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# Prediction of Times to Failure of Censored Units in Progressive Hybrid Censored Samples for the Proportional Hazards Family

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**Abstract.** In this paper, the problem of predicting times to failure of units censored in multiple stages of progressively hybrid censoring for the proportional hazards family is considered. We discuss different classical predictors. The best unbiased predictor ($BUP$), the maximum likelihood predictor ($MLP$) and conditional median predictor ($CMP$) are all derived. As an example, the obtained results are computed for exponential distribution. A numerical example is presented to illustrate the prediction methods discussed here. Using simulation studies, the predictors are compared in terms of bias and mean squared prediction error ($MSPE$).

**Keywords.** Best unbiased predictor; conditional median predictor; maximum likelihood predictor; mean square prediction error; Monte-Carlo simulation; point predictor; progressive hybrid censoring.

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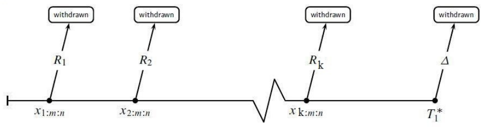


**Figure 1.** Generation process of Type-I progressive hybrid censored order statistics

# 1 Introduction

Quite often, survival data come in a form called "censoring" which occurs when exact survival times are known only for a portion of individuals or units under study. In this paper, we focus on progressive hybrid censoring. Kundu and Joarder (2006) and Childs et al. (2008) proposed, respectively, Type-I and Type-II progressive hybrid censoring procedure by introducing stopping time $T^*$ to a progressive Type-II censored experiment. The termination times are defined by a given (fixed) threshold time $T$ as follows:

(i) $T_1^* = \min\{X_{m:m:n}, T\}$, this procedure is called Type-I progressive hybrid censoring scheme, where $X_{i:m:n}$ is the $i$th progressively Type-II censored order statistic from a sample of size $n$ with censoring scheme $(R_1, R_2, \ldots, R_m)$ and prefixed number of removals $m$. In addition $R_i$ is the number of units that are randomly withdrawn from surviving units in the $i$th stage of censoring. The life testing experiment is stopped when either $m$ failure have been observed or the threshold time $T$ has been exceeded. The number of observations may be zero (when $X_{1:m:n} > T$), see Kundu and Joarder (2006).

(ii) $T_2^* = \max\{X_{m:m:n}, T\}$, this procedure is called Type-II progressive hybrid censoring scheme. The number of observation is between $m$ and $R_m + m$.

For Type-II censored data the first stopping point has been proposed by Epstein (1954) and the second one by Childs et al. (2008). According to the above setting, the number of observation is random. In particular, it's possible to have less than $m$ observations in Type-I progressive hybrid censoring, while we will have at least $m$ observations in Type-II progressive hybrid censoring. In the set up of Type-I progressive hybrid censoring, the life testing experiment is stopped when either $m$ failures have been observed

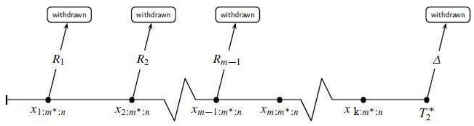


**Figure 2.** Generation process of Type-II progressive hybrid censored order statistics

or threshold time $T$ has been exceeded. Figure 1 depicts the generation procedure of Type-I progressive hybrid censored order statistics. The random variable $\Delta$ represent the removals at the termination time. It's given by

$$\Delta = \begin{cases} R_m, & X_{m:m:n} \leqslant T, \\ n - k - R_1 - \cdots - R_k, & X_{m:m:n} \geqslant T. \end{cases}$$

It is worthwhile to mention that the number of observations may be zero, i.e., for the case when $X_{1:m:n} \geqslant T$. As mentioned before, in Type-II progressive hybrid censoring the number of observations is at least $m$. In fact, more precisely, it is between $m$ and $R_m + m$. The idea of this procedure is to guarantee a minimum number of $m$ observations as well as to come as close as possible to a minimum test duration specified by $T$. If $X_{m:m:n} \geqslant T$, the experiment terminates as the $m$th failure so that the progressive censoring procedure is carried out as initially planned. For $X_{m:m:n} \leqslant T$, we want to come as close as possible from below to the threshold $T$. This means that after the $mth$ failure, all occurring failures are observed until the threshold $T$ is exceeded. Therefore, the censoring scheme is modified as follows:

$$R^* = (R_1, R_2, \ldots, R_{m-1}, 0^{*R_m+1}) \in \zeta_{R_m+m,n}^{R_m+n} \quad ;$$

where $\zeta_{R_m,n}^{R_m}$ is the set of all admissible (Type-II) censoring schemes as

$$\zeta_{R_m,n}^{R_m} = \{(R_1, R_2, \ldots, R_m) \in \mathbb{N}_0^m \ ; \ \Sigma_{i=1}^m R_i = n - m\}.$$

and the notation $0^{*k}$ is used for $k$ successive zeros.
The resulting sample is given by $X_{1:R_m+m:n}^{R^*}, \ldots, X_{k:R_m+m:n}^{R^*}$ where $k$ stands with the inequality $X_{k:R_m+m:n}^{R^*} \leqslant T < X_{k+1:R_m+m:n}^{R^*}, X_{R_m+m+1:R_m+m:n}^{R^*} = \infty$. Figure 2 depicts the generation procedure of Type-II progressive hybrid censored order statistics, where $m^* = R_m + m$ and $\Delta$ is defined as $n - k -$

$\sum_{j=1}^{m-1} R_j$.
A short review on progressive hybrid censoring, distributions and properties has been provided by Balakrishnan and Kundu (2013). For more details of progressive hybrid censoring readers are referred to, for example, Balakrishnan and Cramer (2014) and Lin and Huang (2012).

In this paper we consider various situations that may occur in both Type of progressive hybrid censoring schemes under different circumstances:

**In Progressive Hybrid Censoring Type-I:**

(i) If $X_{m:m:n} \leqslant T$ then censoring method performs similar to the ordinary progressive censoring with predetermined censoring scheme $(R_1, R_2, \ldots, R_m)$.

(ii) If $X_{k:m:n} \leqslant T < X_{k+1:m:n}$; $k < m$ then $R_i$ units are randomly withdrawn at the $i$th stage $i = 1, 2, \ldots, k$ and $R_T$ units are withdrawn at time $T$. Here $R_T$ is the number of survived units at time $T$. Then predetermined censoring scheme changes to $(R_1, R_2, \ldots, R_k, R_T)$ where $R_T = n - k - \sum_{i=1}^{k} R_i$.

**In Progressive Hybrid Censoring Type-II:**

(iii) If $X_{k:m^*:n} \leqslant T < X_{(k+1):m^*:n}$; $k \geqslant m$ then $R_i$ units are randomly withdrawn at the $i$th stage $i = 1, 2, \ldots, m-1$. Denoting $R_T$ as in progressive hybrid censoring Type-I (ii), $R_T$ units are withdrawn at time $T$. Therefore in this case the predetermined censoring scheme changes to $(R_1, R_2, \ldots, R_{m-1}, 0^{*k-m+1}, R_T)$ where $R_T = n - k - \sum_{i=1}^{m-1} R_i$.

(iv) If $X_{m:m:n} > T$ and $X_{k:m:n} \leqslant T < X_{(k+1):m:n}$ censoring method is similar to the ordinary progressive censoring with censoring scheme $(R_1, R_2, \ldots, R_k, 0^{*m-k-1}, R_m^*)$ where $R_m^* = n - m - \sum_{i=1}^{k} R_i$.

Let $F_0(\cdot)$ be a cumulative distribution function (cdf) with a corresponding hazard rate function $r_0(\cdot)$. The family of random variables with hazard rate function of the form $\{\theta r_0(\cdot) : \theta > 0\}$ is called proportional hazard rate $(PHR)$ family and cdf $F_0(\cdot)$ is known as the baseline cdf of that family. Therefore, if $X$ is a member of proportional hazard family with the baseline cdf $F_0(\cdot)$, then cdf of $X$ becomes

$$F(x; \theta) = 1 - [\bar{F}_0(x)]^{\theta} \qquad x \in B,\ \theta > 0\ ; \tag{1}$$

where $\overline{F}_0(\cdot) = 1 - F_0(\cdot)$ is the baseline survival function with support $B$. Note that the baseline cdf $F_0(\cdot)$ corresponds to the case $\theta = 1$. This model is originally proposed by Cox (1972) and has been extensively discussed in statistical and reliability literature. The $PHR$ family includes several well-known lifetime distributions such as exponential, Pareto (Type-I and Type-II), beta, Burr Type-XII and so on, see Ahmadi et al. (2009a, 2009b) and Asgharzadeh and Valiollahi (2009, 2010).
Furthermore as an extension, in $PHR$ model introduced by Cox (1972), $\theta$ is considered as a random variable which is a function of the covariates $z = (z_1, z_2, \ldots, z_k)$. By taking into account this, the resulting model is

$$r(x|\theta(z)) = r(x)\theta(z).$$

Two most commonly used covariate functions in the literatures are the linear

$$\theta(z) = \beta z,$$

and the log linear

$$\theta(z) = \exp(\beta z),$$

models, where $\beta$ may be a vector parameter. When $\theta = \theta(z)$ has the form log linear, the resulting model is often called cox model. Other functions of the covariates are some times used. For further details, see Lawless (2003) and Marshall and Olkin (2007).

From $PHR$ model in (1) the probability density function (pdf) is given by

$$f(x;\theta) = \theta f_0(x)[\bar{F}_0(x)]^{\theta-1}, \qquad x \in B; \tag{2}$$

where $f_0(\cdot)$ is the pdf of $F_0(\cdot)$. In what follows, for simplification, we will use $Y_i$ in place of $X_{i:m^*:n}$ when $X_1, X_2, \ldots, X_n$ denotes the failure times of $n$ independent units placed in a life testing experiment. Assume sample $X_1, X_2, \ldots, X_n$ is drawn from the $PHR$ model given in (1). The aim of this paper is to discuss the prediction of life-length $Y_{j:R_i}$ ($j = 1, 2, \ldots, R_i$ ; $i = 1, 2, \ldots, k$) of all censored units in all $k$ stages of censoring and $Y_{j:R_T}$ ($j = 1, 2, \ldots, R_T$). Here $Y_{j:R_i}$ denotes the $j$th-order statistic out of $R_i$ removed units at stage $i = 1, 2, \ldots, k$ and $Y_{j:R_T}$ denotes the $j$th-order statistic out of $R_T$ removed units at time $T$. Note that we only observe $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)$. We inspired the idea from prediction of times to failure of $Y_{j:R_i}$ at progressive censored data discussed by Basak et al. (2006), Basak and Balakrishnan (2009) and Asgharzadeh and Valiollahi (2010). However, later Asgharzadeh and Valiollahi (2012, 2015) obtained prediction of time to failure in hybrid censored sample. See also Zhang and Shi (2017). We illustrate a brief description of different predictors in Section 2. In Sections 3, 4 and 5 we focus on $BUP$, $MLP$ and $CMP$, respectively. In Section 6 a numerical example and Monte Carlo simulations are provided to validate the prediction methods presented in this paper. Here, we also compare $CMP$ with $BUP$ and $MLP$ in terms of $MSPE$ for exponential distribution. Concluding remarks are given in Section 7.

## 2 Point Predictors

Let $Y_1, Y_2, \ldots, Y_k$ be a progressive hybrid censoring sample with final censoring scheme $(R_1, R_2, \ldots, R_k, R_T)$. Our interest is to predict $Y_{j:R_i}$ $(j = 1, 2, \ldots, R_i)$, $(i = 1, 2, \ldots, k)$ and $Y_{j:R_T}$ $(j = 1, 2, \ldots, R_T)$ based on the observed progressive hybrid right censored sample $\mathbf{Y} = (Y_1, \ldots, Y_k)$. A statistic $T$ which is used to predict $Y_{j:R_i}$ is called a predictor of $Y_{j:R_i}$. $T$ is unbiased predictor if the prediction error $T - Y_{j:R_i}$ has a mean zero. Also a predictor is a linear predictor if it has the form $c_1Y_1 + c_2Y_2 + \cdots + c_mY_m$ for real $c_i$'s. Moreover, the conditional distribution of $Y_{j:R_i}$ given $\mathbf{Y}$ is equal to the conditional distribution of $Y_{j:R_i}$ given $Y_i$ due to a Markovian property of progressive right censored order statistic (see Balakrishnan and Aggarawala, 2000); that is

$$f_{Y_{j:R_i}|\,\mathbf{Y}}(y) = f_{Y_{j:R_i}|Y_i}(y), \qquad i = 1, 2, \ldots, m. \tag{3}$$

In view of (3), $BUP$ of $Y_{j:R_i}$ $(j = 1, 2, \ldots, R_i)$, $(i = 1, 2, \ldots, k)$; $E(Y_{j:R_i}|\mathbf{Y})$ is nothing but $E(Y_{j:R_i}|Y_i)$, hence it depends only on $Y_i$. If the parameter $\theta$ is unknown it has to be estimated. A technique to obtain $BUP$, when the parameter is unknown, is to apply the result obtained by Ishii and Tokeiteki (1978) and mentioned in Takada (1981). It states that an unbiased predictor $Y^*_{j:R_i}$ of $Y_{j:R_i}$ is its $BUP$ if and only if

$$E_\theta((Y_{j:R_i} - Y^*_{j:R_i})\gamma(\mathbf{Y})) = 0, \qquad for\ all\ \theta,$$

where $\gamma(\cdot)$ is an unbiased estimator of zero. As the best of our knowledge, the most popular predictor of censored order-statistics, for a location-scale family

$F$ is the best linear unbiased predictor ($BLUP$). Kaminsky and Nelson (1975) obtained $BLUP$ of censored order-statistics by applying the results of Goldberger (1962) in the context of ordinary Type-II right censored samples. Raqab and Nagaraja (1997) used order statistics $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$ to predict the future order statistics $X_{s:n}$ for $1 \leqslant r < s \leqslant n$.

In the literature, one frequently used predictor is $MLP$ which has been discussed by Kaminsky and Rhodin (1985) for ordinary Type-II right censored samples. $CMP$ is another possible predictor. A statistic $T$ is said to be the $CMP$ of $Y_{j:R_i}$ if it is the median of the conditional distribution of $Y_{j:R_i}$ given $Y_i$. A $CMP$ is a special type of median unbiased predictor ($MUP$). The idea of median unbiasedness is used to define a $MUP$. A statistic $T$ is said $MUP$ of $Y_{j:R_i}$ if for all $\theta$,

$$P_\theta(T \leqslant Y_{j:R_i}) = P_\theta(T \geqslant Y_{j:R_i}).$$

Takada (1991) discussed some properties of MUP in the case of ordinary Type-II right censored samples. He showed that for a location-scale family, a particular $MUP$ is better than the $BLUP$ under Pitman's measure of closeness ($PMC$). It is known that under $PMC$, the predictor $T_1$ is better than $T_2$ for predicting $Y_{j:R_i}$ if

$$P_\theta(|T_1 - Y_{j:R_i}| \leqslant |T_2 - Y_{j:R_i}|) \geqslant \frac{1}{2}, \qquad \text{for all } \theta.$$

Our contribution in Section 3 and 4 is to discuss $BUP$ and $MLP$ of $Y_{j:R_i}$ respectively. We have focused on exponential population there. In Section 5 Takada's $CMP$ of $Y_{j:R_i}$ is considered. In Section 6, a set of numerical simulation is provided to validate all the proposed prediction methods discussed in this paper. We also set comparison between $CMP$, $BUP$ and $MLP$ in terms of $MSPE$ for exponential distribution. Throughout this paper we will use the following notations:

| | |
|---|---|
| $X \overset{d}{=} Y$ : | $X$ and $Y$ are identically distributed |
| $X \sim F$ : | $X$ is distributed as $F$ |
| $\text{Exp}(\theta)$ : | exponential distribution with support $(0, \infty)$ and mean $\frac{1}{\theta}$ |
| $Y_{j:R_i}$ : | $j$th order statistic out of $R_i$ units of $Y$ |

$$\begin{aligned} Y^*_{j:R_i}: &\quad BUP \text{ of } Y_{j:R_i} \\ Y^L_{j:R_i}: &\quad MLP \text{ of } Y_{j:R_i} \\ Y^{CMP}_{j:R_i}: &\quad CMP \text{ of } Y_{j:R_i} \end{aligned}$$

## 3 Best Unbiased Predictor

A statistic $Y^*_{j:R_i}$ which is used to predict $Y_{j:R_i}$ is called BUP of $Y$, if $(Y^*_{j:R_i} - Y_{j:R_i})$ has a mean zero and its prediction error variance, i.e., $var(Y^*_{j:R_i} - Y_{j:R_i})$ is less than or equal to that of any other unbiased predictor of $Y_{j:R_i}$.
Since the conditional distribution of $Y_{j:R_i}$ given $\mathbf{Y}$ is just the distribution of $Y_{j:R_i}$ given $Y_i$, therefore the $BUP$ of $Y_{j:R_i}$ is

$$Y^*_{j:R_i} = \hat{Y}_{BUP} = E(Y_{j:R_i}|Y_i = y_i),$$

see Nayak (2000).
As mentioned before due to the Markovian property of progressive censored order statistic the density of $Y_{j:R_i}$ given $\mathbf{Y} = \mathbf{y}$ is the same as the density of $j$th order statistic out of $R_i$ units from the population with density $\frac{f(y)}{1-F(y_i)}$, $y \geqslant y_i$ (left truncated density at $y_i$). Therefore the conditional density of $Y_{j:R_i}$ given $Y_i$ for $y \geqslant y_i$ is given by (in case (i), (iv)):

$$f(y|y_i;\theta) = j\binom{R_i}{j} f_\theta(y)[F_\theta(y) - F_\theta(y_i)]^{j-1}[1-F_\theta(y)]^{R_i-j}[1-F_\theta(y_i)]^{-R_i}. \quad (4)$$

Using (2), (4) reduces to

$$f(y|y_i;\theta) = j\binom{R_i}{j}\theta\frac{f_0(y)}{\bar{F}_0(y)}[\bar{F}^\theta_0(y_i) - \bar{F}^\theta_0(y)]^{j-1}[\bar{F}^\theta_0(y)]^{R_i-j+1}[\bar{F}_0(y_i)]^{-R_i}\,; y \geqslant y_i. \quad (5)$$

Likewise, for cases (ii) and (iii) $f_{Y_{j:R_i}}(y|y_i)$ takes the form (4) for $i = 1, 2, \ldots, k$, in other cases due to Markovian property of progressive censored order statistic, it is well-known that $f_{Y_{j:R_T}|\mathbf{Y},T}(y|\mathbf{y},T)$ is $f_{Y_{j:R_T}|T}(y|T)$. This means that the density of $Y_{j:R_T}$ given $\mathbf{Y} = \mathbf{y}$ and $T$ is the same as the density of $j$th order statistic out of $R_T$ units from the population with density $\frac{f(y)}{1-F(T)}$, $y \geqslant T$ (left truncated density at $T$). Therefore the conditional density of $Y_{j:R_T}$

given $T$ for $y \geqslant T$ is derived by:

$$f(y|T;\theta) = j\binom{R_T}{j}\theta\frac{f_0(y)}{\bar{F}_0(y)}[\bar{F}_0^{\theta}(T)-\bar{F}_0^{\theta}(y)]^{j-1}[\bar{F}_0^{\theta}(y)]^{R_T-j+1}[\bar{F}_0(T)]^{-R_T};\ \ y \geqslant T. \tag{6}$$

By (5) and (6) we have

$$E(Y_{j:R_i}|Y_i = y_i) = \int_{y_i}^{\infty} yf(y|y_i)dy = \int_0^1 \bar{F}_0{}^{-1}(u^{\frac{1}{\theta}}\bar{F}_0(y_i))\frac{u^{R_i-j}(1-u)^{j-1}}{Beta(R_i-j+1,j)}du. \tag{7}$$

$$E(Y_{j:R_T}|T) = \int_{y_i}^{\infty} yf(y|y_i)dy = \int_0^1 \bar{F}_0^{-1}(u^{\frac{1}{\theta}}\bar{F}_0(T))\frac{u^{R_T-j}(1-u)^{j-1}}{Beta(R_T-j+1,j)}du. \tag{8}$$

We consider exponential distribution as an example in order to illustrate our achievements. Suppose that the lifetimes of the $n$ units put on test are independent and identically distributed as exponential random variables with pdf $\bar{F}_\theta(x) = e^{-\theta x}$ so $\bar{F}_0(x) = e^{-x}$ then we compute $BUP$ of $Y$ as:

$$\begin{aligned} E(Y_{j:R_i}|Y_i = y_i) &= \int_0^1 -\ln(u^{\frac{1}{\theta}}e^{-Y_i})\frac{u^{R_i-j}(1-u)^{j-1}}{Beta(R_i-j+1,j)}du \\ &= y_i + \frac{1}{\theta}E(-\ln U), \end{aligned} \tag{9}$$

where random variable $U$ has beta distribution with parameters $R_i - j + 1$ and $j$. So in this case

$$\begin{aligned} Y^*_{j:R_i} &= y_i + \frac{1}{\theta}E(-\ln U) \\ &= y_i + \frac{1}{\theta}E(Z_{j:R_i}) \\ &= y_i + \frac{1}{\theta}\Sigma_{r=R_i-j+1}^{R_i}\frac{1}{r}, \end{aligned} \tag{10}$$

where $Z_{j:R_i}$ stands for the $j$th order statistic of sample size $R_i$ from standard exponential distribution.
Hence analogously

$$Y^*_{j:R_T} = T + \frac{1}{\theta}\Sigma_{r=R_T-j+1}^{R_T}\frac{1}{r}. \tag{11}$$

If $\theta$ is unknown we can approximate it by using its $MLE$ and plug it into (10) and (11). In case exponential distribution $MLE$ of $\theta$ under progressive hybrid censoring scheme obtained by Childs et al. (2008)

$$\hat{\theta} = \begin{cases} \frac{k}{\sum_{l=1}^{k}(R_l+1)Y_l+TR_T} & \text{for} \quad k \neq m \\ \\ \frac{m}{\sum_{l=1}^{m}(R_l+1)Y_l} & \text{for} \quad k = m. \end{cases}$$

## 4 Maximum Likelihood Predictor

Regarding the prediction context, the maximum likelihood ($ML$) methodology has been the solution of many problems in statistics and reliability analysis. For this, see, Kaminsky and Rhodin (1985), Basak and Balakrishnan (2003) and Basak et al. (2006).

In $MLP$, the principle of maximum likelihood is applied to the joint prediction and estimation of future random variable and an unknown parameter.

Let $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)$ and $Y_{j:R_i}$ have the joint pdf $f(y, \mathbf{y}; \theta)$. We know that both cases (i) and (iv) are similar to the ordinary progressive censoring so the predictive likelihood function ($PLF$) of $Y_{j:R_i}$ and $\theta$ is given by

$$L = L(y, \theta, \mathbf{y}) = f_\theta(y, \mathbf{y}) = f_{\theta, Y_{j:R_i}|\mathbf{Y}}(y|\mathbf{y}) f_{Y_i,\theta}(y) = f_{\theta, Y_{j:R_i}|Y_i}(y|y_i) f_{\mathbf{Y},\theta}(\mathbf{y}).$$

In addition note that in cases (i) and (iv):

$$f_{\mathbf{Y}}(\mathbf{y}; \theta) = c \prod_{l=1}^{m} \frac{f_0(y_l)}{\bar{F}_0(y_l)} \theta (\bar{F}_0(y_l))^{\theta(R_l+1)},$$

and in cases (ii) and (iii)

$$f_{\mathbf{Y},T}(\mathbf{y}; \theta) = c\theta^k \prod_{l=1}^{k} \frac{f_0(y_l)}{\bar{F}_0(y_l)} (\bar{F}_0(y_l))^{\theta(R_l+1)} \bar{F}_0^{R_T\theta}(T),$$

where $k$ is the number of failure before time $T$. We can assume $R_T = 0$ and

$k = m$ in cases (i) and (iv), so we have the general form

$$f_{\mathbf{Y},T}(\mathbf{y};\theta) = c\prod_{l=1}^{k} f(y_l)\bar{F}^{R_l}(y_l)\bar{F}^{R_T}(T)$$
$$= c\theta^k \prod_{l=1}^{k} \frac{f_0(y_l)}{\bar{F}_0(y_l)}(\bar{F}_0(y_l))^{\theta(R_l+1)}\bar{F}_0^{R_T\theta}(T). \tag{12}$$

From (5) and (12), one can write

$$L = L(y,\theta,\mathbf{y}) = c\theta^{m+1}\prod_{l=1}^{m}\frac{f_0(y_l)}{\bar{F}_0(y_l)}(\bar{F}_0(y_l))^{\theta(R_l+1)} j\binom{R_i}{j}\frac{f_0(y)}{\bar{F}_0(y)}$$
$$\times [\bar{F}_0^{\theta}(y_i) - \bar{F}_0^{\theta}(y)]^{j-1}[\bar{F}_0^{\theta}(y)]^{R_i-j+1}[\bar{F}_0(y_i)]^{-R_i\theta}, \quad y \geqslant y_i.$$

So

$$\ln L(y,\theta;\mathbf{y}) = \ln f_\theta(y) + (j-1)\ln[F_\theta(y) - F_\theta(y_i)] + (R_i - j)\ln[1 - F(y,\theta)]$$
$$+ \sum_{l=1}^{m} \ln f_\theta(y_l) + \sum_{l=1;l\neq i} R_l \ln[1 - F_\theta(y_l)], \quad y \geqslant y_i.$$

Again from (5) and (12) we have

$$\ln L(y,\theta;\mathbf{y}) = (m+1)\ln\theta + \ln\left[\frac{f_0(y)}{\bar{F}_0(y)}\right] + (j-1)\ln\left[1 - \left(\frac{\bar{F}_0(y)}{\bar{F}_0(y_i)}\right)^{\theta}\right]$$
$$+ \theta(R_i - j + 1)[\ln\bar{F}_0(y) - \ln\bar{F}_0(y_i)] + \theta\sum_{l=1}^{m}(R_l+1)\ln\bar{F}_0(y_l). \tag{13}$$

Assuming $Y^L_{j:R_i} = t(\mathbf{Y})$ and $\theta^{**} = u(\mathbf{Y})$ are two statistics such that $L(t(\mathbf{y}), u(\mathbf{y});\mathbf{y}) = \sup_{y,\theta} L(y,\theta;\mathbf{y})$, then $t(\mathbf{Y})$ is said to be the $MLP$ of $Y_{j:R_i}$ and $u(\mathbf{Y})$ is the predictive maximum likelihood estimator ($PMLE$) of $\theta$. Since $f$ is continuos then $L$ converges to zero as $y \downarrow y_i$ and $y \uparrow \infty$ also $L > 0$ for $y > y_i$. This means that if there exists a unique solution $Y^L_{j:R_i}$ of the likelihood equation $\dfrac{\partial \ln L}{\partial y} = 0$, then $Y^L_{j:R_i}$ must be the unique $MLP$ of

$Y_{j:R_i}$.

From (13), predictive likelihood equations for y and $\theta$ are given by:

$$\frac{\partial \ln L(y,\theta;\mathbf{y})}{\partial \theta} = \frac{m+1}{\theta} + (R_i - j + 1)[\ln \bar{F}_0(y) - \ln \bar{F}_0(y_i)] \\ + \sum_{l=1}^{m}(R_l+1)\ln \bar{F}_0(y_l) \\ - (j-1)\left(\frac{\bar{F}_0(y)}{\bar{F}_0(y_i)}\right)^{\theta} \frac{\ln\left(\frac{\bar{F}_0(y)}{\bar{F}_0(y_i)}\right)}{1-\left(\frac{\bar{F}_0(y)}{\bar{F}_0(y_i)}\right)^{\theta}} = 0, \tag{14}$$

and

$$\frac{\partial \ln L(y,\theta;\mathbf{y})}{\partial y} = \frac{1}{\bar{F}_0(y)}\left[\frac{f_0'(y)\bar{F}_0(y)+f_0^2(y)}{f_0(y)} + \theta(j-1)\frac{f_0(y)\left(\frac{\bar{F}_0(y)}{\bar{F}_0(y_i)}\right)^{\theta}}{1-\left(\frac{\bar{F}_0(y)}{\bar{F}_0(y_i)}\right)^{\theta}} \right. \\ \left. - \theta(R_i-j+1)f_0(y)\right] = 0. \tag{15}$$

Going back to (i) and (iv), if $\theta$ is known we can find $Y_{j:R_i}$ by solving equation (15), but if $\theta$ is unknown we have to solve (14) and (15) simultaneously. Similarly for cases (ii) and (iii), from (12) we have

$$L(y,\theta,\mathbf{y}) = c\theta^{k+1}\prod_{l=1}^{k}\frac{f_0(y_l)}{\bar{F}_0(y_l)}(\bar{F}_0(y_l))^{\theta(R_l+1)}\binom{R_T}{j}\frac{f_0(y)}{\bar{F}_0(y)} \\ \times [\bar{F}_0^{\theta}(T) - \bar{F}_0^{\theta}(y)]^{j-1}[\bar{F}_0^{\theta}(y)]^{R_T-j+1}, \quad y \geqslant T,$$

consequently we write

$$\ln L(y,\theta;\mathbf{y}) = (k+1)\ln\theta + (j-1)\ln\left[1-(\frac{\bar{F}_0(y)}{\bar{F}_0(T)})^{\theta}\right] + \ln\left(\frac{f_0(y)}{\bar{F}_0(y)}\right)$$

$$
\begin{aligned}
&+\theta\sum_{l=1}^{m}(R_l+1)\ln\bar{F}_0(y_l)+\theta R_T\ln\bar{F}_0(T)\\
&+\theta(R_T-j+1)[\ln\bar{F}_0(y)-\ln\bar{F}_0(T)]. \qquad (16)
\end{aligned}
$$

Now, the expression (16) implies

$$
\begin{aligned}
\frac{\partial\ln L(y,\theta;\mathbf{y},T)}{\partial y} &= \frac{1}{\bar{F}_0(y)}\left[\frac{f_0'(y)\bar{F}_0(y)+f_0^2(y)}{f_0(y)}+\theta(j-1)\frac{f_0(y)\left(\frac{\bar{F}_0(y)}{\bar{F}_0(T)}\right)^{\theta}}{1-\left(\frac{\bar{F}_0(y)}{\bar{F}_0(T)}\right)^{\theta}}\right.\\
&\left.\quad-\theta(R_T-j+1)f_0(y)\right]=0, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\ln L(y,\theta;\mathbf{y},T)}{\partial\theta} &= \frac{k+1}{\theta}+(R_T-j+1)[\ln\bar{F}_0(y)-\ln\bar{F}_0(T)]+R_T\ln\bar{F}_0(T)\\
&\quad+\sum_{j=1}^{k}(R_l+1)\ln\bar{F}_0(y_l)\\
&\quad-(j-1)\left(\frac{\bar{F}_0(y)}{\bar{F}_0(T)}\right)^{\theta}\frac{\ln\left(\frac{\bar{F}_0(y)}{\bar{F}_0(T)}\right)}{1-\left(\frac{\bar{F}_0(y)}{\bar{F}_0(T)}\right)^{\theta}}=0. \qquad (18)
\end{aligned}
$$

As an example, let $F_0$ be standard exponential distribution, then the predictive likelihood equations reduce to:

$$
\begin{aligned}
\frac{\partial\ln L(y,\theta;\mathbf{y})}{\partial\theta} &= \frac{m+1}{\theta}+(R_i-j+1)(y_i-y)\\
&- \sum_{l=1}^{m}(R_l+1)y_l(R_l+1)-(j-1)\frac{(y_i-y)e^{-(y_i-y)\theta}}{1-e^{-(y_i-y)\theta}}=0,
\end{aligned}
$$

and

$$
\frac{\partial\ln L(y,\theta;\mathbf{y})}{\partial y}=-\theta(R_i-j+1)+\theta(j-1)\frac{e^{\theta(y_i-y)}}{1-e^{\theta(y_i-y)}}=0.
$$

In all four cases (i)-(iv), for $j = 1, 2, \ldots, R_i$ and $i = 1, 2, \ldots, k$, $MLP$ of $Y_{j:R_T}$ is obtained by

$$Y^L_{j:R_i} = Y_i + \frac{1}{\hat{\theta}} \ln \frac{R_i}{R_i - j + 1}, \tag{19}$$

where $\hat{\theta}$ stands with the $PMLE$ of $\theta$ is given by

$$\hat{\theta} = \frac{m+1}{\sum_{l=1}^{m}(R_l + 1)Y_l}. \tag{20}$$

In this regard, we again consider cases (ii) and (iii), so we have

$$Y^L_{j:R_T} = T + \frac{1}{\hat{\theta}} \ln \frac{R_T}{R_T - j + 1}, \tag{21}$$

where $\hat{\theta}$ is the $PMLE$ of $\theta$ and is given by

$$\hat{\theta} = \frac{k+1}{\sum_{l=1}^{k}(R_l + 1)Y_l + R_T T}. \tag{22}$$

## 5 Conditional Median Predictor

For the first time, Raqab and Nagaraja (1997) introduced the $CMP$. In their work the predictor $Y^{CMP}_{j:R_i}$ is called $CMP$ of $Y_{j:R_i}$, if it is the median of the conditional distribution of $Y_{j:R_i}$ given $Y_i = y_i$. So, the analytical interpretation would be

$$P_\theta(Y_{j:R_i} \leqslant Y^{CMP}_{j:R_i} | Y_i = y_i) = P_\theta(Y_{j:R_i} \geqslant Y^{CMP}_{j:R_i} | Y_i = y_i).$$

On the other side we know

$$P_\theta(Y_{j:R_i} \leqslant Y^{CMP}_{j:R_i} | Y_i = y_i) = P_\theta \left[ \left( \frac{\bar{F}_0(Y)}{\bar{F}_0(Y_i)} \right)^\theta \geqslant \left( \frac{\bar{F}_0(Y^{CMP}_{j:R_i})}{\bar{F}_0(Y_i)} \right)^\theta | Y_i = y_i \right].$$

By using the fact that expression $\left( \frac{\bar{F}_0(Y)}{\bar{F}_0(Y_i)} \right)^\theta$ given $Y_i = y_i$ has the $Beta(R_i -$

$j+1, j)$ distribution, we have

$$Y_{j:R_i}^{CMP} = \bar{F}_0^{-1}\left(\bar{F}_0(y_i)\,(\text{med}(U))^{\dfrac{1}{\theta}}\right), \tag{23}$$

here $med(U)$ stands for median of $U = \left(\dfrac{\bar{F}_0(Y)}{\bar{F}_0(y_i)}\right)^{\theta}$.
Also, for (ii) and (iii) we have

$$Y_{j:R_T}^{CMP} = \bar{F}_0^{-1}\left(\bar{F}_0(T)(\text{med}(V))^{\dfrac{1}{\theta}}\right), \tag{24}$$

where $V = (\dfrac{\bar{F}_0(Y)}{\bar{F}_0(T)})^{\theta} \sim Beta(R_T - j + 1, j)$.
Remember again that we substitute the $MLE$ $\hat{\theta}$ when the parameter $\theta$ is unknown.

As an example assume $\bar{F}_0(x) = e^{-x}, \quad x > 0$, then we have

$$\begin{aligned} Y_{j:R_i}^{CMP} &= -\ln\left(e^{-Y_i}[\text{med}(U)]^{\frac{1}{\hat{\theta}}}\right) \\ &= Y_i + \frac{1}{\hat{\theta}}[\text{med}(-\ln U)] \\ &= Y_i + \frac{1}{\hat{\theta}}(med(Z_{j:R_i})), \end{aligned} \tag{25}$$

here $Z_{j:R_i}$ denotes $j$th order statistic out of $R_i$ units from a standard exponential distribution. In addition for (ii) and (iii) we can obtain

$$Y_{j:R_T}^{CMP} = T + \frac{1}{\hat{\theta}}(\text{med}(Z_{j:R_i})). \tag{26}$$

# 6 Numerical Computations

In this section, we intend to present the result of numerical study to investigate the performances of the different methods of prediction discussed in previous sections with respect to biases and mean squared prediction error from progressive hybrid Type-I censored data. In this regard, some results

**Table 1.** Progressively hybrid type-I censored data.

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $T=1$ |
|---|---|---|---|---|---|---|---|---|
| $Y_i$ | 0.0123 | 0.0533 | 0.0656 | 0.0944 | 0.1247 | 0.4286 | 0.6615 | |
| $R_i$ | 0 | 0 | 3 | 0 | 0 | 3 | 0 | 6 |

based on Monte-Carlo simulations are presented. In obtaining the numerical results, we used the statistical software R. As a special case of $PHR$ family, we consider the exponential distribution with cdf

$$\bar{F}_\theta(x) = \theta e^{-\theta x}, \qquad x > 0,\ \theta > 0,$$

the baseline cdf is

$$\bar{F}_0(x) = e^{-x}, \quad x > 0.$$

For generating progressive hybrid Type-I censored data, we first generate progressive Type-II censored sample $Y_1, \ldots, Y_m$ according to the algorithm presented in Balakrishnan and Aggarawala (2000). Then if $Y_m < T$ then, above progressive Type-II censored sample is also progressive hybrid Type-I. If $Y_m > T$, then we find $k$ such that $Y_k < T < Y_{k+1}$. In this case, the progressive hybrid Type-I sample becomes $Y_1, \ldots, Y_k$.

We draw $m = 8$ progressively hybrid Type-I censored samples from exponential distribution with parameter $\theta = 1.952$, $n = 19$ and $T = 1$. Also the censoring scheme here is $R = (0,0,3,0,0,3,0,5)$. The sample is stated in Table 1. Moreover, the point predictors $MLP$, $BUP$ and $CMP$ for $Y_{j:R_i}$ $(j = 1,2,\ldots,R_i;\ i = 1,2,\ldots,k)$ and $Y_{j:R_T}$ $(j = 1,2,\ldots,R_T)$ are given in Table 2.
In Table 1 we see that $k = 7$ which means $Y_7 < T < Y_8$.
Table 2 shows that an analytical comparison of predictors is not possible. We use Monte Carlo approximation method to evaluate the biases and $MSPE$ for three predictors $BUP$, $MLP$ and $CMP$ when sample is drawn from exponential distribution. Randomly, we generate 1000 progressively hybrid Type-I censored samples from exponential distribution with parameters $\theta = 0.75$, $\theta = 1$ and $\theta = 2$. We consider threshold time $T = 1$ and use two censoring scheme $R_1 = (0,0,3,0,3,0,0,5)$, $R_2 = (0,0,0,0,5)$. Results obtained from this simulation study are presented in Tables 3 to 8. In these

**Table 2.** Different point predictions.

| $\theta = 1.952$ | $BUP$ | $MLP$ | $CMP$ | $Y_{j:R_i}$ |
|---|---|---|---|---|
| $Y_{1:R_3}$ | 0.1494 | 0.0656 | 0.1237 | 0.1590 |
| $Y_{2:R_3}$ | 0.2751 | 0.1675 | 0.2398 | 0.4020 |
| $Y_{3:R_3}$ | 0.5265 | 0.3418 | 0.4624 | 0.7426 |
| $Y_{1:R_6}$ | 0.2085 | 0.1247 | 0.1828 | 0.3369 |
| $Y_{2:R_6}$ | 0.3342 | 0.2266 | 0.2989 | 1.2434 |
| $Y_{3:R_6}$ | 0.5856 | 0.4009 | 0.5215 | 1.4249 |
| $Y_{1:R_T}$ | 1.0419 | 1.0000 | 1.0290 | 1.0119 |
| $Y_{2:R_T}$ | 1.0922 | 1.0458 | 1.0772 | 1.1125 |
| $Y_{3:R_T}$ | 1.1550 | 1.1019 | 1.1375 | 1.3332 |
| $Y_{4:R_T}$ | 1.2388 | 1.1743 | 1.2172 | 1.3973 |
| $Y_{5:R_T}$ | 1.3650 | 1.2762 | 1.3344 | 1.4604 |
| $Y_{6:R_T}$ | 1.6159 | 1.4504 | 1.5569 | 1.6704 |

tables the $MSPE$s and biases of different predictors of $Y_{j:R_i}$ and $Y_{j:R_T}$ are provided.

# 7 Discussion

In this paper, we have considered different predictor of failure times of units censored in multiple stages of progressively hybrid censored sample. A numerical simulation has been conducted to compare the performances of different point predictors. We generated 1000 random values of $Y_{j:R_i}$ truncated at $Y_i$ from exponential distribution. Bias and $MSPE$ of this $Y_{j:R_i}$ for each predictors are generated and reported. According to tables 3 to 8 one can find that $BUP$ has smaller bias and $MSPE$ than $CMP$ and $CMP$ has smaller bias and $MSPE$ than $MLP$. So it is observed that $BUP$ is better than $CMP$ and $CMP$ is better than $MLP$.

**Table 3.** Biases and MSPEs of point predictors for the censoring scheme $R_1$.

| $\theta = 0.75$ and $T = 1$ | | | | |
|---|---|---|---|---|
| | | $BUP$ | $MLP$ | $CMP$ |
| $Y_{1:R_1}$ | Bias | 0.1368 | 0.4369 | 0.2289 |
| | MSPE | 0.1987 | 0.3664 | 0.2291 |
| $Y_{2:R_1}$ | Bias | 0.3450 | 0.7753 | 0.4720 |
| | MSPE | 0.8205 | 1.2677 | 0.9097 |
| $Y_{3:R_1}$ | Bias | 0.7952 | 1.5769 | 1.0246 |
| | MSPE | 3.2153 | 4.8855 | 3.5656 |
| $Y_{1:R_5}$ | Bias | 0.1525 | 0.4531 | 0.2448 |
| | MSPE | 0.2544 | 0.4331 | 0.2881 |
| $Y_{2:R_5}$ | Bias | 0.3684 | 0.7986 | 0.4949 |
| | MSPE | 0.7985 | 1.2596 | 0.8917 |
| $Y_{3:R_5}$ | Bias | 0.7595 | 1.5420 | 0.9890 |
| | MSPE | 3.2182 | 4.8246 | 3.5496 |
| $Y_{1:R_8}$ | Bias | 0.0677 | 0.2705 | 0.1299 |
| | MSPE | 0.0759 | 0.1434 | 0.0873 |
| $Y_{2:R_8}$ | Bias | 0.1609 | 0.4159 | 0.2353 |
| | MSPE | 0.2356 | 0.3727 | 0.2613 |
| $Y_{3:R_8}$ | Bias | 0.2868 | 0.6204 | 0.3782 |
| | MSPE | 0.5439 | 0.8188 | 0.5956 |
| $Y_{4:R_8}$ | Bias | 0.4717 | 0.9467 | 0.5977 |
| | MSPE | 1.1788 | 1.7926 | 1.2949 |
| $Y_{5:R_8}$ | Bias | 0.7621 | 1.6260 | 1.0041 |
| | MSPE | 3.0673 | 4.9506 | 3.4350 |
| $Y_{1:R_T}$ | Bias | 0.0983 | 0.2105 | 0.1328 |
| | MSPE | 0.0609 | 0.0949 | 0.0681 |
| $Y_{2:R_T}$ | Bias | 0.2029 | 0.3424 | 0.2425 |
| | MSPE | 0.1551 | 0.2272 | 0.1712 |
| $Y_{3:R_T}$ | Bias | 0.3165 | 0.4915 | 0.3619 |
| | MSPE | 0.3165 | 0.4506 | 0.3450 |
| $Y_{4:R_T}$ | Bias | 0.4649 | 0.6898 | 0.5199 |
| | MSPE | 0.5819 | 0.8290 | 0.6319 |
| $Y_{5:R_T}$ | Bias | 0.6974 | 1.003 | 0.7687 |
| | MSPE | 1.1437 | 1.6435 | 1.2436 |
| $Y_{6:R_T}$ | Bias | 1.1592 | 1.6493 | 1.2798 |
| | MSPE | 3.2225 | 4.6450 | 3.5317 |
| $Y_{7:R_T}$ | Bias | 1.2021 | 1.8457 | 1.3561 |
| | MSPE | 4.3246 | 6.1577 | 4.6839 |
| $Y_{8:R_T}$ | Bias | 1.0497 | 1.8834 | 1.2405 |
| | MSPE | 4.6839 | 7.0791 | 5.0882 |
| $Y_{9:R_T}$ | Bias | 0.5165 | 0.8015 | 0.5579 |
| | MSPE | 0.6837 | 0.9649 | 0.7127 |
| $Y_{10:R_T}$ | Bias | 0.9380 | 1.2975 | 0.9914 |
| | MSPE | 1.6036 | 2.1990 | 1.6727 |

**Table 4.** Biases and MSPEs of point predictors for the censoring scheme $R_2$.

| $\theta = 0.75$ | and | $T = 1$ | | |
|---|---|---|---|---|
| | | $BUP$ | $MLP$ | $CMP$ |
| $Y_{1:R_5}$ | Bias | 0.0458 | 0.2518 | 0.1090 |
| | MSPE | 0.0703 | 0.12599 | 0.0774 |
| $Y_{2:R_5}$ | Bias | 0.1458 | 0.4178 | 0.2214 |
| | MSPE | 0.2281 | 0.3627 | 0.2492 |
| $Y_{3:R_5}$ | Bias | 0.2479 | 0.6163 | 0.3408 |
| | MSPE | 0.4765 | 0.7490 | 0.5170 |
| $Y_{4:R_5}$ | Bias | 0.4339 | 0.9693 | 0.5620 |
| | MSPE | 1.1610 | 1.8089 | 1.2586 |
| $Y_{5:R_5}$ | Bias | 0.7366 | 1.7071 | 0.9827 |
| | MSPE | 3.6209 | 5.6630 | 3.9425 |
| $Y_{1:R_T}$ | Bias | 0.1348 | 0.2019 | 0.1554 |
| | MSPE | 0.0652 | 0.0875 | 0.0709 |
| $Y_{2:R_T}$ | Bias | 0.3210 | 0.4116 | 0.3448 |
| | MSPE | 0.2337 | 0.2973 | 0.2486 |
| $Y_{3:R_T}$ | Bias | 0.5095 | 0.6301 | 0.5368 |
| | MSPE | 0.4843 | 0.6187 | 0.5121 |
| $Y_{4:R_T}$ | Bias | 0.7419 | 0.9036 | 0.7748 |
| | MSPE | 0.8828 | 1.1440 | 0.9316 |
| $Y_{5:R_T}$ | Bias | 1.1252 | 1.3515 | 1.1690 |
| | MSPE | 1.9420 | 6.9540 | 2.0420 |
| $Y_{6:R_T}$ | Bias | 1.8534 | 2.2221 | 1.9303 |
| | MSPE | 5.4555 | 2.4944 | 5.7523 |
| $Y_{7:R_T}$ | Bias | 2.0870 | 2.5418 | 2.1711 |
| | MSPE | 6.6571 | 8.6936 | 7.005 |
| $Y_{8:R_T}$ | Bias | 2.4374 | 3.0599 | 2.5391 |
| | MSPE | 8.4680 | 11.7869 | 8.9441 |
| $Y_{9:R_T}$ | Bias | 1.4608 | 2.5083 | 1.5982 |
| | MSPE | 3.1412 | 7.7821 | 3.5949 |

**Table 5.** Biases and MSPEs of point predictors for the censoring scheme $R_1$.

| $\theta = 1$ and $T = 1$ | | | | |
|---|---|---|---|---|
| | | BUP | MLP | CMP |
| $Y_{1:R_1}$ | Bias | 0.04210 | 0.3238 | 0.1285 |
| | MSPE | 0.1056 | 0.1991 | 0.1158 |
| $Y_{2:R_1}$ | Bias | 0.1209 | 0.5215 | 0.2393 |
| | MSPE | 0.3972 | 0.6158 | 0.4253 |
| $Y_{3:R_1}$ | Bias | 0.2845 | 1.0111 | 0.4999 |
| | MSPE | 1.6639 | 2.4236 | 1.7681 |
| $Y_{1:R_5}$ | Bias | 0.0531 | 0.3349 | 0.1396 |
| | MSPE | 0.1219 | 0.2245 | 0.1349 |
| $Y_{2:R_5}$ | Bias | 0.1126 | 0.5132 | 0.2310 |
| | MSPE | 0.4596 | 0.6754 | 0.4868 |
| $Y_{3:R_5}$ | Bias | 0.3518 | 1.0784 | 0.5672 |
| | MSPE | 1.7798 | 2.6696 | 1.9229 |
| $Y_{1:R_8}$ | Bias | 0.0118 | 0.1903 | 0.0666 |
| | MSPE | 0.0352 | 0.0698 | 0.0385 |
| $Y_{2:R_8}$ | Bias | 0.0551 | 0.2796 | 0.1205 |
| | MSPE | 0.1159 | 0.1839 | 0.1245 |
| $Y_{3:R_8}$ | Bias | 0.1023 | 0.3961 | 0.1828 |
| | MSPE | 0.2607 | 0.3891 | 0.2774 |
| $Y_{4:R_8}$ | Bias | 0.1392 | 0.5575 | 0.2502 |
| | MSPE | 0.5586 | 0.7968 | 0.5851 |
| $Y_{5:R_8}$ | Bias | 0.2612 | 1.0219 | 0.4743 |
| | MSPE | 1.7428 | 2.5295 | 1.8376 |
| $Y_{1:R_T}$ | Bias | 0.0589 | 0.1564 | 0.0889 |
| | MSPE | 0.0250 | 0.0453 | 0.0291 |
| $Y_{2:R_T}$ | Bias | 0.1355 | 0.2567 | 0.1701 |
| | MSPE | 0.0749 | 0.1209 | 0.0848 |
| $Y_{3:R_T}$ | Bias | 0.2188 | 0.3714 | 0.2588 |
| | MSPE | 0.1711 | 0.2549 | 0.1882 |
| $Y_{4:R_T}$ | Bias | 0.3314 | 0.5293 | 0.3800 |
| | MSPE | 0.3165 | 0.4769 | 0.3482 |
| $Y_{5:R_T}$ | Bias | 0.4641 | 0.7389 | 0.5298 |
| | MSPE | 0.6099 | 0.9225 | 0.6705 |
| $Y_{6:R_T}$ | Bias | 0.7611 | 1.2274 | 0.8805 |
| | MSPE | 1.7756 | 2.6623 | 1.961 |
| $Y_{7:R_T}$ | Bias | 0.8795 | 1.4711 | 1.0237 |
| | MSPE | 2.4874 | 3.6653 | 2.704 |
| $Y_{8:R_T}$ | Bias | -0.0509 | 0.8235 | 0.1523 |
| | MSPE | 1.1802 | 1.6007 | 1.1357 |

**Table 6.** Biases and MSPEs of point predictors for the censoring scheme $R_2$.

| $\theta = 1$ and $T = 1$ | | | | |
|---|---|---|---|---|
| | | $BUP$ | $MLP$ | $CMP$ |
| $Y_{1:R_5}$ | Bias | 0.0294 | 0.2046 | 0.0832 |
| | MSPE | 0.0419 | 0.0796 | 0.0461 |
| $Y_{2:R_5}$ | Bias | 0.0584 | 0.2896 | 0.1227 |
| | MSPE | 0.1231 | 0.1873 | 0.1288 |
| $Y_{3:R_5}$ | Bias | 0.1067 | 0.4198 | 0.1856 |
| | MSPE | 0.2919 | 0.414 | 0.3018 |
| $Y_{4:R_5}$ | Bias | 0.1895 | 0.6446 | 0.2984 |
| | MSPE | 0.6829 | 0.9622 | 0.7070 |
| $Y_{5:R_5}$ | Bias | 0.3241 | 1.149 | 0.5333 |
| | MSPE | 2.2402 | 3.0859 | 2.3087 |
| $Y_{1:R_T}$ | Bias | 0.111 | 0.1800 | 0.1326 |
| | MSPE | 0.0452 | 0.0634 | 0.0497 |
| $Y_{2:R_T}$ | Bias | 0.1939 | 0.2858 | 0.2182 |
| | MSPE | 0.1012 | 0.3328 | 0.1104 |
| $Y_{3:R_T}$ | Bias | 0.3283 | 0.4502 | 0.3564 |
| | MSPE | 0.2422 | 0.1425 | 0.2603 |
| $Y_{4:R_T}$ | Bias | 0.5464 | 0.7101 | 0.5804 |
| | MSPE | 0.5177 | 0.7172 | 0.5547 |
| $Y_{5:R_T}$ | Bias | 0.8149 | 1.0458 | 0.8608 |
| | MSPE | 1.1200 | 1.5304 | 1.1933 |
| $Y_{6:R_T}$ | Bias | 1.3423 | 1.7275 | 1.4252 |
| | MSPE | 3.3585 | 4.4793 | 3.5772 |
| $Y_{7:R_T}$ | Bias | 1.1803 | 1.6669 | 1.2727 |
| | MSPE | 2.2093 | 3.5065 | 2.4124 |
| $Y_{8:R_T}$ | Bias | 1.2488 | 1.9268 | 1.3654 |
| | MSPE | 3.3043 | 4.9799 | 3.5136 |

**Table 7.** and MSPEs of point predictors for the censoring scheme $R_1$.

| $\theta = 2$ and $T = 1$ | | | | |
|---|---|---|---|---|
| | | BUP | MLP | CMP |
| $Y_{1:R_3}$ | Bias | -0.0052 | 0.1607 | 0.0457 |
| | MSPE | 0.0322 | 0.0546 | 0.0326 |
| $Y_{2:R_3}$ | Bias | 0.0023 | 0.2378 | 0.0721 |
| | MSPE | 0.1186 | 0.1584 | 0.1174 |
| $Y_{3:R_3}$ | Bias | 0.0105 | 0.4371 | 0.1375 |
| | MSPE | 0.4778 | 0.5974 | 0.4708 |
| $Y_{1:R_5}$ | Bias | 0.0071 | 0.1731 | 0.0581 |
| | MSPE | 0.0341 | 0.0607 | 0.0357 |
| $Y_{2:R_5}$ | Bias | -0.0063 | 0.2291 | 0.0634 |
| | MSPE | 0.1045 | 0.1401 | 0.1021 |
| $Y_{3:R_5}$ | Bias | 0.0408 | 0.4673 | 0.1677 |
| | MSPE | 0.4842 | 0.6239 | 0.4832 |
| $Y_{1:R_8}$ | Bias | 0.0009 | 0.1005 | 0.0315 |
| | MSPE | 0.0128 | 0.0219 | 0.0133 |
| $Y_{2:R_8}$ | Bias | -0.0009 | 0.1243 | 0.0356 |
| | MSPE | 0.0330 | 0.0435 | 0.0324 |
| $Y_{3:R_8}$ | Bias | -0.0035 | 0.1604 | 0.0414 |
| | MSPE | 0.0730 | 0.0864 | 0.0707 |
| $Y_{4:R_8}$ | Bias | -0.010 | 0.2233 | 0.0518 |
| | MSPE | 0.1605 | 0.1802 | 0.1538 |

**Table 8.** Biases and MSPEs of point predictors for the censoring scheme $R_2$.

| $\theta = 2$ and $T = 1$ | | | | |
|---|---|---|---|---|
| | | BUP | MLP | CMP |
| $Y_{1:R_5}$ | Bias | -0.0007 | 0.0977 | 0.0295 |
| | MSPE | 0.0109 | 0.0186 | 0.0108 |
| $Y_{2:R_5}$ | Bias | 0.0055 | 0.1355 | 0.0416 |
| | MSPE | 0.034 | 0.0446 | 0.0331 |
| $Y_{3:R_5}$ | Bias | 0.0219 | 0.1979 | 0.0663 |
| | MSPE | 0.0837 | 0.1027 | 0.0815 |
| $Y_{4:R_5}$ | Bias | 0.0312 | 0.2871 | 0.09245 |
| | MSPE | 0.1921 | 0.2262 | 0.1860 |
| $Y_{5:R_5}$ | Bias | 0.0581 | 0.5219 | 0.1757 |
| | MSPE | 0.6461 | 0.7552 | 0.6251 |

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