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Testing Skew-Laplace Distribution Using Density-based Empirical Likelihood Approach

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Abstract. In this paper, we first describe the skew-Laplace distribution and its properties. We then introduce a goodness of fit test for this distribution according to the density-based empirical likelihood ratio concept. Asymptotic consistency of the proposed test is demonstrated. The critical values and Type I error of the test are obtained by Monte Carlo simulations. Moreover, the empirical distribution function (EDF) tests are considered for the skew-Laplace distribution to show they do not have acceptable Type I error in comparison with the proposed test. Results show that the proposed test has an excellent Type I error which does not depend on the unknown parameters. The results obtained from simulation studies designed to investigate the power of the test are presented, too. The applicability of the proposed test in practice is demonstrated by real data examples.

Keywords. Density-based empirical likelihood; likelihood ratio; skew-Laplace distribution; goodness of fit tests; Type I error.

MSC 2010: 62G10; 62G20; 62G30.

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1 Introduction

The family of Skew-Laplace (SL) distributions extends the Laplace probability distribution function by adding a shape parameter which is used to model the skewness of the data. Therefore, in practical situations in which some skewness presents, the skew-Laplace distribution is more flexible to model real data. Some main properties of this distribution were investigated by Kotz et al. (2001).

The skew-Laplace distribution has been used in Economics, Engineering, Finance and Biology. Also, it has been used to describe the logarithm of particle sizes, see Fieller et al. (1992). The skew-Laplace distribution has been used to analyze bacterial sizes in axenic cultures by Julia and Vives-Rego (2005). Because of these attractive applications for this distribution, developing a corresponding goodness of fit test with some satisfactory properties is essential.

Some well-known goodness of fit tests are based on comparison between the EDF and the cumulative distribution function (CDF), namely, Cramer-von Mises, Kolmogorov-Smirnov, Anderson-Darling, Kuiper and Watson;(See D' Agostino and Stephens (1986), Chapter 4 for more details.)

Puig and Stephens (2007), applied Anderson-Darling and Cramer-von Mises tests to examine the null hypothesis of a skew-Laplace distribution.

The purpose of this article is to develop an empirical likelihood (EL) approach to the problem of testing skew-Laplace distribution. The maximum empirical likelihood method is a recently developed nonparametric technique by Owen (1988) for conducting estimation and hypothesis testing. This method has several merits. First, it is able to avoid mis-specification problems that can be associated with parametric methods. Second, the use of the empirical likelihood method enables us to fully employ the information available from the data in an asymptotically efficient way.

Vexler and Gurevich (2010) applied the main idea of empirical likelihood method and proposed a density-based empirical likelihood ratio (DBELR) goodness of fit for normality and uniformity. Vexler et al. (2011) used the same method and construct a DBELR goodness of fit test for the Inverse Gaussian distribution. Ning and Ngunkeng (2013) and Safavinejad et al. (2014) followed similar idea to develop a test of fit for skew normal and Rayleigh distributions, respectively. Recently, the empirical likelihood methods have been extended to many statistical problems see for example, Vexler et al. (2011), Vexler et al. (2011), Vexler and Gurevich (2011), Gurevich and Vexler (2011), Shan et al. (2011), Vexler and Yu (2011), Yu et al. (2011), Vexler et al. (2012a), Vexler et al. (2012b), and Vexler et al. (2014b). Attractive applications of the empirical likelihood method in epidemiology are presented in Vexler (2014a).

Vexler and Gurevich (2010) introduced a general test statistic based on ELR and applied it for the special cases of the normal, exponential and uniform distributions. Further, Vexler et al. (2011) used the DBELR test statistic for the special case of the IG distribution and proposed a DBELR test for the IG distribution. Finally, they performed a simulation study about the performance of their test and concluded that the DBELR test for the IG distribution has a good performance. In this article, we used the DBELR statistic for the special case of SL distribution and investigate the properties of the test statistic such as consistency, type-I error and power. Here, we show that the proposed test has an excellent type-I error which does not depend on the unknown parameters. In the present paper, it is shown that the competitor tests do not have an acceptable type-I error and therefore just the DBELR test can be applied in practice. By two real data examples, the applicability of the DBELR test in practice is shown.

In this paper, we utilize the empirical likelihood technique and propose a density-based empirical likelihood ratio test for the skew-Laplace distribution. In addition to developing a new test of fit, we analyze its sampling properties by undertaking a detailed power comparison of the DBELR test and five other competitors. The outline of this paper is as follows. In Section 2, we express some properties of skew-Laplace distribution. The DBELR test statistic will be presented too. Section 3 is devoted to Monte Carlo simulation study and associated results. In Section 4, applicability of the proposed test with real data examples is shown. Section 5 provides a brief conclusion.

2 Statistical Methodology

A random variable X has a skew-Laplace distribution with parameters θ_1, θ_2 and μ denoted by $X \sim SL(\theta_1, \theta_2, \mu)$, if its probability density function (pdf) has the following form

$$f(x;\theta_1,\theta_2,\mu) = \begin{cases} \dfrac{1}{\theta_1+\theta_2}\exp\{\frac{x-\mu}{\theta_1}\}, & x \leqslant \mu, \\ \dfrac{1}{\theta_1+\theta_2}\exp\{-\frac{x-\mu}{\theta_2}\}, & x > \mu, \end{cases} \tag{1}$$

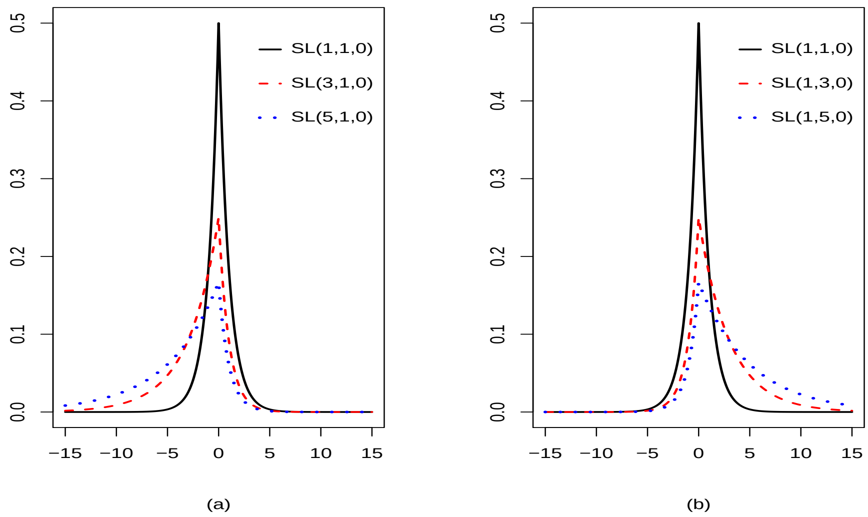

Figure 1. PDF of skew-Laplace distribution when (a) $\theta_1 \geqslant \theta_2$ and (b) $\theta_1 \leqslant \theta_2$.

where $\theta_1, \theta_2 > 0$ and μ is a real number. Hence, the distribution function of X is

$$F(x;\theta_1,\theta_2,\mu) = \begin{cases} \dfrac{\theta_1}{\theta_1+\theta_2}\exp\{\dfrac{x-\mu}{\theta_1}\}, & x \leqslant \mu, \\ 1 - \dfrac{\theta_2}{\theta_1+\theta_2}\exp\{-\dfrac{x-\mu}{\theta_2}\}, & x > \mu. \end{cases} \quad (2)$$

If θ_1 is greater than θ_2, it suggests that the right tails are thinner and thus, there is less population to the right side of μ than to the left. If θ_2 is greater than θ_1 then the left tails are thinner and thus, there exists less population to the left side of μ than to the right. If $\theta_1 = \theta_2$, the distribution is the classical symmetric Laplace pdf. When θ_1 or θ_2 tends to 0, then $f(x;\theta_1,\theta_2,\mu)$ tends to the two parameter exponential or negative-exponential distribution. This situations are exhibited in Figure 1.

The moment generating function of a random variable X with density function (1) is as follows.

$$M(t) = \frac{\exp(\mu t)}{(1+\theta_1 t)(1-\theta_2 t)}. \tag{3}$$

Thus, the cumulant generating function has the following form.

$$K(t) = \mu t - \log(1+\theta_1 t) - \log(1-\theta_2 t). \tag{4}$$

According to (3) and (4), the mean, variance, skewness and kurtosis of the skew-Laplace distribution are derived as

$$E(X) = \mu + \theta_2 - \theta_1,$$

$$Var(X) = \theta_1^2 + \theta_2^2,$$

$$Ske(X) = \frac{2(\theta_2^3 - \theta_1^3)}{(\theta_1^2+\theta_2^2)^{3/2}},$$

$$Kur(X) = 3 + \frac{6(\theta_2^4 + \theta_1^4)}{(\theta_1^2+\theta_2^2)^2}.$$

The following theorem gives the maximum likelihood estimator (MLE) of the unknown parameters. This theorem was presented by Puig and Stephens (2007).

Theorem 1. *Let $X_1, \dots, X_n$ be a given random sample from skew-Laplace distribution with parameters θ_1, θ_2 and μ. The MLEs of the parameters are derived as*

$$\hat{\mu} = X_j,$$

$$\hat{\theta}_1 = \frac{1}{2}\left\{\Delta(\hat{\mu}) - \bar{X} + \hat{\mu} + \sqrt{\Delta^2(\hat{\mu}) - (\bar{X}-\hat{\mu})^2}\right\},$$

$$\hat{\theta}_2 = \frac{1}{2}\left\{\Delta(\hat{\mu}) + \bar{X} - \hat{\mu} + \sqrt{\Delta^2(\hat{\mu}) - (\bar{X}-\hat{\mu})^2}\right\},$$

where $\Delta(\mu) = \sum_{i=1}^n |X_i - \mu|/n$ and X_j is any sample value where the function $\psi(\mu) = \Delta(\mu) + \sqrt{\Delta^2(\mu) - (\bar{X}-\mu)^2}$ attains its unique minimum.

Proof. See the proof of Theorem 1 by Puig and Stephens (2007). □

2.1 The DBELR Test Statistic

Suppose $X_1, \ldots, X_n$ be independent and identically distributed random variables from a distribution function F with density function f. Our aim is to test the following hypothesis

$$H_0 : f(x) = f(x, \theta_1, \theta_2, \mu),$$

versus

$$H_1 : f(x) \neq f(x, \theta_1, \theta_2, \mu).$$

In goodness of fit test problems, the density function under the alternative hypothesis is unknown, and we must estimate it. In this paper, we use the main idea of classical EL method developed by Owen (1988), Owen (1990) to approximate the most powerful test statistics stated by the Neyman-Pearson Lemma.

This likelihood methodology is based on cumulative distribution function. However, according to the Neyman-Pearson Lemma, the most powerful tests depend on density functions. Some modifications are therefore essential for applying this method.

Following the idea of Vexler and Gurevich (2010), we rewrite the EL function under H_1 as follows,

$$L_f = \prod_{i=1}^{n} f_{H_1}(X_i) = \prod_{i=1}^{n} f_{H_1}(X_{(i)}) = \prod_{i=1}^{n} f_i, \tag{5}$$

where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are the order statistics based on $X_1, \ldots, X_n$.

According to EL method mentioned above, we must derive values of f_is that maximize L_f subject to constraint $\int f(u)du = 1$, corresponding to the alternative hypothesis. First, We express the following lemma by Vexler and Gurevich (2010) to formalize this constraint.

***Lemma* 1.** *Let $f(x)$ be a density function, then*

$$\sum_{j=1}^{n}\int_{X_{(j-m)}}^{X_{(j+m)}} f(x)dx = 2m\int_{X_{(1)}}^{X_{(n)}} f(x)dx - \sum_{k=1}^{m-1}(m-k)\int_{X_{(n-k)}}^{X_{(n-k+1)}} f(x)dx - \sum_{k=1}^{m-1}(m-k)\int_{X_{(k)}}^{X_{(k+1)}} f(x)dx, \qquad (6)$$

where $X_{(j-m)} = X_{(1)}$, if $j-m \leqslant 1$, $X_{(j+m)} = X_{(n)}$, if $j+m \geqslant n$.

Proof. See the proof of Proposition 2.1 by Vexler and Gurevich (2010). □

Since $\int_{X_{(1)}}^{X_{(n)}} f(x)dx < \int_{-\infty}^{+\infty} f(x)dx$, Lemma 1. shows that,

$$\Delta_m = \frac{1}{2m}\sum_{j=1}^{n}\int_{X_{(j-m)}}^{X_{(j+m)}} f(x)dx \leqslant 1.$$

To approximate condition $\Delta_m \leqslant 1$, we can apply the mean value theorem to the integration. Hence, the empirical form of $\Delta_m \leqslant 1$ is obtained with

$$\hat{\Delta}_m = \frac{1}{2m}\sum_{j=1}^{n}(X_{(j+m)} - X_{(j-m)})f_j. \qquad (7)$$

Consequently, we must obtain values of f_js that maximize (5) under empirical condition (7). By applying Lagrange multiplier method, f_js are derived as follows.

$$f_j = \frac{2m}{n(X_{(j+m)} - X_{(j-m)})}, \qquad j = 1, \ldots, n.$$

Utilizing the density based EL technique, the DBELR test statistic can be constructed as follows

$$T_{mn} = \frac{\prod_{j=1}^{n} \frac{2m}{n(X_{(j+m)} - X_{(j-m)})}}{\prod_{i=1}^{n} f_{H_0}(X_i;\underline{\theta})},$$

where $\underline{\theta} = (\theta_1, \theta_2, \mu)$.

Following the algorithm of Theorem 1, we can apply the maximum likelihood method to estimate the unknown parameters θ_1, θ_2 or μ. Therefore, the DBELR test statistic is

$$T_{mn} = \frac{\prod_{j=1}^{n} \dfrac{2m}{n(X_{(j+m)} - X_{(j-m)})}}{\prod_{i=1}^{n} f_{H_0}(X_i; \hat{\underline{\theta}})}, \tag{8}$$

where $\hat{\underline{\theta}}$ is the vector of maximum likelihood estimator of $\underline{\theta}$.

The test statistic T_{mn} depends on the values of m. In practice, if we want to use the test statistic $T_m n$, a guide for the choice of m for a fixed n would be needed to users. The problem of the choice of the optimal value of m is still an open problem and different values of m may be have different results. It is well known that the optimal value of m is depend on the considered alternative and we can not determine an optimal m for all alternatives. This problem restricts the applicability of the proposed test statistic to real data applications. To solve this problem, we use the idea of Vexler and Gurevich (2010), and modify our test as follows,

$$T_n = \frac{\min_{1 \leqslant m < n^{\delta}} \prod_{j=1}^{n} \dfrac{2m}{n(X_{(j+m)} - X_{(j-m)})}}{\prod_{i=1}^{n} f_{H_0}(X_i; \hat{\underline{\theta}})}, \tag{9}$$

where $0 < \delta < 1$. The null hypothesis will be rejected if

$$\log(T_n) > C_\alpha, \tag{10}$$

where C_α is a test-threshold.

2.2 Asymptotic Properties

To examine asymptotic properties of the test statistic (9) under the null and alternative hypotheses, first we define

$$g_i(x, \underline{\theta}) = \frac{\partial}{\partial \theta_i} \log f_{H_0}(x; \underline{\theta}), \qquad i = 1, 2, 3.$$

We assume the following conditions hold:
(A1) $E(\log f(X_1))^2 < \infty$.
(A2) Under the null hypothesis, $\mid \hat{\underline{\theta}} - \underline{\theta} \mid \xrightarrow{P} 0$, as $n \longrightarrow \infty$, where

$$\mid \hat{\underline{\theta}} - \theta \mid = \max \left(|\hat{\theta}_1 - \theta_1|, |\hat{\theta}_2 - \theta_2|, |\hat{\mu} - \mu| \right).$$

(A3) Under alternative hypothesis, $\mid \hat{\underline{\theta}} - \underline{a} \mid \xrightarrow{P} 0$, as $n \longrightarrow \infty$, where $\underline{a}$ is a vector with finite components.
(A4) There are open intervals $\Theta_0 \subseteq \mathcal{R}^3$, and $\Theta_a \subseteq \mathcal{R}^3$, containing $\underline{\theta}$ and $\underline{a}$ respectively. There also exists a function $h(x)$ such that $\mid g_i(x, \eta) \mid \leqslant h(x)$ for all $x \in \mathcal{R}$ and $\eta \in \Theta_0 \cup \Theta_a$ and $E\left(h(X_1)\right) < \infty$.

***Theorem* 2.** *Assume that* (**A1**) $-$ (**A4**) *hold. Then, under* H_0*,*

$$\frac{1}{n} \log(T_n) \xrightarrow{P} 0,$$

and, under H_1*,*

$$\frac{1}{n} \log(T_n) \xrightarrow{P} E \log \left\{ \frac{f_{H_1}(X_1)}{f_{H_0}(X_1; \underline{a})} \right\} > 0,$$

as $n \longrightarrow \infty$.

Proof. The proof of this theorem is similar to Proposition 2.2 of Vexler and Gurevich (2010) and is therefore omitted. □

3 Simulation Study

3.1 Competitor Tests

The EDF tests are those based on a measure of discrepancy between the empirical and hypothesized distribution. These tests can be further subdivided into those belong to supremum and square class of the discrepancies. The most crucial and widely known EDF tests are Kolmogorov-Smirnov, Anderson-Darling, Cramer-von Mises, Kuiper and Watson tests. Let F be the hypothesized distribution and $Z_{(i)} = F(X_{(i)})$, where $X_{(i)}$ is the ith order statistic of a random sample of size n. The competing test statistics based on EDF are as follows.

1. Cramer-von Mises test statistic W^2,

$$W^2 = \sum_{i=1}^{n} \left(Z_{(i)} - \frac{2i-1}{2n} \right)^2 + \frac{1}{12n}. \tag{11}$$

2. Kolmogorov-Smirnov test statistic D,

$$D = \max(D^+, D^-), \tag{12}$$

where

$$D^+ = \max_{1 \leqslant i \leqslant n} \left\{ \frac{i}{n} - Z_{(i)} \right\}, \qquad D^- = \max_{1 \leqslant i \leqslant n} \left\{ Z_{(i)} - \frac{i-1}{n} \right\},$$

3. Anderson-Darling test statistic A^2,

$$A^2 = -\frac{1}{n} \sum_{i=1}^{n} (2i-1) \big[\log(Z_{(i)}) + \log(1 - Z_{(n+1-i)}) \big] - n. \tag{13}$$

4. Watson test statistic U^2,

$$U^2 = W^2 - n \left(\bar{Z} - \frac{1}{2} \right)^2, \tag{14}$$

where W^2 is Cramer-von Mises test statistic and $\bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_{(i)}$.

5. Kuiper test statistic V,

$$V = D^+ + D^-. \tag{15}$$

3.2 Critical values and Type I error

Deriving the exact distribution of the test statistic T_n under the null hypothesis is complicated. Thus, we have used Monte Carlo simulations to obtain the critical values of our test. We conduct a simulation study with 50000 replications under the skew-Laplace distribution with parameters

Table 1. Critical values of $\log(T_n)$ statistic.

	α							
n	0.01	0.025	0.05	0.1	0.125	0.15	0.25	0.3
5	2.242	2.180	2.104	1.946	1.861	1.820	1.672	1.620
10	3.946	3.657	3.425	3.168	3.075	3.013	2.763	2.667
15	5.350	4.965	4.609	4.204	4.066	3.960	3.605	3.463
20	6.486	5.941	5.480	4.971	4.831	4.657	4.178	4.007
25	7.396	6.719	6.177	5.597	5.386	5.177	4.645	4.419
30	8.153	7.382	6.818	6.118	5.877	5.675	5.039	4.804
35	8.871	7.968	7.327	6.567	6.343	6.064	5.386	5.136
40	9.327	8.358	7.663	6.790	6.534	6.278	5.558	5.249
45	9.857	8.806	8.044	7.141	6.817	6.561	5.770	5.449
50	10.026	9.045	8.241	7.278	6.965	6.694	5.842	5.499

$\theta_1 = 1, \theta_2 = 1, \mu = 0$ and choose $(1-\alpha)$th percentiles to be the critical values corresponding to the significance level α. The critical values of the proposed test statistic for different values of sample size and significance level α are presented in Table 1.

It is obvious that the proposed test statistic is invariant with respect to the location transformations, thus the empirical distribution of the proposed test just depends on the values of θ_1 and θ_2 and the sample size n. Hence, the test is not accurate and critical values depend on the parameters θ_1 and θ_2. As a consequence, the actual size of the proposed test differs from the nominal significance level. To investigate the difference between the actual size of the test and the nominal significance level $\alpha = 0.05$, we have simulated 30000 samples of size $n = 10, 20, 30, 40, 50$ from the $SL(\theta_1, \theta_2, 0)$ with different values of θ_1 and θ_2. For each sample, the test statistic is calculated to compare with the corresponding critical value. The percentage of rejecting the null hypothesis will be the actual size of the proposed test. The results of this simulation study are presented in Table 2. We observe that the actual size of the proposed test is acceptable, specially, for large value of n. Therefore, the proposed test can be used in practice.

In order to obtain the actual size of the competitor tests, 30000 repetitions of the samples from $SL(\theta_1, \theta_2, 0)$ were simulated for different values of θ_1 and θ_2. For each sample, we compute the test statistics based on the equations (11)-(15). The results of this simulation study are presented in Tables 3-7.

Table 2. The actual sizes of T_n test at nominal significance level $\alpha = 0.05$.

		n				
θ_1	θ_2	10	20	30	40	50
0.5	0.5	0.051	0.051	0.052	0.050	0.050
1	1	0.050	0.050	0.053	0.050	0.052
2	2	0.051	0.052	0.052	0.049	0.054
5	5	0.051	0.049	0.050	0.051	0.051
1	2	0.066	0.052	0.051	0.046	0.050
1	5	0.092	0.067	0.049	0.041	0.048
2	0.5	0.084	0.063	0.050	0.047	0.048
2	5	0.067	0.058	0.050	0.047	0.050
3	2	0.056	0.052	0.052	0.049	0.051
3	5	0.056	0.053	0.050	0.048	0.051
5	2	0.069	0.058	0.051	0.048	0.049
8	5	0.054	0.050	0.052	0.048	0.052
5	10	0.062	0.053	0.049	0.047	0.051

Table 3. The actual sizes of D test at nominal significance level $\alpha = 0.05$.

		n				
θ_1	θ_2	10	20	30	40	50
0.5	0.5	0.051	0.047	0.049	0.050	0.048
1	1	0.049	0.050	0.048	0.052	0.048
2	2	0.048	0.050	0.047	0.050	0.050
5	5	0.047	0.045	0.047	0.051	0.049
1	2	0.061	0.069	0.074	0.080	0.080
1	5	0.090	0.114	0.129	0.147	0.148
2	0.5	0.080	0.106	0.116	0.130	0.136
2	5	0.062	0.079	0.085	0.097	0.101
3	2	0.054	0.055	0.056	0.061	0.060
3	5	0.051	0.058	0.057	0.070	0.069
5	2	0.061	0.079	0.088	0.096	0.097
8	5	0.054	0.057	0.059	0.064	0.068
5	10	0.055	0.065	0.074	0.077	0.084

Table 4. The actual sizes of V test at nominal significance level $\alpha = 0.05$.

		n				
θ_1	θ_2	10	20	30	40	50
0.5	0.5	0.051	0.048	0.051	0.053	0.048
1	1	0.049	0.050	0.048	0.051	0.049
2	2	0.048	0.050	0.047	0.050	0.051
5	5	0.046	0.047	0.047	0.050	0.051
1	2	0.060	0.069	0.074	0.075	0.074
1	5	0.090	0.114	0.130	0.142	0.144
2	0.5	0.078	0.104	0.114	0.127	0.129
2	5	0.061	0.078	0.083	0.090	0.093
3	2	0.053	0.056	0.058	0.059	0.058
3	5	0.052	0.058	0.060	0.065	0.066
5	2	0.062	0.080	0.085	0.092	0.089
8	5	0.051	0.055	0.062	0.062	0.065
5	10	0.054	0.064	0.069	0.072	0.078

Table 5. The actual sizes of A^2 test at nominal significance level $\alpha = 0.05$.

		n				
θ_1	θ_2	10	20	30	40	50
0.5	0.5	0.050	0.049	0.05	0.050	0.049
1	1	0.050	0.050	0.050	.051	0.047
2	2	0.049	0.050	0.049	0.050	0.051
5	5	0.041	0.045	0.048	0.051	0.048
1	2	0.063	0.082	0.079	0.073	0.072
1	5	0.116	0.179	0.159	0.154	0.147
2	0.5	0.094	0.153	0.132	0.128	0.132
2	5	0.064	0.101	0.090	0.089	0.092
3	2	0.050	0.060	0.057	0.059	0.057
3	5	0.048	0.063	0.060	0.064	0.062
5	2	0.066	0.103	0.092	0.090	0.086
8	5	0.050	0.060	0.061	0.060	0.063
5	10	0.049	0.073	0.074	0.070	0.074

Table 6. The actual sizes of U^2 test at nominal significance level $\alpha = 0.05$.

		n				
θ_1	θ_2	10	20	30	40	50
0.5	0.5	0.051	0.048	0.050	0.052	0.048
1	1	0.052	0.049	0.050	0.053	0.048
2	2	0.049	0.050	0.049	0.053	0.052
5	5	0.046	0.046	0.047	0.052	0.051
1	2	0.062	0.070	0.078	0.081	0.080
1	5	0.093	0.122	0.147	0.159	0.165
2	0.5	0.081	0.112	0.127	0.142	0.147
2	5	0.061	0.081	0.089	0.100	0.105
3	2	0.053	0.057	0.059	0.062	0.061
3	5	0.051	0.058	0.061	0.069	0.069
5	2	0.063	0.084	0.092	0.101	0.102
8	5	0.050	0.055	0.062	0.064	0.068
5	10	0.054	0.066	0.075	0.078	0.085

Table 7. The actual sizes of W^2 test at nominal significance level $\alpha = 0.05$.

		n				
θ_1	θ_2	10	20	30	40	50
0.5	0.5	0.051	0.048	0.050	0.052	0.048
1	1	0.051	0.050	0.049	0.052	0.048
2	2	0.050	0.050	0.049	0.050	0.051
5	5	0.046	0.045	0.048	0.052	0.049
1	2	0.064	0.076	0.083	0.086	0.087
1	5	0.103	0.136	0.160	0.178	0.186
2	0.5	0.088	0.123	0.138	0.154	0.166
2	5	0.065	0.089	0.095	0.106	0.115
3	2	0.054	0.058	0.059	0.063	0.064
3	5	0.050	0.061	0.062	0.071	0.071
5	2	0.066	0.090	0.098	0.107	0.108
8	5	0.053	0.058	0.063	0.065	0.072
5	10	0.058	0.070	0.079	0.081	0.091

It can be seen that when the parameters θ_1 and θ_2 are equal, the true size of the competitor tests is approximately matches the nominal significance level $\alpha = 0.05$. But in the case that the parameters θ_1 and θ_2 are not equal, the actual size of the competitor tests is mostly greater than the nominal significance level and the difference between the actual sizes and the nominal sizes are substantial. Hence, the competitor tests can not control well the Type I error. Consequently, we can not recommend these tests for examining the skew-Laplace distribution. The superiority of our test to its competitors is to have an acceptable actual sizes which makes it to be useful in practice.

3.3 Power Study

In this section, since the proposed test can control well the Type I error, we conduct a simulation study to investigate the power of the proposed test against various alternative distributions. But since the competitor tests do not control the Type I error, power of these tests are not acceptable and we can not use these tests in power comparison. We consider the following alternatives in our power study. These alternatives can be classified into two groups; symmetric and asymmetric alternatives.

- *Symmetric alternatives:*
  1. Laplace distribution, denoted by $Lap(a, b)$,
  2. Normal distribution, denoted by $N(\mu, \sigma)$,
  3. Logistic distribution, denoted by $Logis(a, b)$,
  4. Uniform distribution, denoted by $U(a, b)$,
  5. Beta distribution, denoted by $B(a, b)$,
  6. Student distribution, denoted by $t(n)$,
  7. Cauchy distribution, denoted by $C(\mu, \sigma)$.

- *Asymmetric alternatives:*
  1. Exponential distribution, denoted by $\text{Exp}(\theta)$,

2. Gamma distribution, denoted by $\Gamma(\mu, \lambda)$,
3. Lognormal distribution, denoted by $LN(\mu, \sigma)$,
4. Weibull distribution, denoted by $W(\theta, \lambda)$,
5. Gumbel distribution denoted by $EV(a, b)$,
6. Inverse Gaussian distribution, denoted by $IG(\mu, \lambda)$,
7. Skew normal distribution, denoted by $SN(\mu, \sigma, \lambda)$,
8. Beta distribution, denoted by $B(a, b)$.

In order to obtain the power of the proposed test, 30000 repetitions of samples with size $n = 20, 30$ and 40 were simulated from the mentioned alternative distributions to compare the values of the test statistic with the critical values corresponding to significance level $\alpha = 0.05$. The simulated results are given in Table 8.

It is evident from Table 8 that the proposed test has a good power against Beta family, which indicates the proposed test is more sensitive to distinguish a Beta distribution from a skew Laplace distribution. We can see that the power of our test against the skew normal distribution is also good. Generally, we can conclude that the proposed test has a good power against some alternatives such as Beta and Weibull(0.5), and against some alternatives such as exponential and t(3), its power is low.

Since the proposed test statistic (9) includes $\delta \in (0, 1)$, we take $\delta = 0.5$ in power comparisons. Moreover, we investigate the performance of the proposed test for different values of δ. A selection of results that obtained in this power study are given in Table 9. It shows that the power values of the test do not significantly depend on the different values of δ. Because, the operator min in the test statistic T_n has the functional ability to find the preferable values of parameter m that in turn is a component of the test statistic T_{mn}. According to the argument of Vexler et al. (2011), we can assume that the preferable values of m are mostly located below or around $n^{0.5}$.

Table 8. Powers of the proposed test against various alternatives at significance level $\alpha = 0.05$.

		n		
	Alternatives	20	30	40
	Lap(0, 1)	0.050	0.050	0.052
	N(0, 1)	0.193	0.284	0.375
	Logis(0, 1)	0.115	0.143	0.174
Symmetric	U(0, 1)	0.690	0.929	0.992
	B(2, 2)	0.471	0.720	0.882
	t(10)	0.140	0.171	0.217
	t(3)	0.056	0.051	0.048
	C(0, 1)	0.179	0.286	0.383
	Exp(1)	0.083	0.055	0.053
	Γ(0.5, 1)	0.364	0.462	0.567
	Γ(2, 1)	0.131	0.137	0.170
	LN(0, 1)	0.071	0.056	0.057
	W(0.5, 1)	0.772	0.898	0.959
	W(2, 1)	0.239	0.346	0.473
	IG(1, 0.5)	0.154	0.169	0.209
Asymmetric	IG(1, 1)	0.054	0.038	0.037
	IG(1, 2)	0.077	0.063	0.067
	SN(0, 1, 0.5)	0.197	0.286	0.374
	SN(0, 1, 1)	0.198	0.279	0.354
	SN(0, 1, 2)	0.186	0.250	0.329
	EV(0, 1)	0.137	0.162	0.209
	B(2, 1)	0.339	0.473	0.633
	B(3, 2)	0.389	0.582	0.762

3.4 Approximating the p-value

In this section, through the Monte Carlo simulations, we provide the following empirical procedure to approximate the p-value of T_n.

1. Find the MLEs of the parameters $(\hat{\theta}_1, \hat{\theta}_2, \hat{\mu})$ following the algorithm of Theorem 1 based on original data $x_1, \ldots, x_n$.

2. Simulate the sample $y_1, \ldots, y_n$ from $SL(\hat{\theta}_1, \hat{\theta}_2, 0)$, due to the sampling distribution of the test statistic being invariant to the changes of the location parameter.

Table 9. Power evaluation of the statistic T_n with different values of δ at level $\alpha = 0.05$.

Distribution	n	$\delta = 0.4$	$\delta = 0.5$	$\delta = 0.6$
t(3)	20	0.051	0.056	0.053
	30	0.050	0.051	0.052
	40	0.048	0.048	0.049
Logis(0, 1)	20	0.101	0.115	0.119
	30	0.109	0.143	0.151
	40	0.144	0.174	0.182
LN(0, 1)	20	0.096	0.071	0.037
	30	0.094	0.056	0.034
	40	0.096	0.057	0.028
IG(1, 2)	20	0.076	0.077	0.073
	30	0.055	0.063	0.066
	40	0.056	0.067	0.066

3. Calculate the test statistic T_n based on $x_1, \ldots, x_n$ and denote it by T_n^1.
4. Calculate the test statistic T_n based on $y_1, \ldots, y_n$ and denote it by T_n^{1B}.
5. Repeat the step 4, N times to obtain $T_n^{1B}, \ldots, T_n^{NB}$.
6. The approximated p-value (ap-value) of T_n will be obtained as follows

$$ap - value = \frac{1}{N} \sum_{i=1}^{N} I\left(T_n^{iB} > T_n^1\right),$$

where $I(\cdot)$ is the indicator function.

4 Data Examples

In this section, we use the DBELR goodness of fit test given in (9) to examine the skew-Laplace distribution to the real data.

Example 1. Table 10 includes the Otis IQ scores for 52 non-white males hired by a large insurance company in 1971. This data set is taken from

Table 10. The Otis IQ scores for 52 non-white male.

91	102	100	117	122	115	97	109	108	104	108	118	103	123
123	103	106	102	118	100	103	107	108	107	97	95	119	102
108	103	102	112	99	116	114	102	111	104	122	103	111	101
91	99	121	97	109	106	102	104	107	95				

Table 11. Critical value, test statistic, and the approximated p-value.

	Critical value	test statistic	ap-value
T_n	8.310	10.936	0.006

Robert (1988).

Figure 2, depicts the histogram of the IQ scores with a skew-Laplace fit. Using Theorem 1, the MLE of the unknown parameters of an assumed skew-Laplace distribution for the data are

$$\hat{\theta}_1 = 3.832, \quad \hat{\theta}_2 = 8.486, \quad \hat{\mu} = 102.000.$$

The critical value, the p-value and The value of the proposed test statistic T_n, are presented in Table 11. The approximated p-value of the proposed test is 0.006 and the null hypothesis that the IQ scores follow a skew-Laplace distribution is therefore rejected.

Example 2. Table 12 presents the 50 Australian female athletes body mass index (BMI) which are taken from Australian Institute of Sport data by Cook and Weisberg (1994). Histogram of the considered data set is presented in Figure 3.

Following Theorem 1, the MLE of the unknown parameters of skew-Laplace distribution are obtained as

$$\hat{\theta}_1 = 1.512, \quad \hat{\theta}_2 = 3.258, \quad \hat{\mu} = 20.120.$$

According to the approximated p-value of the test statistc T_n given in Table 13, the skew-Laplace assumption is not rejected at significance level of 0.05. Hence, the skew-Laplace distribution can provide a reasonable fitting for the data.

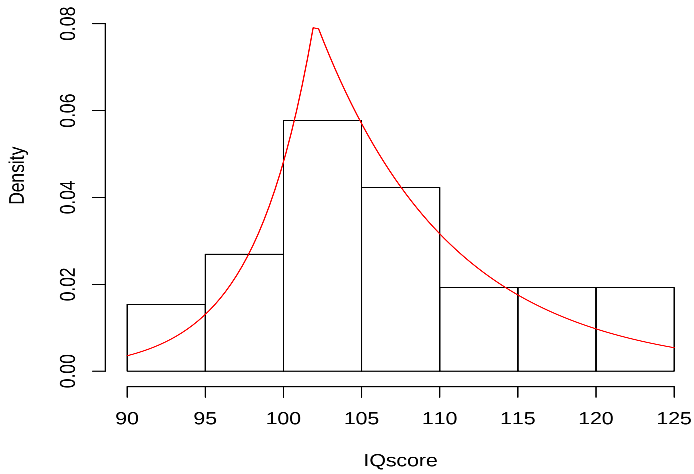

Figure 2. Histogram of the IQ scores with a fitted skew-Laplace density function.

Table 12. The BMI values for the 50 females.

24.47	23.99	26.24	20.04	25.72	25.64	19.87	23.35	22.42	20.42
22.13	25.17	23.72	21.28	20.87	19.00	22.04	20.12	21.35	28.57
26.95	28.13	26.85	25.27	31.93	16.75	19.54	20.42	22.76	20.12
22.35	19.16	20.77	19.37	22.37	17.54	19.06	20.30	20.15	25.36
22.12	21.25	20.53	17.06	18.29	18.37	18.93	17.79	17.05	20.31

Table 13. Critical value, test statistic, and the approximated p-value.

	Critical value	test statistic	ap-value
T_n	8.148	6.823	0.144

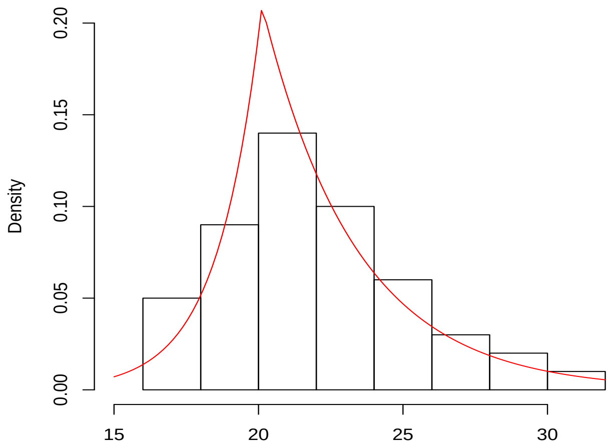

Figure 3. Histogram of body mass index (BMI) of 50 females with a fitted skew-Laplace density function.

5 Conclusions

In this paper, we proposed a density-based empirical likelihood ratio goodness of fit test for the skew-Laplace distribution. In the context of parametric statistical inference, according to the Neyman-Pearson Lemma, the likelihood ratio test is a uniformly most powerful test. But in goodness of fit test problems, the distribution under alternative hypothesis is unknown and needed to be estimated. In this paper, we used the idea of the empirical likelihood technique to estimate the alternative distribution and constructed a DBELR test for the skew-Laplace distribution. We also demonstrated the asymptotic consistency of the proposed test. Through the simulation study, we obtain the Type I error of the proposed test and show that the proposed test can control well the Type I error for a given nominal level. Moreover, we showed that the competitor tests can not control well the Type I error and difference between the actual size and the nominal size of these tests is substantial and therefore we can not use these tests in practice. Note that the superiority of our test to its competitors is to have an acceptable actual sizes which makes it to be useful in practice. We then obtain the power values of our test against the two group of alternatives; symmetric and asymmetric; and showed that it has an acceptable power. Finally, we illustrated how the proposed test can be applied in real examples.

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