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پژوهشکده‌ی آمار

A Flexible Skew-Generalized Normal Distribution

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Abstract. In this paper, we consider a flexible skew-generalized normal distribution. This distribution is denoted by $FSGN(\lambda_1, \lambda_2; \theta)$. It contains the normal, skew-normal (Azzalini, 1985), skew generalized normal (Arellano-Valle et al., 2004) and skew flexible-normal (Gómez et al., 2011) distributions as special cases. Some important properties of this distribution are established. Also, the practical usefulness of $FSGN$ is illustrated via a well known real data set.

Keywords. Skew normal distribution; skew generalized normal distribution; skew flexible normal distribution; flexible skew generalized normal distribution.

MSC 2010: 62E15.

1 Introduction

The standard skew normal distribution as a generalization of the normal distribution, introduced by Azzalini (1985) at first. A random variable X has a standard skew normal distribution with parameter $\lambda \in \mathbb{R}$, denoted by $SN(\lambda)$, if its pdf is

$$f(x; \lambda) = 2\phi(x)\,\Phi(\lambda x), \qquad x \in \mathbb{R},$$

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where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal pdf (probability density function) and cdf (cumulative distribution function), respectively. This distribution has been studied and generalized by some researchers. Arellano-Valle et al. (2004) considered a generalization of $SN(\lambda)$ by the name of skew-generalized normal distribution defined in the following form

$$f(x;\lambda_1,\lambda_2) = 2\phi(x)\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right), \qquad x \in \mathbb{R},$$

where $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \geqslant 0$. This distribution is denoted by $X \sim SGN(\lambda_1,\lambda_2)$ and for the special case $\lambda_1^2 = \lambda_2$, is called skew-curved normal which is denoted by $X \sim SCN(\lambda_1)$. They derived the main properties of the SGN distribution. Gómez et al. (2011) considered an extension of the skew-normal model through the inclusion of an additional parameter which can lead to both uni and bimodal distributions. A random variable X has a skew-flexible-normal distribution if its pdf is

$$f(x;\lambda,\theta) = \frac{1}{1-\Phi(\theta)}\phi(|x|+\theta)\,\Phi(\lambda x), \qquad x \in \mathbb{R},$$

and a random variable X having the above density is denoted by $X \sim SFN(\lambda,\theta)$. They presented various basic properties of this family of distributions and provided a stochastic representation which is useful for obtaining theoretical properties and to simulate from this distribution. Moreover, they investigated the singularity of the Fisher information matrix and considered maximum likelihood estimation for a random sample with no covariates.

In this paper we introduce a new flexible generalization of standard skew-normal distribution named *A Flexible Skew-Generalized Normal Distribution*, denoted by $FSGN(\lambda_1,\lambda_2;\theta)$, that the standard skew-normal ($SN(\lambda)$), the skew-generalized normal ($SGN(\lambda_1,\lambda_2)$) and the skew-flexible-normal ($SFN(\lambda,\theta)$) distributions are special cases of this distribution.

This paper is arranged as follows. In the next section, we present the definition and some simple properties of $FSGN(\lambda_1,\lambda_2;\theta)$. In Section 3, some important theorems concerning several useful properties are given. The moment generating function and some important theorems about the moments of this distribution are derived in Section 4 and in Section 5, by a real data set, we illustrate the practical usefulness of this distribution.

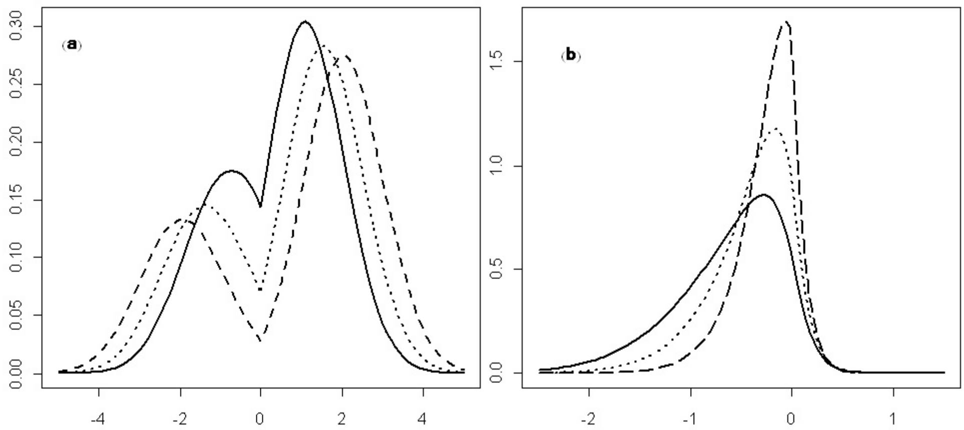

Figure 1. Some shapes of $FSGN(\lambda_1, \lambda_2; \theta)$: (a) $FSGN(0.5, 1; -1)$ (solid line), $FSGN(0.5, 1; -1.5)$ (dotted line), $FSGN(0.5, 1; -2)$ (dashed line); (b) $FSGN(-5, 2; 0.5)$ (solid line), $FSGN(-5, 2; 1.5)$ (dotted line), $FSGN(-5, 2; 3)$ (dashed line).

2 A Flexible Skew-Generalized Normal Distribution

In this section, we introduce a flexible generalization of standard skew-normal distribution and present some simple properties of this distribution.

Definition 1. A random variable X is said to have a flexible skew-generalized normal distribution with parameters λ_1, λ_2 and θ, denoted by $FSGN(\lambda_1, \lambda_2; \theta)$, if its pdf is

$$f(x, \lambda_1, \lambda_2; \theta) = \{1 - \Phi(\theta)\}^{-1}\phi(|x| + \theta)\Phi\left(\frac{\lambda_1 x}{\sqrt{1 + \lambda_2 x^2}}\right), \qquad x \in \mathbb{R},$$

where $\theta, \lambda_1 \in \mathbb{R}$, $\lambda_2 \geqslant 0$ and $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal pdf and cdf, respectively.

By integration, it is easy to show that $\int_{-\infty}^{+\infty} f(x, \lambda_1, \lambda_2; \theta)dx = 1$, then $f(x, \lambda_1, \lambda_2; \theta)$ is a density function. In Definition 1, for a special case, if $\lambda_1^2 = \lambda_2$, the resulting density is called *flexible skew-curved normal distribution* and is denoted by $FSCN(\lambda_1; \theta)$. Figure 1 shows the shapes of $FSGN(\lambda_1, \lambda_2; \theta)$ for some different values of the parameters.

Definition 2. A location-scale flexible skew-generalized normal distribution is defined as the distribution of $Y = \mu + \sigma X$, where $X \sim FSGN(\lambda_1, \lambda_2; \theta)$, $\mu \in \mathbb{R}$ and $\sigma > 0$. Its density is given by

$$f(y;\Theta) = \frac{1}{\sigma\{1-\Phi(\theta)\}}\phi\left(\left|\frac{y-\mu}{\sigma}\right| + \theta\right)\Phi\left\{\frac{\lambda_1\,(y-\mu)}{\sqrt{\sigma^2 + \lambda_2\,\{y-\mu)^2}}\right\}, \quad y \in \mathbb{R},$$

where $\Theta = (\mu, \sigma; \lambda_1, \lambda_2; \theta)$. We denote this extension of distribution by $FSGN(\mu, \sigma; \lambda_1, \lambda_2; \theta)$.

Some simple properties of $FSGN(\lambda_1, \lambda_2; \theta)$ are presented as follows.

Theorem 1. *a.* $FSGN(\lambda_1, 0; 0) = SN(\lambda_1)$.

b. $FSGN(\lambda_1, \lambda_2; 0) = SGN(\lambda_1, \lambda_2)$.

c. $FSGN(\lambda_1, 0; \theta) = FSN(\lambda_1, \theta)$.

d. $f(x, 0, \lambda_2; \theta) = \frac{1}{2\{1-\Phi(\theta)\}}\phi(|x| + \theta)$ *for all* $x \in \mathbb{R}$ *and* $\lambda_2 \geqslant 0$.

e. $\lim_{\lambda_1 \to +\infty} f(x, \lambda_1, \lambda_2; \theta) = \{1 - \Phi(\theta)\}^{-1}\phi(x + \theta)I(x \geqslant 0)$ *for fixed* λ_2.

f. $\lim_{\lambda_1 \to -\infty} f(x, \lambda_1, \lambda_2; \theta) = \{1 - \Phi(\theta)\}^{-1}\phi(x - \theta)I(x < 0)$ *for fixed* λ_2.

g. If $X \sim FSGN(\lambda_1, \lambda_2; \theta)$, *then* $-X \sim FSGN(-\lambda_1, \lambda_2; \theta)$.

h. $f(x, \lambda_1, \lambda_2; \theta) + f(x, -\lambda_1, \lambda_2; \theta) = \{1 - \Phi(\theta)\}^{-1}\phi(|x| + \theta)$ *for all* $x \in \mathbb{R}$.

Proof. The proof is easy. □

3 Some Important Properties of $FSGN(\lambda_1, \lambda_2; \theta)$

In this Section, we derive some important properties of $FSGN(\lambda_1, \lambda_2; \theta)$ distribution.

Theorem 2. *Let* $X \sim FSGN(\lambda_1, \lambda_2; \theta)$ *and* $Y \sim N(-\theta, 1)$. *Then* $|X| \overset{d}{=} \{Y | Y \geqslant 0\}$.

Proof. If $Y \sim N(-\theta, 1)$, then

$$f_{Y|Y\geqslant 0}(y) = \{1 - \Phi(\theta)\}^{-1}\phi(y + \theta)I(y \geqslant 0),$$

and if $X \sim FSGN(\lambda_1, \lambda_2; \theta)$, then

$$\begin{aligned} f_{|X|}(x) &= f_X(x) + f_X(-x) \\ &= \{1-\Phi(\theta)\}^{-1}\phi(x+\theta)\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right) \\ &\quad +\{1-\Phi(\theta)\}^{-1}\phi(x+\theta)\Phi\left(-\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right) \\ &= \{1-\Phi(\theta)\}^{-1}\phi(x+\theta)I(x \geqslant 0). \end{aligned}$$

□

We need the next lemma introduced by Ellison (1964) to present the next theorem.

***Lemma* 1.** *If $X \sim N(\mu, \sigma^2)$, then $E\{\Phi(X)\} = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$.*

Proof. Suppose that $Y \sim N(0,1)$ and X, Y are independent. Then

$$E\{\Phi(X)\} = \Pr(Y \leqslant X) = \Pr(Y - X \leqslant 0),$$

and since $Y - X \sim N\left(-\mu, 1+\sigma^2\right)$, thus

$$E\{\Phi(X)\} = \Pr\left(\frac{Y-X+\mu}{\sqrt{1+\sigma^2}} \leqslant \frac{\mu}{\sqrt{1+\sigma^2}}\right) = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right).$$

□

***Theorem* 3.** *If $X|Y = y \sim SFN(y, \theta)$ and $Y \sim N(\lambda_1, \lambda_2)$, then $X \sim FSGN(\lambda_1, \lambda_2; \theta)$.*

Proof.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} \frac{1}{1-\Phi(\theta)}\phi(|x|+\theta)\,\Phi(yx)\,\frac{1}{\lambda_2}\phi\left(\frac{y-\lambda_1}{\sqrt{\lambda_2}}\right)dy \\ &= \int_{-\infty}^{+\infty} \frac{1}{1-\Phi(\theta)}\phi(|x|+\theta)\,\Phi\left(\sqrt{\lambda_2}xz+\lambda_1 x\right)\phi(z)\,dz \\ &= \frac{1}{1-\Phi(\theta)}\phi(|x|+\theta)\,E\left\{\Phi\left(\sqrt{\lambda_2}xZ+\lambda_1 x\right)\right\} \\ &= \{1-\Phi(\theta)\}^{-1}\phi(|x|+\theta)\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right), \end{aligned}$$

where $Z \sim N(0,1)$ and the last equality is obtained by Lemma 1. □

Theorem 4. *If* $X|Y = y \sim FSCN(y;\theta)$ *and* $Y \sim N(0,1)$, *then*

$$f_X(x) = \frac{1}{2\{1-\Phi(\theta)\}}\phi(|x|+\theta), \qquad x \in \mathbb{R},$$

and $Y|X = x \sim SCN(x)$.

Proof.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} \frac{1}{1-\Phi(\theta)}\phi(|x|+\theta)\Phi\left(\frac{yx}{\sqrt{1+y^2x^2}}\right)\phi(y)\,dy \\ &= \frac{1}{2\{1-\Phi(\theta)\}}\phi(|x|+\theta)\int_{-\infty}^{+\infty} 2\phi(y)\,\Phi\left(\frac{yx}{\sqrt{1+y^2x^2}}\right)dy \\ &= \frac{1}{2\{1-\Phi(\theta)\}}\phi(|x|+\theta), \qquad x \in \mathbb{R}. \end{aligned}$$

Also, we have

$$\begin{aligned} f_{Y|X}(y) &= \frac{\frac{1}{1-\Phi(\theta)}\phi(|x|+\theta)\Phi\left(\frac{yx}{\sqrt{1+y^2x^2}}\right)\phi(y)}{\frac{1}{2\{1-\Phi(\theta)\}}\phi(|x|+\theta)} \\ &= 2\phi(y)\,\Phi\left(\frac{xy}{\sqrt{1+x^2y^2}}\right), \end{aligned}$$

then $Y|X = x \sim SCN(x)$. □

Theorem 5. *Let* $X \sim FSGN(\lambda_1, \lambda_2; \theta)$. *If* $\theta \neq 0$, *then the density function* $f(x, \lambda_1, \lambda_2; \theta)$ *is not differentiable at* $x = 0$.

Proof. Notice that the right derivative at $x = 0$ is

$$\frac{\phi(\theta)}{1-\Phi(\theta)}\left(\frac{\lambda_1}{2\pi} - \frac{\theta}{2}\right),$$

and the left derivative at $x = 0$ is

$$\frac{\phi(\theta)}{1-\Phi(\theta)}\left(\frac{\lambda_1}{2\pi} + \frac{\theta}{2}\right),$$

so, the density function $f(x, \lambda_1, \lambda_2; \theta)$ is not differentiable at $x = 0$. □

***Theorem* 6.** *Let $X \sim FSGN(\lambda_1, \lambda_2; \theta)$. If $\theta \geqslant 0$, then X is a strongly unimodal random variable.*

Proof. By differentiation from $f(x, \lambda_1, \lambda_2; \theta)$ with respect to x and equating to zero, we obtain

$$x = \frac{\lambda_1}{(1+\lambda_2 x^2)^{\frac{3}{2}}} \times \frac{\phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}{\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)} - \theta, \qquad \text{for} \quad x \geqslant 0, \tag{1}$$

$$x = \frac{\lambda_1}{(1+\lambda_2 x^2)^{\frac{3}{2}}} \times \frac{\phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}{\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)} + \theta, \qquad \text{for} \quad x < 0. \tag{2}$$

To prove that X is a strongly unimodal random variable, it is enough to show that $f(x, \lambda_1, \lambda_2; \theta)$ is a logconcave function of x, for all $x \in \mathbb{R}$ (see Ibragimov, 1956). Thus

$$\frac{d^2}{dx^2} \log f(x, \lambda_1, \lambda_2; \theta) = -\frac{\lambda_1^2}{(1+\lambda_2 x^2)^3} \times \frac{\phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}{\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}$$

$$\times \left\{ \frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}} + \frac{\phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}{\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)} \right\} - \frac{3\lambda_1 \lambda_2 x}{(1+\lambda_2 x^2)^{\frac{5}{2}}} \times \frac{\phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}{\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)} - 1.$$

Since $\phi(t) + t\Phi(t) > 0$, for all $t \in \mathbb{R}$ (see Azzalini, 1986) and because of equations (1) and (2) we have $\frac{d^2}{dx^2} \log f(x, \lambda_1, \lambda_2; \theta)$ is a negative expression and the proof is completed. □

***Theorem* 7.** *Let $X \sim FSGN(\lambda_1, \lambda_2; \theta)$. If $\theta < 0$, then X is a bimodal random variable.*

Proof. As for equations (1) and (2), it is easy to show that if $\lambda_1 = 0$, then $x = \theta$ for $x < 0$, and $x = -\theta$ for $x \geqslant 0$. If $\lambda_1 \to +\infty$ (for fixed λ_2) then $x \to \theta$ for $x \geqslant 0$ and $\lambda_1 \to -\infty$ (for fixed λ_2) then $x \to -\theta$ for $x < 0$. Thus X is a bimodal random variable. □

4 Moments of $FSGN(\lambda_1, \lambda_2; \theta)$

In this section, we discuss the moments of $FSGN(\lambda_1, \lambda_2; \theta)$. We show that the even moments have a closed form, but there is no explicit expression for the odd moments of this distribution. We give the moment generating function of $FSGN(\lambda_1, \lambda_2; \theta)$ that has to be computed numerically. Furthermore, we present the moment generating function of X^2 which has a closed form and the even moments of $FSGN(\lambda_1, \lambda_2; \theta)$ can be calculated using it.

***Theorem* 8.** *Let* $X \sim FSGN(\lambda_1, \lambda_2; \theta)$. *Then*

$$E\left(X^{2r}\right) = \frac{1}{\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}} \sum_{i=0}^{2r} \binom{2r}{i} \theta^{2r-i} 2^{\frac{i-1}{2}} \Gamma\left(\frac{i+1}{2}\right) \left\{1 - F_{Z_i}\left(\theta^2\right)\right\},$$

where $F_{Z_i}(\cdot)$ *is the cdf of* $Z_i \sim \chi^2_{\left(\frac{i+1}{2}\right)}$.

Proof. By Theorem 2, we know that if $f_Y(y) = \frac{\phi(y+\theta)I(y\geqslant 0)}{1-\Phi(\theta)}$, then $X^{2r} \overset{d}{=} Y^{2r}$. Thus, we have $E\left(X^{2r}\right) = E\left(Y^{2r}\right)$. And,

$$\begin{aligned}
E\left(Y^{2r}\right) &= \int_0^\infty \frac{y^{2r}}{\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}} e^{-\frac{1}{2}(y+\theta)^2} dy \\
&= \int_{\theta^2}^\infty \frac{\left(\sqrt{z}-\theta\right)^{2r}}{2\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}\sqrt{z}} e^{-\frac{1}{2}z} dz \\
&= \frac{1}{\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}} \sum_{i=0}^{2r} \binom{2r}{i} \theta^{2r-i} \int_{\theta^2}^\infty \frac{1}{2} e^{-\frac{1}{2}z} z^{\frac{i-1}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}} \sum_{i=0}^{2r} \binom{2r}{i} \theta^{2r-i} 2^{\frac{i-1}{2}} \Gamma\left(\frac{i+1}{2}\right) \left\{1 - F_{Z_i}\left(\theta^2\right)\right\}.
\end{aligned}$$

□

Theorem 9. *Let* $X \sim FSGN(\lambda_1, \lambda_2; \theta)$. *Then*

$$\begin{aligned}
E\left(X^{2r+1}\right) &= \frac{1}{\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}} \sum_{i=0}^{2r+1} \binom{2r+1}{i} \theta^{2r+1-i} 2^{\frac{i-1}{2}} \Gamma\left(\frac{i+1}{2}\right) \\
&\quad \times \left[2\int_{\theta^2}^{\infty} f_{Z_i}(z)\, \Phi\left\{ \frac{\lambda_1\left(\sqrt{z}-\theta\right)}{\sqrt{1+\lambda_2\left(\sqrt{z}-\theta\right)^2}} \right\} dz \right. \\
&\quad \left. - \left\{1 - F_{Z_i}\left(\theta^2\right)\right\} \right],
\end{aligned}$$

where $f_{Z_i}(\cdot)$ *and* $F_{Z_i}(\cdot)$ *are the pdf and cdf of* $Z_i \sim \chi^2_{\left(\frac{i+1}{2}\right)}$, *respectively.*

Proof.

$$\begin{aligned}
E\left(X^{2r+1}\right) &= \int_{-\infty}^{+\infty} \frac{x^{2r+1}}{1-\Phi(\theta)} \phi(|x|+\theta) \Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right) dx \\
&= 2\int_{0}^{\infty} \frac{x^{2r+1}}{1-\Phi(\theta)} \phi(x+\theta) \Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right) dx \\
&\quad - \int_{0}^{\infty} \frac{x^{2r+1}}{1-\Phi(\theta)} \phi(x+\theta) dx.
\end{aligned}$$

By Theorem 2 and Theorem 8, it is easy to show that

$$\begin{aligned}
\int_0^{\infty} \frac{x^{2r+1}}{1-\Phi(\theta)} \phi(x+\theta) dx = \frac{1}{\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}} \sum_{i=0}^{2r+1} \binom{2r+1}{i} \theta^{2r+1-i} 2^{\frac{i-1}{2}} \\
\times \Gamma\left(\frac{i+1}{2}\right) \left\{1 - F_{Z_i}\left(\theta^2\right)\right\}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\int_0^\infty \frac{x^{2r+1}}{1-\Phi(\theta)}\phi(x+\theta)\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)dx \\
=& \int_{\theta^2}^\infty \frac{\left(\sqrt{z}-\theta\right)^{2r+1}}{2\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}\sqrt{z}} e^{-\frac{1}{2}z}\Phi\left\{\frac{\lambda_1\left(\sqrt{z}-\theta\right)}{\sqrt{1+\lambda_2\left(\sqrt{z}-\theta\right)^2}}\right\}dz \\
=& \frac{1}{\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}}\sum_{i=0}^{2r+1}\left[\binom{2r+1}{i}\theta^{2r+1-i}\right. \\
&\left.\times\int_{\theta^2}^\infty \frac{1}{2}e^{-\frac{1}{2}z}z^{\frac{i-1}{2}}\Phi\left\{\frac{\lambda_1\left(\sqrt{z}-\theta\right)}{\sqrt{1+\lambda_2\left(\sqrt{z}-\theta\right)^2}}\right\}dz\right] \\
=& \frac{1}{\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}}\sum_{i=0}^{2r+1}\left[\binom{2r+1}{i}\theta^{2r+1-i}2^{\frac{i-1}{2}}\Gamma\left(\frac{i+1}{2}\right)\right. \\
&\left.\times\int_{\theta^2}^\infty f_{Z_i}(z)\,\Phi\left\{\frac{\lambda_1\left(\sqrt{z}-\theta\right)}{\sqrt{1+\lambda_2\left(\sqrt{z}-\theta\right)^2}}\right\}dz\right],
\end{aligned}$$

where $f_{Z_i}(\cdot)$ and $F_{Z_i}(\cdot)$ are the pdf and cdf of $Z_i \sim \chi^2_{\left(\frac{i+1}{2}\right)}$, respectively. Thus

$$\begin{aligned}
E\left(X^{2r+1}\right) =& \frac{1}{\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}}\sum_{i=0}^{2r+1}\binom{2r+1}{i}\theta^{2r+1-i}2^{\frac{i-1}{2}}\Gamma\left(\frac{i+1}{2}\right) \\
&\times\left[2\int_{\theta^2}^\infty f_{Z_i}(z)\,\Phi\left\{\frac{\lambda_1\left(\sqrt{z}-\theta\right)}{\sqrt{1+\lambda_2\left(\sqrt{z}-\theta\right)^2}}\right\}dz - \left\{1-F_{Z_i}\left(\theta^2\right)\right\}\right].
\end{aligned}$$

□

***Theorem* 10.** *Let $X \sim FSGN(\lambda_1,\lambda_2;\theta)$. Then the moment generating function of X is*

$$\begin{aligned}
M_X(t) =& \frac{e^{\frac{1}{2}\left\{(\theta-t)^2-\theta^2\right\}}}{2\left\{1-\Phi(\theta)\right\}}\int_{(\theta-t)^2}^\infty f_Z(z)\,\Phi\left[\frac{\lambda_1\left\{\sqrt{z}-(\theta-t)\right\}}{\sqrt{1+\lambda_2\left\{\sqrt{z}-(\theta-t)\right\}^2}}\right]dz \\
&+\frac{e^{\frac{1}{2}\left\{(\theta+t)^2-\theta^2\right\}}}{2\left\{1-\Phi(\theta)\right\}}\int_{(\theta+t)^2}^\infty f_Z(z)\,\Phi\left[\frac{\lambda_1\left\{\sqrt{z}-(\theta+t)\right\}}{\sqrt{1+\lambda_2\left\{\sqrt{z}-(\theta+t)\right\}^2}}\right]dz,
\end{aligned}$$

where $f_Z(\cdot)$ *is the pdf of* $Z \sim \chi^2_{(1)}$.

Proof.

$$
\begin{aligned}
M_X(t) &= \int_{-\infty}^{+\infty} \frac{e^{tx}}{1-\Phi(\theta)}\phi(|x|+\theta)\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)dx \\
&= e^{\frac{1}{2}\left\{(\theta-t)^2-\theta^2\right\}}\int_0^{\infty}\frac{\phi\{x+(\theta-t)\}}{1-\Phi(\theta)}\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)dx \\
&+ e^{\frac{1}{2}\left\{(\theta+t)^2-\theta^2\right\}}\int_0^{\infty}\frac{\phi\{x+(\theta+t)\}}{1-\Phi(\theta)}\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)dx \\
&= \frac{e^{\frac{1}{2}\left\{(\theta-t)^2-\theta^2\right\}}}{1-\Phi(\theta)}\int_{(\theta-t)^2}^{\infty}\frac{1}{2\sqrt{2\pi}}z^{-\frac{1}{2}}e^{-\frac{z}{2}}\Phi\left[\frac{\lambda_1\left\{\sqrt{z}-(\theta-t)\right\}}{\sqrt{1+\lambda_2\left\{\sqrt{z}-(\theta-t)\right\}^2}}\right]dz \\
&+ \frac{e^{\frac{1}{2}\left\{(\theta+t)^2-\theta^2\right\}}}{1-\Phi(\theta)}\int_{(\theta+t)^2}^{\infty}\frac{1}{2\sqrt{2\pi}}z^{-\frac{1}{2}}e^{-\frac{z}{2}}\Phi\left[\frac{\lambda_1\left\{\sqrt{z}-(\theta+t)\right\}}{\sqrt{1+\lambda_2\left\{\sqrt{z}-(\theta+t)\right\}^2}}\right]dz,
\end{aligned}
$$

then by taking $Z \sim \chi^2_{(1)}$, the proof is completed. □

Theorem 11. *Let* $X \sim FSGN(\lambda_1, \lambda_2; \theta)$, *the moment generating function of* X^2 *is*

$$
M_{X^2}(t) = \frac{1-\Phi\left(\frac{\theta}{\sqrt{1-2t}}\right)}{1-\Phi(\theta)} \times \frac{e^{\frac{t\theta^2}{1-2t}}}{\sqrt{1-2t}}.
$$

Proof.

$$
\begin{aligned}
M_{X^2}(t) &= \int_{-\infty}^{+\infty}\frac{e^{tx^2}}{1-\Phi(\theta)}\phi(|x|+\theta)\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)dx \\
&= \int_0^{\infty}\frac{e^{tx^2}}{1-\Phi(\theta)}\phi(x+\theta)dx = \int_0^{\infty}\frac{e^{-\frac{1}{2}\left(x^2+2\theta x+\theta^2-2tx^2\right)}}{\sqrt{2\pi}\left\{1-\Phi(\theta)\right\}}dx \\
&= \frac{e^{-\frac{1}{2}\theta^2}}{1-\Phi(\theta)}\int_0^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left\{(1-2t)x^2+2\theta x\right\}}dx \\
&= \frac{e^{-\frac{1}{2}\theta^2}}{\left\{1-\Phi(\theta)\right\}\sqrt{1-2t}}\int_0^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(x+\frac{\theta}{\sqrt{1-2t}}\right)^2}dx \\
&= \frac{1-\Phi\left(\frac{\theta}{\sqrt{1-2t}}\right)}{1-\Phi(\theta)} \times \frac{e^{\frac{t\theta^2}{1-2t}}}{\sqrt{1-2t}}.
\end{aligned}
$$

□

5 Data Illustration

In this section, we present a well known real data set to illustrate the applications of $FSGN$, that they are used in many papers about univariate skew models. We fit the SN, SGN, SFN, and $FSGN$ models on the data set and show that the $FSGN$ model fits the data set better than other sub models. In fact, we used the maximum likelihood estimation for our model selection among four models. If $X \sim FSGN(\mu, \sigma; \lambda_1, \lambda_2; \theta)$, based on n observations $x_1, x_2, \ldots, x_n$, the log likelihood function of the density of X is

$$\begin{aligned}\log L(\Theta) &= -n \log\{1 - \Phi(\theta)\} - n \log \sigma - \frac{n}{2} \log(2\pi) \\ &\quad - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (|x_i - \mu| + \theta\sigma)^2 + \sum_{i=1}^{n} \log \Phi \left\{ \frac{\lambda_1 (x_i - \mu)}{\sqrt{\sigma^2 + \lambda_2 (x_i - \mu)^2}} \right\},\end{aligned}$$

where $\Theta = (\mu, \sigma; \lambda_1, \lambda_2; \theta)$. We estimate parameters by numerically maximizing the log likelihood function.

Example 1. This example considers the data concerning the heights (in centimeters) of 100 Australian athletes, given in Cook and Weisberg (1994). The data are given in Table 1. By numerically maximizing the log likelihood functions and estimating parameters, we fit the SN, SGN, SFN, and $FSGN$ models on this data set. The obtained numerical results are presented in Table 2. The graphs of histogram of the data and fitted densities are plotted in Figure 2. Obviously, for the data set, the $FSGN$ model fits better than the three sub models, as expected.

Table 1. The heights (in cm) of 100 Australian athletes.

195.5	193.4	179.3	180.9	178.2	174	180.5	171.1	170.8	177.5
189.7	188.7	175.3	179.5	177.3	176	173.3	172.7	163	162.5
177.8	169.1	174	178.9	174.1	172.2	173.5	175.6	166.1	172.5
185	177.9	183.3	182.1	173.6	182.7	181	171.6	176	166.7
184.6	177.5	184.7	186.3	173.7	180.5	175	172.3	163.9	175
174	179.6	180.2	176.8	178.7	179.8	170.3	171.4	173	157.9
186.2	181.3	180.2	172.6	183.3	179.6	165	178	177	158.9
173.8	179.7	176	176	174.4	171.7	169.8	162	168	156.9
171.4	185.2	156	169.9	173.3	170	174.1	167.3	172	148.9
179.9	177.3	179.7	183	168.6	170	175	162	167.9	149

Table 2. MLEs for heights (in cm) of 100 Australian athletes.

Distribution	SN	SGN	SFN	$FSGN$
$\hat{\mu}$	174.583	170.320	176.001	169.4176
$\hat{\sigma}$	8.2009	9.2476	17.5778	8.0199
$\hat{\lambda}_1$	0.004	4.381	−0.4702	3.9714
$\hat{\lambda}_2$	–	24.184	–	15.5304
$\hat{\theta}$	–	–	2.2222	−0.4657
log likelihood	−352.318	−347.2427	−348.8878	−347.0899

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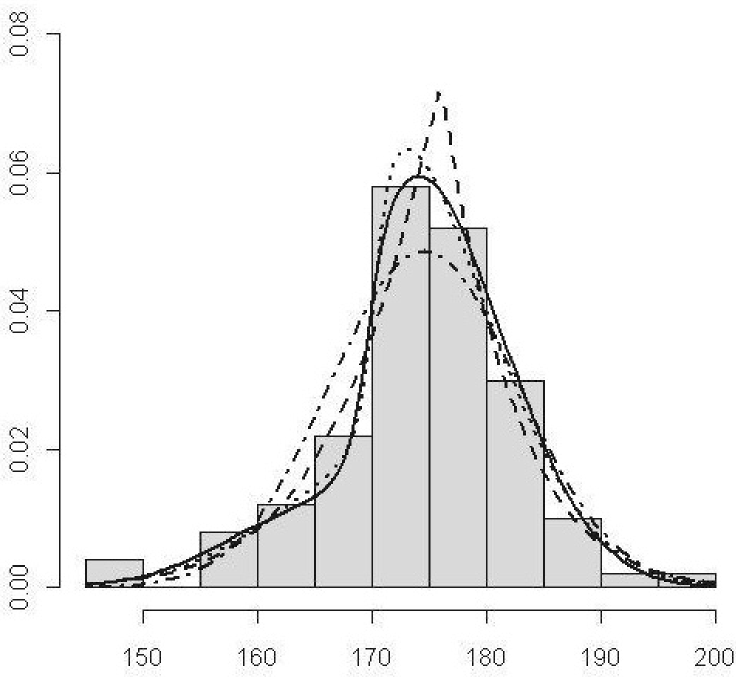

Figure 2. Histogram of heights (in cm) of 100 Australian athletes. The lines represent distributions fitted using MLE: $FSGN$ (solid line), SFN (dashed line), SGN (dotted line), SN (dotted-dashed line).

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