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Shrinkage Testimation in Exponential Distribution based on Records under Asymmetric Squared Log Error Loss

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Abstract. In the present paper, we study shrinkage testimation for the unknown scale parameter $\theta > 0$ of the exponential distribution based on record data under the asymmetric squared log error loss function. A minimum risk unbiased estimator within the class of the estimators of the form cT_m is derived, where T_m is the maximum likelihood estimate of θ. Some shrinkage testimators are proposed and their risks are computed. The relative efficiencies of the shrinkage testimators with respect to a minimum risk unbiased estimator of the form cT_m under the squared log error loss function are calculated for the comparison purposes. An illustrative example is also presented.

Keywords. Digamma function; exponential distribution; records; shrinkage testimators.

MSC 2010: 62F03; 62F10.

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1 Introduction

Let $\{X_i,\ i \geqslant 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables having a cumulative distribution function (c.d.f.) F and a probability density function (p.d.f.) f. An observation X_j is said to be an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. By convention X_1 is a record value. An analogous definition deals with lower record values. Data of this type arise in a wide variety of practical situations. Examples of application areas include industrial stress testing, meteorological analysis, sporting and athletic events, and oil and mining surveys; see Arnold et al. (1998) for these types of applications. We denote the mth upper record value by R_m. The joint density of the first m-records $\mathbf{R} = (R_1, \ldots, R_m)$ is given by

$$f_{R_1,\ldots,R_m}(r_1,\ldots,r_m) = f(r_m) \prod_{i=1}^{m-1} \frac{f(r_i)}{1-F(r_i)}, \qquad r_1 < r_2 < \cdots < r_m. \quad (1)$$

Also, the marginal p.d.f. of the mth record, R_m, is given by

$$f_{R_m}(x) = \frac{[-\log(1-F(x))]^{m-1}}{(m-1)!} f(x). \quad (2)$$

Throughout the paper, we denote by $\mathrm{Exp}(\theta)$ an exponential distribution with p.d.f.

$$f(x;\theta) = \frac{1}{\theta} \exp\left(-\left\{ \frac{x}{\theta} \right\} \right), \quad x > 0, \qquad \theta > 0. \quad (3)$$

If $\mathbf{R} = (R_1, \ldots, R_m)$ be the first m-records samples from the $\mathrm{Exp}(\theta)$-distribution, then from (1) and (3), the likelihood function of θ based on $\mathbf{R} = (R_1, \ldots, R_m)$ at $\mathbf{r} = (r_1, \ldots, r_m)$ is given by

$$L(\theta|\mathbf{r}) = \frac{1}{\theta^m} \exp\left(-\left\{ \frac{r_m}{\theta} \right\} \right), \qquad \theta > 0.$$

Then, the MLE of θ, denoted by T_m, can be derived from the equation $\frac{\partial L(\theta|\mathbf{r})}{\partial \theta} = 0$, which is given by $T_m = R_m/m$. Also, by substituting the p.d.f. and the c.d.f. of the $\mathrm{Exp}(\theta)$-distribution in (2), the marginal p.d.f. of the mth record, R_m, is given by

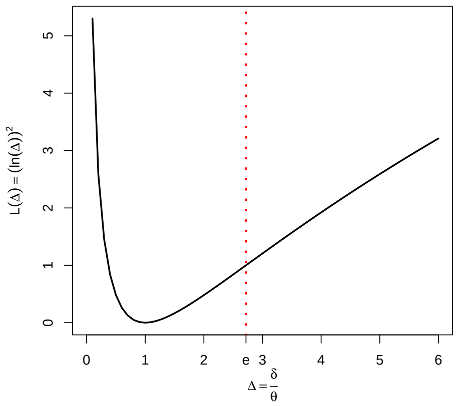

Figure 1. Plot of the SLEL function

$$f_{R_m}(x) = \frac{x^{m-1}\ \exp(-\{\frac{x}{\theta}\})}{\Gamma(m)\theta^m}, \qquad \theta > 0,$$

which implies that $R_m \sim Gamma(m, \theta)$ and then $2mT_m/\theta = 2R_m/\theta \sim \chi^2_{2m}$.

Brown (1990) proposed Squared Log Error Loss (SLEL) function for estimating the scale parameter θ as

$$L(\theta, \delta) = (\ln \delta - \ln \theta)^2 = \left[\ln \frac{\delta}{\theta}\right]^2, \tag{4}$$

where both θ and δ are positive; see also Pal and Ling (1996). This loss is not symmetric and convex; it is convex for $\Delta = \frac{\delta}{\theta} \leqslant e$ (Euler's number) and concave otherwise, but has a unique minimum at $\Delta = 1$. Also when $\Delta > 1$, this loss increases sublinearly, while when $0 < \Delta < 1$, it rises rapidly to infinity at zero; see Figure 1. The SLEL function is useful in situations where underestimation is more serious than overestimation; see Sanjari Farsipour and Zakerzadeh (2005) and Kiapour and Nematollahi (2011).

According to Thompson (1968), a shrinkage estimator for the parameter θ when a prior point guess value θ_0 of θ is available, is given by

$$\hat{\theta}_S = kT_m + (1-k)\theta_0, \qquad 0 \leqslant k \leqslant 1, \tag{5}$$

where k is a shrinkage factor. The value of k near to zero (one) implies strong belief in the guess value θ_0 (sample values). It seems that for the values of θ near to θ_0, the shrinkage estimators should have performance better than the usual estimator T_m. Then, a preliminary test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is performed for that θ_0 is near to θ or not. For testing the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, a likelihood ratio test (LRT) statistic is $U = 2mT_m/\theta \sim \chi^2_{2m}$ that has a rejection region of the form $U > q_2$ or $U < q_1$, where $q_1 = \chi^2_{\alpha/2,2m}$ and $q_2 = \chi^2_{1-\alpha/2,2m}$ are left quantiles of the chi-square distribution with $2m$ degrees of freedom.

To this end, we propose some shrinkage testimators for the scale parameter of the Exp(θ)-distribution based on record data and study the performance of these testimators with respect to a minimum risk unbiased estimator within the class cT_m under the SLEL function. A real data set is used for illustrating the results. Finally, we end the paper with some remarks.

2 A Minimum Risk Unbiased Estimator of the Form cT_m

Consider a class of estimators for the MLE of θ, T_m, of the form cT_m. If $U = 2mT_m/\theta \sim \chi^2_{2m}$, then the risk of cT_m under the SLEL function is

$$\begin{aligned} R(\theta, cT_m) &= E\left[\ln\left(\frac{cT_m}{\theta}\right)\right]^2 = E\left[\ln\left(\frac{c}{2m}U\right)\right]^2 \\ &= \ln^2\left(\frac{c}{2m}\right) + E[\ln^2(U)] + 2\ln\left(\frac{c}{2m}\right)E[\ln(U)]. \end{aligned} \tag{6}$$

Following Sanjari Farsipour and Zakerzadeh (2005), we have

$$E[\ln(U)] = \ln(2) + \Psi(m), \qquad E[\ln^2(U)] = [\ln(2) + \Psi(m)]^2 + \Psi'(m), \tag{7}$$

where $\Psi(m) = \frac{d}{dm}\ln\Gamma(m) = \frac{\Gamma'(m)}{\Gamma(m)}$ is the digamma function, $\Psi'(m) = \frac{d}{dm}\Psi(m)$ is the trigamma function and $\Gamma(m)$ denotes the complete gamma function given by

$$\Gamma(m) = \int_0^\infty t^{m-1}e^{-t}dt.$$

Upon substituting Equation (7) into Equation (6), we have

$$\begin{aligned} R(\theta, cT_m) &= \ln^2\left(\frac{c}{2m}\right) + [\ln(2) + \Psi(m)]^2 + \Psi'(m) \\ &+ 2\ln\left(\frac{c}{2m}\right)[\ln(2) + \Psi(m)]. \end{aligned} \tag{8}$$

The risk function in (8) is a convex function of c and is minimized at the point $c = c_1$ given by

$$c_1 = me^{-\Psi(m)}.$$

Therefore, $c_1 T_m$ is a minimum risk estimator of θ under the class cT_m with finite risk as follows

$$R(\theta, c_1 T_m) = \Psi'(m). \tag{9}$$

Following the definition of Lehmann (1951) an estimator δ of θ is said to be risk unbiased if it satisfies

$$E[L(\theta, \delta)] \leqslant E[L(\theta', \delta)], \qquad \forall \theta' \neq \theta. \tag{10}$$

Under the SLEL, we have

$$E\left[\ln^2 \frac{\delta}{\theta}\right] - E\left[\ln^2 \frac{\delta}{\theta'}\right] = (\ln^2 \theta - \ln^2 \theta') - 2(\ln\theta - \ln\theta')E[\ln\delta].$$

If we consider $E[\ln\delta] = \ln\theta$, we conclude that

$$E\left[\ln^2 \frac{\delta}{\theta}\right] - E\left[\ln^2 \frac{\delta}{\theta'}\right] = -(\ln\theta - \ln\theta')^2 < 0.$$

Therefore, an estimator δ of θ is risk unbiased under the SLEL if it satisfies in the condition $E[\ln\delta] = \ln\theta$. Now, the estimator $c_1 T_m$ satisfies the risk unbiased condition, as follows:

$$\begin{aligned} E[\ln(c_1 T_m)] &= \ln(c_1) + E[\ln(T_m)] = \ln m - \Psi(m) + E\left[\ln\left(\frac{\theta U}{2m}\right)\right] \\ &= \ln m - \Psi(m) + E[\ln(U)] - \ln 2 - \ln m + \ln\theta \\ &= \ln\theta \qquad\qquad \text{(using (7))} \end{aligned}$$

Then, the estimator c_1T_m is a minimum risk unbiased estimator under the class of cT_m.

Note that using Theorem 3.1 of Sanjari Farsipour and Zakerzadeh (2005), the estimator $cT_m + d$ is admissible, provided

(i) $0 \leqslant c < c_1, \quad d > 0,$

(ii) $c = c_1, \quad d = 0,$

where $c_1 = me^{-\Psi(m)}$ which is the value of c that minimizes the risk function of cT_m. Therefore, the estimator c_1T_m is admissible in the class of estimators of the form cT_m.

3 Some Shrinkage Testimators

In this section, we propose three shrinkage testimators and calculate their risks under the SLEL function. We construct our shrinkage testimators based on acceptance or rejection of $H_0 : \theta = \theta_0$. The general form of the proposed shrinkage testimator is $kT_m + (1-k)\theta_0$, if $H_0 : \theta = \theta_0$ is accepted or c_1T_m, otherwise. If $H_0 : \theta = \theta_0$ is accepted at the level of α, then we have

$$\Pr\left(q_1 \leqslant \frac{2mT_m}{\theta_0} \leqslant q_2\right) = 1 - \alpha.$$

Therefore, the proposed shrinkage testimators can be written as

$$\hat{\theta}_{st}^{(i)} = \begin{cases} k_iT_m + (1-k_i)\theta_0 & t_1 \leqslant T_m \leqslant t_2 \\ c_1T_m & T_m < t_1 \text{ or } T_m > t_2, \end{cases} \tag{11}$$

where $t_1 = q_1\theta_0/2m$, $t_2 = q_2\theta_0/2m$ and $k_i,\ i = 1,2,3$ are shrinkage factors corresponding to the shrinkage testimators $\hat{\theta}_{st}^{(i)},\ i = 1,2,3$. In the sequel, we propose three shrinkage testimator for θ.

3.1 Shrinkage Testimator $\hat{\theta}_{st}^{(1)}$

The risk of the shrinkage estimator (5) under the SLEL function is

$$R(\theta, \hat{\theta}_S) = E\left[\ln\frac{\hat{\theta}_S}{\theta}\right]^2 = E\left[\ln\left(\frac{kT_m + (1-k)\theta_0}{\theta}\right)\right]^2$$

$$= E\left[\ln\left((1-k)\theta^\star + \frac{kW}{m}\right)\right]^2$$
$$= \int_0^\infty \left[\ln\left((1-k)\theta^\star + \frac{kw}{m}\right)\right]^2 g_W(w)dw, \qquad (12)$$

where $\theta^\star = \theta_0/\theta$ and $W = mT_m/\theta$ with p.d.f. $g_W(w) = w^{m-1}e^{-w}/\Gamma(m)$. The value of $k_1 = k_{min}$ which minimizes (12) can be obtained numerically and gives us the shrinkage testimator $\hat{\theta}_{st}^{(1)}$.

3.2 Shrinkage Testimator $\hat{\theta}_{st}^{(2)}$

If $H_0 : \theta = \theta_0$ is accepted, then following Waikar et al. (1984), the inequality $q_1 \leqslant 2mT_m/\theta_0 \leqslant q_2$ implies that

$$0 \leqslant k_2 = \frac{1}{q_2 - q_1}\left(\frac{2mT_m}{\theta_0} - q_1\right) \leqslant 1.$$

The value of k_2 can be used for constructing the shrinkage testimator $\hat{\theta}_{st}^{(2)}$.

3.3 Shrinkage Testimator $\hat{\theta}_{st}^{(3)}$

If $H_0 : \theta = \theta_0$ is accepted, then following Prakash and Singh (2008), the inequality $q_1 \leqslant 2mT_m/\theta_0 \leqslant q_2$ implies that $q_1 \leqslant 2m \leqslant q_2$ and then $q_1/(2m) \leqslant 1$. For small values of shrinkage factor, we can take $q_1/(2m) \approx 1$. Hence,

$$\frac{2m}{q_2 - q_1}\left(\frac{2mT_m/\theta_0}{2m} - \frac{q_1}{2m}\right) \approx \frac{2m}{q_2 - q_1}\left(\frac{T_m}{\theta_0} - 1\right).$$

Therefore, the shrinkage factor k_3 for constructing the shrinkage testimator $\hat{\theta}_{st}^{(3)}$ is given by

$$k_3 = \frac{2m}{q_2 - q_1}\left|\frac{T_m}{\theta_0} - 1\right|,$$

where the absolute is for avoiding from negative values.

3.4 The Risks of Shrinkage Testimators $\hat{\theta}_{st}^{(i)}$

The risk of the shrinkage testimator $\hat{\theta}_{st}^{(i)},\ i = 1, 2, 3$ given in (11) under the LSEL function is

$$
\begin{aligned}
R(\theta, \hat{\theta}_{st}^{(i)}) &= E\left[\ln^2\left(\frac{\hat{\theta}_{st}^{(i)}}{\theta}\right)\right] \\
&= E\left[\ln^2\left(\frac{k_i T_m + (1-k_i)\theta_0}{\theta}\right) I(t_1 \leqslant T_m \leqslant t_2)\right] \\
&+ E\left[\ln^2\left(\frac{c_1 T_m}{\theta}\right) I(T_m < t_1 \text{ or } T_m > t_2)\right] \\
&= E\left[\ln^2\left((1-k_i)\theta^\star + \frac{k_i W}{m}\right) I(w_1 \leqslant W \leqslant w_2)\right] \\
&+ E\left[\ln^2\left(\frac{c_1 W}{m}\right)\right] - E\left[\ln^2\left(\frac{c_1 W}{m}\right) I(w_1 \leqslant W \leqslant w_2)\right],
\end{aligned}
$$

where $W = mT_m/\theta$, $w_1 = q_1\theta^\star/2$ and $w_2 = q_2\theta^\star/2$. Using (9), we get

$$
R(\theta, \hat{\theta}_{st}^{(i)}) = \int_{w_1}^{w_2} \left\{ \ln^2\left((1-k_i)\theta^\star + \frac{k_i w}{m}\right) - \ln^2\left(\frac{c_1 w}{m}\right)\right\} g(w)dw + \Psi'(m),
$$

which can be computed numerically using the statistical package `R` version `3.1.2`.

Using a derivation similar to the above, we have

$$
E\left[\ln\left(\frac{\hat{\theta}_{st}^{(i)}}{\theta}\right)\right] = \int_{w_1}^{w_2} \left\{ \ln\left((1-k_i)\theta^\star + \frac{k_i w}{m}\right) - \ln\left(\frac{c_1 w}{m}\right)\right\} g(w)dw. \quad (13)
$$

For checking the condition of risk unbiasedness for $\hat{\theta}_{st}^{(i)}$, we should prove that the expression given in (13) is zero, which is difficult to investigate theoretically, then we investigate it numerically. Figure 2, shows the plot of (13) for shrinkage testimator $\hat{\theta}_{st}^{(1)}$ (solid line), $\hat{\theta}_{st}^{(2)}$ (dashed line) and $\hat{\theta}_{st}^{(3)}$ (dot) for selected values of $m = 2, 4$ and $\alpha = 0.01, 0.05, 0.1$ with respect to $\theta^\star$ (more figures are provided, but not presented here). From Figure 2, we observe that the expression (13) may be negative, zero or positive, then we can state that the testimator $\hat{\theta}_{st}^{(i)}$ may be negatively risk biased, risk unbiased or positively risk biased.

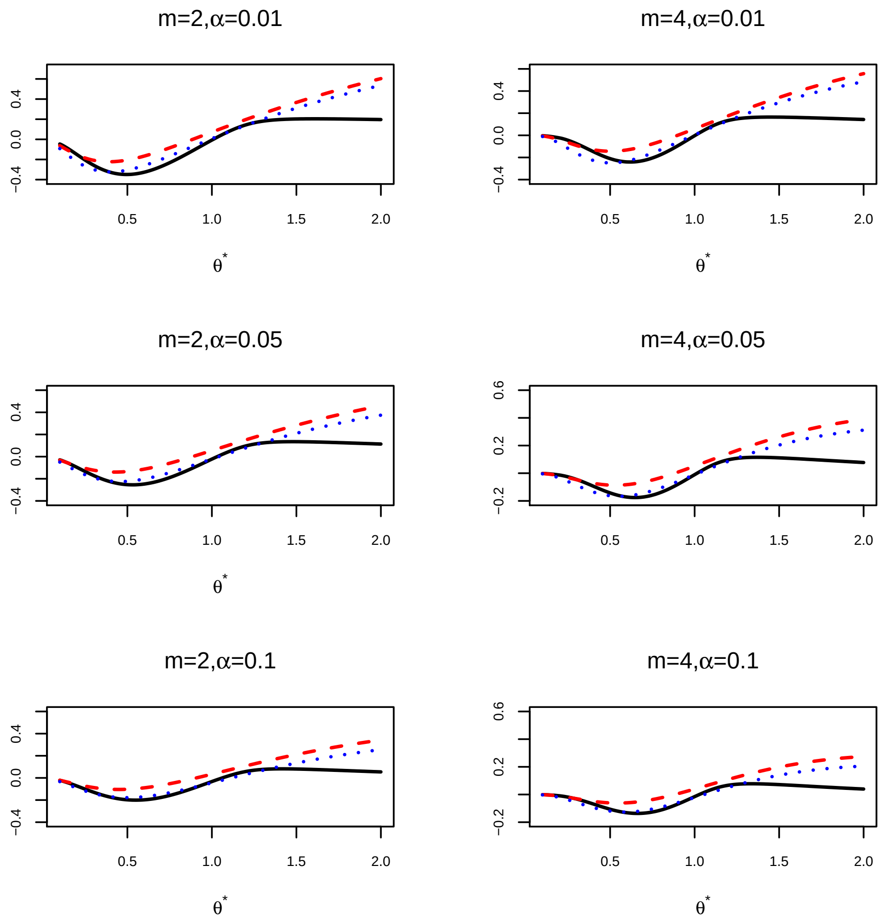

Figure 2. The plot of the expression (13) for shrinkage testimator $\hat{\theta}_{st}^{(1)}$ (solid line), $\hat{\theta}_{st}^{(2)}$ (dashed line) and $\hat{\theta}_{st}^{(3)}$ (dot) for selected values of $m = 2, 4$ and $\alpha = 0.01, 0.05, 0.1$ with respect to $\theta^{\star}$.

4 Comparison between Shrinkage Testimators and a Minimum Risk Unbiased Estimator

In this section, we evaluate the performance of the proposed shrinkage testimators and the minimum risk unbiased estimator. For comparison, the relative efficiency (R.E.) of shrinkage testimator $\hat{\theta}_{st}^{(i)},\ i = 1, 2, 3$ with respect to the minimum risk unbiased estimator $c_1 T_m$ is calculated as

$$RE(\hat{\theta}_{st}^{(i)}, c_1 T_m) = \frac{R(\theta, c_1 T_m)}{R(\theta, \hat{\theta}_{st}^{(i)})}, \qquad i = 1, 2, 3. \tag{14}$$

Tables 1-3 give the relative efficiency (14) for the selected values of $m = 2(1)5$, $\alpha = 0.01, 0.05, 0.1$ and $\theta^\star = \theta_0/\theta = 0.4(0.2)1.8$. From these tables, we observe that no testimator performs uniformly better than the estimator $c_1 T_m$. The testimator $\hat{\theta}_{st}^{(1)}$ perform better than the estimator $c_1 T_m$ in $0.8 \leqslant \theta^\star \leqslant 1.8$. Also, the testimators $\hat{\theta}_{st}^{(2)}$ and $\hat{\theta}_{st}^{(3)}$ have good performance for $0.6 \leqslant \theta^\star \leqslant 1.4$ and $0.8 \leqslant \theta^\star \leqslant 1.4$, respectively.

For all testimators, the relative efficiency attains maximum at the point $\theta^\star = 1$. For fixed m, as the value of α increases, the relative efficiency decreases for the testimators $\hat{\theta}_{st}^{(1)}$, $\hat{\theta}_{st}^{(2)}$ and $\hat{\theta}_{st}^{(3)}$ in $0.6 \leqslant \theta^\star \leqslant 1.8$, $0.6 \leqslant \theta^\star \leqslant 1.4$ and $0.6 \leqslant \theta^\star \leqslant 1.6$, respectively. The shrinkage testimator $\hat{\theta}_{st}^{(1)}$ perform better than other shrinkage testimators when $0.8 \leqslant \theta^\star \leqslant 1.8$.

5 A Real Example

Consider a data set discussed by Dunsmore (1983). A rock crushing machine is kept working as long as the size of the crushed rock is larger than the rocks crushed before. Otherwise it is reset. The following data show the sizes of the crushed rocks up to the third reset of the machine:

```
9.3 0.6 24.4 18.1 6.6 9.0 14.3 6.6 13 2.4 5.6 33.8.
```

The Kolmogorov-Smirnov (K-S) test was used for checking the validity of the exponential distribution based on the parameter $\theta = 11.975$. It is observed that the K-S distance is K-S= 0.2069 with a corresponding p-value= 0.6835, which implies that the exponential distribution have a good fit to the above

Table 1. Relative efficiency between $\hat{\theta}_{st}^{(1)}$ and $c_1 T_m$

		$\theta^\star$							
	α	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
	0.01	0.9833	1.9467	5.0499	9.5744	6.3681	4.0207	3.0760	2.5961
$m = 2$	0.05	0.9026	1.4103	2.4175	3.1995	2.7920	2.2128	1.8635	1.6526
	0.1	0.8814	1.2173	1.7593	2.1167	1.9803	1.7143	1.5224	1.3966
	0.01	0.8309	1.4632	4.1203	10.2077	5.5089	3.1907	2.3946	2.0062
$m = 3$	0.05	0.8107	1.1706	2.2094	3.3140	2.6379	1.9333	1.5840	1.3933
	0.1	0.8189	1.0659	1.6646	2.1656	1.9146	1.5602	1.3526	1.2336
	0.01	0.7866	1.2230	3.4745	10.5717	4.8151	2.6822	2.006	1.6817
$m = 4$	0.05	0.7925	1.0353	2.0250	3.3769	2.4752	1.7296	1.4053	1.2398
	0.1	0.8123	0.9747	1.5738	2.1919	1.8387	1.4400	1.2390	1.1349
	0.01	0.7806	1.0847	3.0158	10.8050	4.2798	2.3430	1.7569	1.4775
$m = 5$	0.05	0.8017	0.9532	1.8710	3.4163	2.3273	1.5795	1.2840	1.1414
	0.1	0.8268	0.9179	1.4933	2.2083	1.7657	1.3474	1.1602	1.0719
	.01	0.7899	0.9978	2.6773	10.9664	3.8625	2.1011	1.5830	1.3377
$m = 6$	.05	0.8209	0.9013	1.7433	3.4431	2.1974	1.4655	1.1974	1.0450
	0.1	0.8484	0.8821	1.4232	2.2193	1.6988	1.2750	1.1037	1.0303

Table 2. Relative efficiency between $\hat{\theta}_{st}^{(2)}$ and $c_1 T_m$

		$\theta^\star$							
	α	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
	0.01	1.1153	2.0508	3.7914	5.1386	4.3388	2.9813	2.0581	1.5007
$m = 2$	0.05	1.0436	1.5365	2.1740	2.5010	2.2775	1.8346	1.4428	1.1547
	0.1	1.0089	1.3239	1.6746	1.8356	1.7266	1.4858	1.2449	1.0494
	0.01	0.9059	1.5886	3.1420	4.6664	3.7103	2.2956	1.4779	1.0343
$m = 3$	0.05	0.9153	1.3220	2.0059	2.4242	2.0983	1.5508	1.1407	0.8760
	0.1	0.9240	1.2009	1.6030	1.8073	1.6307	1.3102	1.0381	0.8454
	0.01	0.8157	1.3430	2.7396	4.3580	3.2849	1.8770	1.1551	0.7903
$m = 4$	0.05	0.8598	1.1885	1.8774	2.3649	1.9540	1.3487	0.9503	0.7159
	0.1	0.8875	1.1175	1.5432	1.7837	1.5489	1.1777	0.9005	0.7234
	0.01	0.7734	1.1911	2.4628	4.1423	2.9709	1.5918	0.9492	0.6412
$m = 5$	0.05	0.8370	1.0984	1.7757	2.3191	1.8352	1.1979	0.8206	0.6142
	0.1	0.8744	1.0581	1.4925	1.7847	1.4789	1.0748	0.8039	0.6450
	0.01	0.7552	1.0882	2.2583	3.9825	2.7251	1.3837	0.8065	0.5414
$m = 6$	0.05	0.8313	1.0342	1.6923	2.2829	1.7347	1.0810	0.7273	0.5454
	0.1	0.8738	1.0144	1.4487	1.7493	1.4180	0.9929	0.7331	0.5922

Table 3. Relative efficiency between $\hat{\theta}_{st}^{(3)}$ and c_1T_m

	α	$\theta^\star$							
		0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
	0.01	0.9855	1.9553	4.1208	6.4123	5.4912	3.6261	2.4394	1.7592
$m=2$	0.05	0.9221	1.4345	2.2102	3.7509	2.5994	2.1127	1.6666	1.3397
	0.1	0.8993	1.2281	1.6566	1.9345	1.8964	1.6674	1.4150	1.2052
	0.01	0.7612	1.4755	3.5033	6.3231	4.8621	2.7946	1.7509	1.2207
$m=3$	0.05	0.7788	1.2001	2.0505	2.7665	2.4691	1.8171	1.3370	1.0344
	0.1	0.8012	1.0861	1.5822	1.9469	1.8379	1.5008	1.2017	0.9899
	0.01	0.6562	1.2086	3.0762	6.2553	4.3426	2.2675	1.3683	0.9416
$m=4$	0.05	0.7121	1.0495	1.9152	2.7706	2.3367	1.5912	1.1238	0.8572
	0.1	0.7568	0.9879	1.5147	1.9522	1.7739	1.3637	1.0537	0.8582
	0.01	0.6008	1.0377	2.7568	6.2048	3.9189	1.9092	1.1279	0.7733
$m=5$	0.05	0.6812	0.9453	1.7998	2.7715	2.2138	1.4183	0.9780	0.7447
	0.1	0.7397	0.9169	1.4540	1.9550	1.7116	1.2528	0.9481	0.7730
	0.01	0.5713	0.9189	2.5059	6.1664	3.5701	1.6512	0.9639	0.6618
$m=6$	0.05	0.6703	0.8697	1.7002	2.7712	2.1023	1.2831	0.8733	0.6689
	0.1	0.7382	0.8641	1.3994	1.9565	1.6529	1.1626	0.8704	0.7157

data. The observed upper record values $\mathbf{r} = (r_1, r_2, r_3)$ are obtained to be

$$\mathbf{r} = (9.3, 24.4, 33.8).$$

Then, the MLE is $T_3 = \frac{r_3}{3} = 11.27$. Also, we have $c_1 = 3e^{-\Psi(3)} = 1.19$, which indicates that a minimum risk unbiased estimator c_1T_3 is 13.44. We consider the estimation of θ when the guess value is $\theta_0 = 11$. Using the ML estimate of θ, the estimate of $\theta^\star$ is $\hat{\theta}^\star = \frac{\theta_0}{T_3} = 0.98$ and therefore the value of shrinkage factor k_1 founded by minimizing the risk of shrinkage estimator $\hat{\theta}_S$ given in (5) is 0.001. The test statistic for testing the null hypothesis $H_0: \ \theta = 11$ is $\chi^2 = 6.15$. If we consider $\alpha = 0.05$, then the left quantiles of a chi-square distribution with 6 degree of freedom are $q_1 = 1.24$ and $q_2 = 14.45$. This implies that the null hypothesis is accepted. Then the values of shrinkage factors k_2 and k_3 are as follows:

$$\begin{aligned} k_2 &= \frac{1}{14.45 - 1.24}\left(\frac{2(3)11.27}{11} - 1.24\right) = 0.37 \\ k_3 &= \frac{2(3)}{14.45 - 1.24}\left|\frac{11.27}{11} - 1\right| = 0.009. \end{aligned}$$

Table 4. Risks and relative efficiency of estimators

estimator	c_1T_3	$\hat{\theta}_{st}^{(1)}$	$\hat{\theta}_{st}^{(2)}$	$\hat{\theta}_{st}^{(3)}$
Risk	0.39493	0.12019	0.15141	0.12021
R.E.		3.28555	2.60817	3.28494

Using the values of shrinkage factors $k_i,\ i = 1, 2, 3$, we obtain the risks of shrinkage testimators $\hat{\theta}_{st}^{(i)}$ given in (11) and relative efficiency of them with respect to c_1T_3 which are summarized in Table 4.

From Table 4, we observe that all of the shrinkage testimators are better than the estimator c_1T_3. Also, the shrinkage testimator $\hat{\theta}_{st}^{(1)}$ is more efficient than other shrinkage testimators, however it is comparable with the testimator $\hat{\theta}_{st}^{(3)}$.

6 Concluding Remarks

The problem of shrinkage testimation under the squared log error loss function on the basis of observed exponential records is considered. Some shrinkage testimators are provided and their risks are computed. Comparisons are made between these testimators and a minimum risk unbiased estimator within the class of estimators of the form cT_m. The results show that the shrinkage testimators are more efficient when the experimenter has a priori that the guess value θ_0 is in the vicinity of θ. Also, the shrinkage testimator $\hat{\theta}_{st}^{(1)}$ corresponding to the shrinkage factor k_1, which founded by minimizing the risk of the shrinkage estimator $\hat{\theta}_S$, performs better than other shrinkage testimators for more values of $\theta^\star = \frac{\theta_0}{\theta}$. Finally, we presented a real data set to illustrate the results. Note that the shrinkage factor k_1 which constructs the testimator $\hat{\theta}_{st}^{(1)}$, depends upon the unknown parameter θ, hence an estimate $\hat{k}_1$ of k_1 can be obtained by replacing the parameter θ to ML estimator.

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