot)$ are given by (1) and (2), respectively, and

$$C_{r:n} = \frac{n!}{(r-1)!\,(n-r)!}.$$

$f_{r:n}(x)$ can be written as

$$f_{r:n}(x) = \sum_{i=0}^{n-r} d_i(n,r) f\{x;\theta(r+i)\} \qquad (5)$$

where

$$d_i(n,r) = (-1)^i n \frac{\binom{n-1}{r-1}\binom{n-r}{i}}{r+i} \qquad (6)$$

The ath moment of the rth order statistic, $E(X^a_{r:n})$, denoted by $\alpha^{(a)}_{r:n}$, $1 \leqslant r \leqslant n$ and $a = 0, 1, 2, \ldots$, is given by

$$\alpha^{(a)}_{r:n} = \int_0^\infty x^a f_{r:n}(x;\theta)dx. \qquad (7)$$

The exact explicit expression for the single moments of order statistics from EG distribution is given by the following theorem:

Theorem 1 *For $1 \leqslant r \leqslant n$, $a \geqslant 0$ and θ is a real value, then*

$$\alpha_{r:n}^{(a)} = \theta C_{r:n} \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^{i+j} \binom{n-r}{i} \binom{\theta(r+i)-1}{j} \binom{j}{k} \frac{\Gamma(a+k+2)}{(j+1)^{a+k+2}}. \tag{8}$$

Proof. From (1), (5) and (7) we get

$$\alpha_{r:n}^{(a)} = \sum_{i=0}^{n-r} d_i(n,r)\theta(r+i) \int_0^{\infty} x^{a+1} e^{-x} \{1 - e^{-x}(x+1)\}^{\theta(r+i)-1} dx.$$

Using the binomial expansion we have

$$\alpha_{r:n}^{(a)} = \sum_{i=0}^{n-r} d_i(n,r)\theta(r+i) \sum_{j=0}^{\infty} (-1)^j \binom{\theta(r+i)-1}{j} \int_0^{\infty} x^{a+1} e^{-(j+1)x} (x+1)^j dx.$$

Thus,

$$\begin{aligned} \alpha_{r:n}^{(a)} &= \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^j d_i(n,r)\theta(r+i) \\ &\quad \times \binom{\theta(r+i)-1}{j} \binom{j}{k} \int_0^{\infty} x^{a+k+1} e^{-(j+1)x} dx \\ &= \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^j d_i(n,r)\theta(r+i) \binom{\theta(r+i)-1}{j} \binom{j}{k} \frac{\Gamma(a+k+2)}{(j+1)^{a+k+2}}. \end{aligned}$$

Then we get the result (8) by using (6), hence the theorem is proved. If θ is a positive integer number, then the relation (8) becomes

$$\begin{aligned} \alpha_{r:n}^{(a)} &= \theta C_{r:n} \sum_{i=0}^{n-r} \sum_{j=0}^{\theta(r+i)-1} \sum_{k=0}^{j} (-1)^{i+j} \binom{n-r}{i} \binom{\theta(r+i)-1}{j} \binom{j}{k} \\ &\quad \times \frac{\Gamma(a+k+2)}{(j+1)^{a+k+2}}, \qquad a = 0, 1, 2, \ldots \end{aligned} \tag{9}$$

The single moments of order statistics from gamma distribution $G(2,1)$ can be obtained from (9) by setting $\theta = 1$.

Table 1. Values of β_1 and β_2 of $X_{r:n}$ for different values of θ

n	r	$\theta = 0.1$		$\theta = 0.5$		$\theta = 1.5$		$\theta = 3$	
		β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2
2	1	6.01152	55.7806	1.80963	7.77588	1.04913	4.66801	0.83161	4.14982
	2	2.61306	12.4684	1.41421	6.00000	1.13866	5.16830	1.07670	5.03505
4	1	13.3702	266.712	1.77428	7.71715	0.85373	4.02074	0.58068	3.51307
	2	5.10736	41.7955	1.35670	5.70018	0.77613	3.95516	0.63447	3.70419
	3	2.87074	15.0541	1.16317	5.03569	0.83087	4.17702	0.75479	4.04373
	4	1.93098	8.27580	1.20736	5.34612	1.07659	5.03569	1.05490	5.00999
6	1	32.3889	1447.16	1.72844	7.39703	0.75907	3.74508	0.46170	3.28325
	2	9.90026	143.107	1.34822	5.70410	0.65718	3.64917	0.48760	3.40164
	3	4.66034	35.0048	1.13630	4.87955	0.64666	3.67537	0.53458	3.51097
	4	2.86780	15.3364	1.00628	4.52655	0.68067	3.79044	0.60683	3.67894
	5	2.09612	9.46421	0.98248	4.52497	0.78035	4.08620	0.73838	4.02201
	6	1.66034	6.98011	1.13866	5.16830	1.06063	5.01274	1.05203	5.01759

The measures of skewness and kurtosis of the distribution of the rth order statistic can be evaluated from the following expressions:

$$\beta_1 = \frac{\alpha_{r:n}^{(3)} - 3\alpha_{r:n}\alpha_{r:n}^{(2)} + 2\alpha_{r:n}^3}{(\alpha_{r:n}^{(2)} - \alpha_{r:n}^2)^{\frac{3}{2}}}$$

and

$$\beta_2 = \frac{\alpha_{r:n}^{(4)} - 4\alpha_{r:n}\alpha_{r:n}^{(3)} + 6\alpha_{r:n}^2\alpha_{r:n}^{(2)} - 3\alpha_{r:n}^4}{(\alpha_{r:n}^{(2)} - \alpha_{r:n}^2)^2}.$$

Table 1 presents the values of these measures for different choices of r, n and θ.

In Table 1, we present the values of β_1 and β_2 for the cases in which $n = 2,\ 4,\ 6$ and $\theta = 0.1, 0.5, 1.5, 3$ with $1 \leqslant r \leqslant n$. For fixed r and n, the value of β_1 decreases as θ increases. For fixed and $\theta \geqslant 0.5$ the value of β_2 increases as r increases and for fixed n and $\theta \leqslant 0.5$ the value of β_2 decreases as r increases. Also, when $\theta \leqslant 0.5$ for fixed r and n, the value of β_2 decreases as θ increases.

2.2 Double Moments

The joint p.d.f. of $X_{r:n}$ and $X_{s:n}(1 \leqslant r < s \leqslant n)$ is given by (see David, 1981)

$$f_{r,s:n}(x,y) = C_{r,s:n}\{F(x;\theta)\}^{r-1}\{F(y;\,\theta) - F(x;\theta)\}^{s-r-1}\{1 - F(y;\theta)\}^{n-s} \times f(x;\theta)\,f(y;\theta), \qquad y > x > 0, \tag{10}$$

where $f(\cdot)$ and $F(\cdot)$ are given by (1) and (2), respectively, and

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

$f_{r,s:n}(x,y)$ can be written as

$$f_{r,s:n}(x,y) = C_{r,s:n} \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{n-s}{i} \binom{s-r-1}{j} \times \{F(x;\theta)\}^{r+j-1} \{F(y;\theta)\}^{s-r-1-j+i} f(x;\theta) f(y;\theta). \quad (11)$$

The double moments of $X_{r:n}^a$ and $X_{s:n}^b E(X_{r:n}^a X_{s:n}^b)$, denoted by $\alpha_{r,s:n}^{(a,b)}$, $1 \leqslant r < s \leqslant n$ and $a, b = 0, 1, 2, \ldots$ is given by

$$\alpha_{r,s:n}^{(a,b)} = \int_0^\infty \int_x^\infty x^a y^b f_{r,s:n}(x,y) dy\ dx. \quad (12)$$

The exact explicit expression for the double moments of order statistics from the EG distribution is given by the following theorem:

Theorem 2 *For $1 \leqslant r < s \leqslant n, a, b \geqslant 0$ and θ is a real value, then*

$$\alpha_{r,s:n}^{(a,b)} = C_{r,s:n} \theta^2 \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{u=0}^{b+l+1} \sum_{v=0}^{\infty} \sum_{p=0}^{v} (-1)^{i+j+k+v} \binom{n-s}{i} \times \binom{s-r-1}{j} \binom{\theta(s-r-j+i)-1}{k} \binom{k}{l} \binom{\theta\,(r+j)-1}{v} \times \binom{v}{p} \frac{\Gamma(b+l+2)\Gamma(a+u+p+2)}{u!(k+1)^{b+l-u+2}(k+v+2)^{a+u+p+2}}. \quad (13)$$

Proof. From (1), (2) and (12) we get

$$\alpha_{r,s:n}^{(a,b)} = \theta^2 C_{r,s:n} \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{n-s}{i} \binom{s-r-1}{j} \int_0^\infty \int_x^\infty x^{a+1} y^{b+1} e^{-x} e^{-y} \times \{1-e^{-x}(x+1)\}^{\theta(r+j)-1} \{1-e^{-y}(y+1)\}^{\theta(s-r-j+i)-1} dydx$$

$$= \theta^2 C_{r,s:n} \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{n-s}{i} \binom{s-r-1}{j}$$

$$\int_0^\infty x^{a+1} e^{-x} \{1 - e^{-x}(x+1)\}^{\theta(r+j)-1} I_1(x) dx, \tag{14}$$

where

$$I_1(x) = \int_x^\infty y^{b+1} e^{-y} [1 - e^{-y}(y+1)]^{\theta(s-r-j+i)-1} dy$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} (-1)^k \binom{\theta\,(s-r-j+i)-1}{k} \binom{k}{l}$$

$$\sum_{u=0}^{b+l+1} \frac{x^u e^{-(k+1)x}}{(k+1)^{b+l+2-u}} \cdot \frac{\Gamma(b+l+2)}{u!}. \tag{15}$$

Substituting $I_1(x)$ in (14) we get

$$\alpha_{r,s:n}^{(a,b)} = \theta^2 C_{r,s:n} \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{u=0}^{b+l+1} (-1)^{i+j+k} \binom{n-s}{i}$$

$$\times \binom{s-r-1}{j} \binom{\theta(s-r-j+i)-1}{k} \binom{k}{l}$$

$$\times \frac{\Gamma(b+l+2)}{u!(k+1)^{b+l+2-u}} \int_0^\infty x^{a+u+1} e^{-(k+2)x} \{1 - e^{-x}(x+1)\}^{\theta(r+j)-1} dx$$

$$= \theta^2 C_{r,s:n} \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{u=0}^{b+l+1} \sum_{v=0}^{\infty} \sum_{p=0}^{v} (-1)^{i+j+k+v}$$

$$\times \binom{n-s}{i} \binom{s-r-1}{j} \binom{\theta(s-r-j+i)-1}{k} \binom{k}{l}$$

$$\times \binom{\theta(r+j)-1}{v} \binom{v}{p} \frac{\Gamma(b+l+2)}{u!(k+1)^{b+l+2-u}} \int_0^\infty x^{a+u+p+1} e^{-(k+v+2)x} dx.$$

The integral in the last equation is a gamma function, hence the theorem is proved.

If θ is a positive integer number, then the relation (13) becomes

$$\alpha_{r,s:n}^{(a,b)} = C_{r,s:n}\theta^2 \sum_{i=0}^{n-s}\sum_{j=0}^{s-r-1}\sum_{k=0}^{\theta(s-r-j+i)-1}\sum_{l=0}^{k}\sum_{u=0}^{b+l+1}\sum_{v=0}^{\theta(r+j)-1}\sum_{p=0}^{v}(-1)^{i+j+k+v} \times \binom{n-s}{i}\binom{s-r-1}{j}\binom{\theta(s-r-j+i)-1}{k}\binom{k}{l}\binom{\theta(r+j)-1}{v}\binom{v}{p} \times \frac{\Gamma(b+l+2)\Gamma(a+u+p+2)}{u!(k+1)^{b+l-u+2}(k+v+2)^{a+u+p+2}}, \qquad a,b=0,1,2,\ldots \tag{16}$$

The double moments of order statistics from gamma distribution $G(2,1)$ can be obtained from (16) by setting $\theta = 1$.

3 Recurrence Relations Between Moments of Order Statistics

With $f(\cdot)$ and$F(\cdot)$ are given in (1) and (2), respectively. Then by making use of the characterization differential equation in (3), we establish in this section recurrence relations for the single and double moments of order statistics as follows:

3.1 Recurrence Relations for Single Moments

Theorem 3 *For* $1 < r \leqslant n-1,\ a \geqslant 0$ *and* $L = \frac{1}{\theta} = 1, 2, 3, \ldots, r-1$

$$\alpha_{r:n}^{(a+1)} + \alpha_{r:n}^{(a)} = \frac{n!(r-L)!}{L(a+2)(r-1)!(n-L)!}\left\{\alpha_{r-L+1:n-L}^{(a+2)} - \alpha_{r-L:n-L}^{(a+2)}\right\} + \frac{r}{L(a+2)}\left\{\alpha_{r:n}^{(a+2)} - \alpha_{r+1:n}^{(a+2)}\right\}. \tag{17}$$

Proof. From (3) and (4), we get

$$E(X_{r:n}^{a+1} + X_{r:n}^{a}) = \int_0^\infty (x^{a+1} + x^a) f_{r:n}(x)dx,$$

$$E(X_{r:n}^{a+1}+X_{r:n}^{a}) = C_{r:n}\int\limits_0^\infty x^a(x+1)\{F(x)\}^{r-1}\{1-F(x)\}^{n-r}f(x)dx$$

$$= C_{r:n}\theta\Bigg[\int\limits_0^\infty x^{a+1}\{F(x)\}^{r-L}\{1-F(x)\}^{n-r}dx$$

$$-\int\limits_0^\infty x^{a+1}\{F(x)\}^{r}\{1-F(x)\}^{n-r}dx\Bigg].$$

Integrating by parts, we get the result, hence the theorem is proved.

For $\theta = 1, 1 < r \leqslant n-1,\ a \geqslant 0$, and by using the identity (see Arnold et al., 1992),

$$i\mu_{i+1:n}^{(m)} + (n-i)\mu_{i:n}^{(m)} = n\mu_{i:n-1}^{(m)}$$

the recurrence relation in (17) is valid for gamma distribution $G(2,1)$, which takes the form

$$\alpha_{r:n}^{(a+1)} + \alpha_{r:n}^{(a)} = \frac{n-r+1}{a+2}\left\{\alpha_{r:n}^{(a+2)} - \alpha_{r-1:n}^{(a+2)}\right\},$$

which is a special case when $m=2$ from the recurrence relation

$$\mu_{r:n}^{(k)} = \mu_{r-1:n-1}^{(k)} + \frac{k}{n}\sum_{t=0}^{m-1}\frac{\mu_{r:n}^{(t+k-m)}}{t!},$$

that was derived by Joshi (1979).

3.2 Recurrence Relations for Double Moments

Theorem 4 *For* $L+1 < r+1 < s \leqslant n, a, b \geqslant 0$ *and* $L = \frac{1}{\theta} = 1, 2, 3, \ldots, r-1$

$$\alpha_{r,s:n}^{(a+1,b)} + \alpha_{r,s:n}^{(a,b)} = \frac{n!(r-L)!}{L(a+2)(r-1)!(n-L)!}\left\{\alpha_{r-L+1,s-L:n-L}^{(a+2,b)} - \alpha_{r-L,s-L:n-L}^{(a+2,b)}\right\}$$
$$+\frac{r}{L(a+2)}\left\{\alpha_{r,s:n}^{(a+2,b)} - \alpha_{r+1,s:n}^{(a+2,b)}\right\}. \quad (18)$$

Proof. From (3) and (10), we obtain

$$E(X_{r,s:n}^{a+1}Y_{r,s:n}^{b} + X_{r,s:n}^{a}Y_{r,s:n}^{b}) = \int_0^\infty \int_0^y (x^{a+1}y^b + x^a y^b) f_{r,s:n}(x,y)dxdy$$
$$= C_{r,s:n}\int_0^\infty y^b\{1-F(y)\}^{n-s} I(y) f(y)dy, \quad (19)$$

where

$$I(y) = \int_0^y x^a(x+1)\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}f(x)dx \quad (20)$$
$$= \theta\Big[\int_0^y x^{a+1}\{F(x)\}^{r-L}\{F(y)-F(x)\}^{s-r-1}dx$$
$$- \int_0^y x^{a+1}\{F(x)\}^{r}\{F(y)-F(x)\}^{s-r-1}dx\Big]$$
$$= \frac{\theta}{a+2}\Big[(s-r-1)\int_0^y x^{a+2}\{F(x)\}^{r-L}\{F(y)-F(x)\}^{s-r-2}f(x)dx$$
$$-(r-L)\int_0^y x^{a+2}\{F(x)\}^{r-L-1}\{F(y)-F(x)\}^{s-r-1}f(x)dx$$
$$-(s-r-1)\int_0^y x^{a+2}\{F(x)\}^{r}\{F(y)-F(x)\}^{s-r-2}f(x)dx$$
$$+ r\int_0^y x^{a+2}\{F(x)\}^{r-1}\{F(y)-F(x)\}^{s-r-1}f(x)dx\Big]. \quad (21)$$

Integrating by parts and substituting the above expression of $I(y)$ in (19), then simplifying the resulting equation, we obtain (18). Hence the theorem is proved.

For $\theta = 1$, $2 < r+1 < s \leqslant n$, and a, $b \geqslant 0$, the recurrence relation in (18) is valid for gamma distribution $G(2,1)$, which takes the form

$$\alpha_{r,s:n}^{(a+1,b)}+\alpha_{r,s:n}^{(a,b)} = \frac{n}{a+2}\left\{\alpha_{r,s-1:n-1}^{(a+2,b)} - \alpha_{r-1,s-1:n-1}^{(a+2,b)}\right\}+\frac{r}{a+2}\left\{\alpha_{r,s:n}^{(a+2,b)} - \alpha_{r+1,s:n}^{(a+2,b)}\right\}.$$

4 MGF of Order Statistics

Let $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$ be the order statistics from the EG distribution given in (1). Then MGF for the single moments of order statistics are given as follows:

4.1 MGF for Single Moments

The MGF of the rth order statistics $X_{r:n}$ denoted by $M_{r:n}(t)$ is given (see David, 1981, and Arnold et al., 1992) by

$$M_{r:n}(t) = E(e^{tX_{r:n}}) = \int_0^\infty e^{tx} f_{r:n}(x)dx \tag{22}$$

where $f_{r:n}(x)$ is defined in (4).

The exact explicit expression for the MGF for single moments of order statistics from EG distribution is given by the following theorem:

Theorem 5 *For $1 \leqslant r \leqslant n$ and θ is a real value, then*

$$M_{r:n}(t) = \theta C_{r:n} \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^{i+j} \binom{n-r}{i} \binom{\theta(r+i)-1}{j} \binom{j}{k} \frac{\Gamma(k+2)}{(1-t+j)^{k+2}}. \tag{23}$$

Proof. From (1), (5) and (22) we get

$$M_{r:n}(t) = \theta \sum_{i=0}^{n-r} d_i(n,r)(r+i) \int_0^\infty x e^{-(1-t)x} \{1 - e^{-x}(x+1)\}^{\theta(r+i)-1} dx.$$

Using the binomial expansion we have

$$M_{r:n}(t) = \theta \sum_{i=0}^{n-r} d_i(n,r)(r+i) \sum_{j=0}^{\infty} (-1)^j \binom{\theta(r+i)-1}{j} \int_0^{\infty} x e^{-(1-t+j)x}(x+1)^j dx$$
$$= \theta \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^j d_i(n,r)(r+i) \binom{\theta(r+i)-1}{j} \binom{j}{k} \frac{\Gamma(k+2)}{(1-t+j)^{k+2}}.$$

Then we get the result (23) by using (6), hence the theorem is proved.

If θ is a positive integer number, then the relation (23) becomes

$$M_{r:n}(t) = \theta C_{r:n} \sum_{i=0}^{n-r} \sum_{j=0}^{\theta(r+i)-1} \sum_{k=0}^{j} (-1)^{i+j} \binom{n-r}{i} \binom{\theta(r+i)-1}{j} \times \binom{j}{k} \frac{\Gamma(k+2)}{(1-t+j)^{k+2}}. \quad (24)$$

The MGF for single moments of order statistics from gamma distribution $G(2,1)$ can be obtained from (24) by setting $\theta = 1$.

4.2 Recurrence Relations for MGF

We establish a recurrence relation for the MGF for single moments of order statistics by making use of the following characterization differential equation:

$$\theta F(x;\theta) = \{x^{-1}(e^x - 1) - 1\} f(x;\theta)\}, \quad (25)$$

the MGF of X is denoted by $M_X(t)$, which its ith derivative is $M_X^{(i)}(t)$.

The following theorem gives the recurrence relations for the MGF of order statistics.

Theorem 6 *For* $1 \leqslant r \leqslant n-1$, *then*

(i)
$$M_{r+1:n}(t) - tM'_{r+1:n}(t) = -\tfrac{t^2}{r\theta} M_{r:n}(t+1) + (\tfrac{t^2}{r\theta} + 1) M_{r:n}(t) + \left(\frac{t^2}{r\theta} - t\right) M'_{r:n}(t). \quad (26)$$

$$\text{(ii)} \quad -(i-1)M^{(i)}_{r+1:n}(t) - tM^{(i+1)}_{r+1:n}(t) = \tfrac{-t^2}{r\theta} M^{(i)}_{r:n}(t+1) + i\left(\tfrac{-2t}{r\theta}\right) M^{(i-1)}_{r:n}(t+1)$$
$$+\binom{i}{2}\left(\frac{-2}{r\theta}\right)M^{(i-2)}_{r:n}(t+1) + \left\{\frac{t^2}{r\theta} + \frac{2it}{r\theta}\right.$$
$$\left.+(1-i)\right\}M^{(i)}_{r:n}(t) + \left(\frac{2}{r\theta}\right)\left\{it\right.$$
$$\left.+\binom{i}{2}\right\}M^{(i-1)}_{r:n}(t)$$
$$+\binom{i}{2}\left(\frac{2}{r\theta}\right)M^{(i-2)}_{r:n}(t)$$
$$+\left(\frac{t^2}{r\theta} - t\right)M^{(i+1)}_{r:n}(t), \quad i \geqslant 2, \quad (27)$$

and consequently,

$$\text{(iii)} \quad \alpha^{(2)}_{r+1:n} = \frac{2}{r\theta}\sum_{j=3}^{\infty}\frac{1}{j!}\alpha^{(j)}_{r:n} + \left(1 + \frac{1}{r\theta}\right)\alpha^{(2)}_{r:n}, \quad (28)$$

$$\text{(iv)} \quad (i-1)\alpha^{(i)}_{r+1:n} = \binom{i}{2}\left(\frac{2}{r\theta}\right)\sum_{j=2}^{\infty}\frac{1}{j!}\alpha^{(i-2+j)}_{r:n} + (i-1)\alpha^{(i)}_{r:n}, \quad i \geqslant 2. \quad (29)$$

Proof. It is clear that

$$M_{r:n}(t+1) = C_{r:n}\int_0^\infty e^{(t+1)x}\{F(x;\theta)\}^{r-1}\{1-F(x;\theta)\}^{n-r}f(x;\theta)dx$$
$$= C_{r:n}\int_0^\infty xe^{tx}\left(\frac{e^x-1}{x} - 1\right)\{F(x;\theta)\}^{r-1}\{1-F(x;\theta)\}^{n-r}$$
$$\times f(x;\theta)dx + M'_{r:n}(t) + M_{r:n}(t). \quad (30)$$

From (25) and (30), we get

$$M_{r:n}(t+1) = C_{r:n}\theta\int_0^\infty xe^{tx}\{F(x;\theta)\}^{r-1+1}\{1-F(x;\theta)\}^{n-r}dx + M'_{r:n}(t) + M_{r:n}(t).$$

Integrating by parts, we obtain

$$M_{r:n}(t+1) = \frac{d}{dt} C_{r:n}\theta \Bigg\{ \frac{n-r}{t} \int_0^\infty e^{tx}\{F(x;\theta)\}^r\{1-F(x;\theta)\}^{n-r-1} f(x;\theta)dx$$
$$+ \frac{-r}{t} \int_0^\infty e^{tx}\{F(x;\theta)\}^{r-1}\{1-F(x;\theta)\}^{n-r} f(x;\theta)dx \Bigg\}$$
$$+ M'_{r:n}(t) + M_{r:n}(t).$$

Simplifying, we get (26).

Differentiating equation (26) with respect to t, we have

$$M'_{r+1:n}(t) - tM''_{r+1:n}(t) - M'_{r+1:n}(t) = \frac{-t^2}{r\theta} M'_{r:n}(t+1) - \frac{2t}{r\theta} M_{r:n}(t+1)$$
$$+ \left(\frac{t^2}{r\theta} + \frac{2t}{r\theta}\right) M'_{r:n}(t) + \frac{2t}{r\theta} M_{r:n}(t)$$
$$+ \left(\frac{t^2}{r\theta} - t\right) M''_{r:n}(t). \qquad (31)$$

Setting $t = 0$ in (32), we get

$$-\alpha^{(2)}_{r+1:n} = \frac{-2}{r\theta} M_{r:n}(1) + \frac{2}{r\theta} - \alpha^{(2)}_{r:n} + \frac{2}{r\theta}\alpha_{r:n}.$$

After some simplification, we get (28). Also by differentiating equation (26), with respect to t, i times, we get (27). Setting $t = 0$ in (27), we get (29). Thus the theorem is proved.

5 Applications

The results established in this paper and some similar generalizations can be used to determine the mean, variance and the coefficients of skewness and kurtosis. The moments can also be used for finding best linear unbiased estimators (BLUE's) of location and scale parameters and conditional moments. Some of the results are then used to characterize the distribution. Shawky and Bakoban (2009) has discussed order statistics from EG distribution and associated inference. Based on the moments of order statistics, the BLUE's of the location and scale parameters of EG distribution under Type-II censoring were obtained. The variances and covariances of these estimators were also presented.

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