مجلّه‌ی پژوهش‌های آماری ایران

J. Statist. Res. Iran **17** (2020): 95–111

دوره‌ی ۱۷، شماره‌ی ۱، بهار و تابستان ۱۳۹۹، صص ۹۵–۱۱۱

DOI: 10.52547/jsri.17.1.95

DOR: 20.1001.1.25385771.2020.17.1.5.6

Power Normal-Geometric Distribution: Model, Properties and Applications

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Received: 2021/19/07 Approved: 2022/15/05

Abstract. In this paper, we introduce a new skewed distribution of which normal and power normal distributions are two special cases. This distribution is obtained by taking geometric maximum of independent identically distributed power normal random variables. We call this distribution as the power normal–geometric distribution. Some mathematical properties of the new distribution are presented. Maximum likelihood estimates of parameters are obtained via an EM algorithm. Simulation experiments have been presented to evaluate the performance of the maximum likelihood. We analyze two data sets for illustrative purposes. Finally, we derive a bivariate version of the proposed distribution.

Keywords. Geometric distribution, power normal distribution, EM algorithm.

MSC 2010: 62F10, 62N05.

1 Introduction

The normal distribution is one of the most important statistical distributions used in many sciences. This distribution is a symmetric distribution. There is data in many different fields of science that is asymmetric and normal distribution is not able to model it. Due to this reason, Azzalini (1985) introduced the univariate skew-normal (SN) distribution with following probability density function (pdf)

$$\phi(z;\lambda) = 2\phi(z)\Phi(\lambda z),$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cumulative distribution function (cdf) of a standard normal distribution and $-\infty < \lambda < \infty$ is the skewness parameter. It can be seen that the maximum likelihood estimators may not always exist. To overcome this problem Gupta and Gupta (2008) introduced a new distribution called the power normal (PN) distribution as an alternative to the Azzalini's skew normal distribution with cdf given by

$$F_{PN}(x;\mu,\sigma,\alpha) = \left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}; \qquad -\infty < x < \infty.$$

The corresponding pdf is

$$f_{PN}(x;\mu,\sigma,\alpha) = \frac{\alpha}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha-1}, \tag{1}$$

where $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the shape parameter and $\alpha > 0$ is the skewness parameter. When $\alpha = 1$, it becomes a normal distribution function with mean $\mu \in \mathbb{R}$ and standard deviation σ.

In this paper, we introduce a new four parameter skewed normal distribution which extends power normal distribution. The procedure used here is based on compound the power normal and geometric distributions. It can be positively and negatively skewed. Various properties of the proposed distribution are presented. We propose an efficient EM type algorithm to compute the MLEs of the parameters. Furthermore, we will define a bivariate distribution which has power normal marginals. Compounding method used here, is described in Marshall and Olkin (1997) and Adamidis and Loukas (1998) redor, more generally, by Chahkandi and Ganjali (2009), Barreto-Souza et al. (2011), Morais and Barreto-Souza (2011), Mahmoudi and Jafari (2012), Tahmasebi and Jafari (2015), Roozegar and Nadarajah (2016), Mahmoudi and Jafari (2017), Mahmoudi and mahmoodian (2017), Muhammad (2017), Bahrami and Yarmoghaddam (2019) and Roozegar et al. (2020). The main reasons for introducing the new compound distribution are: (i) This class of distributions is an important model that can be used in a variety of problems in modeling lifetime data. (ii) This distribution is a suitable model in a complementary risk problem base in the presence of latent risks which arise in several areas. (iii) It provides a reasonable parametric fit to skewed data that cannot be properly fitted by other distributions. (iv) This class contains some lifetime models as special cases.

The paper is organized as follows. In Section 2, the PNG distribution is introduced and studied some properties. The moment generating function and moments are obtaind in section 3. Estimation of the parameters by maximum likelihood method is presented in Section 4. In Section 5 a simulation study is given. Two data sets have been analyzed using the PNG model in Section 6. In section 7, a bivariate derivation of the proposed model is obtaind. Finally, Section 8 concludes the paper.

2 The Power Normal–geometric Distribution and Some Properties

Let $X_1, X_2, \cdots, X_N$ be independent identically distributed PN with pdf in Equation 1 random variables. N is a geometric distribution (truncated at zero) with the following probability mass function

$$P(N = n) = (1 - \theta)\theta^{n-1}, \quad n = 1, \dots, \ 0 < \theta < 1.$$

Moreover, N is independent of X_i's.

Definition 1. A random variable X is said to have a PNG distribution with parameters $(\mu, \sigma, \alpha, \theta)$ and is denoted by $X \sim PNG(\mu, \sigma, \alpha, \theta)$, if it has the following definition

$$X = \max(X_1, X_2, \cdots, X_N).$$

Before going to derive the cdf and the pdf we will briefly mention how it can happen in practice.

Consider a system having N number of independent and identical components attached in parallel. Here N is a random variable. If X_i denote the lifetime of the i-th component of the system, then the random variable X denoting the lifetime of the system is defined by $X = \max(X_1, X_2, \cdots, X_N)$.

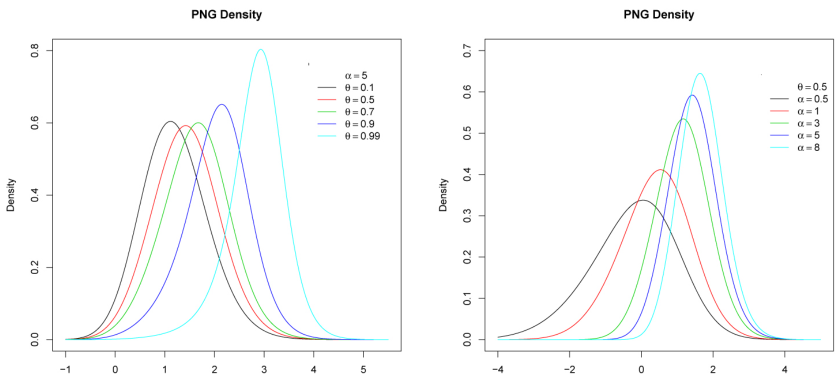

Figure 1. The pdf of PNG distribution for different values α, θ, $\mu = 0$ and $\sigma = 1$.

The cdf of $X \sim PNG(\mu, \sigma, \alpha, \theta)$ for $x \in \mathbb{R}$, is given by

$$\begin{aligned} F(x;\mu,\sigma,\alpha,\theta) &= P(X \leq x) = \sum_{k=1}^{\infty} P(X \leq x \mid N = k) P(N = k) \\ &= \sum_{k=1}^{\infty} \left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha k} \theta^k (1-\theta) \\ &= \frac{(1-\theta)\left[\Phi(\frac{x-\mu}{\sigma})\right]^{\alpha}}{1-\theta\left[\Phi(\frac{x-\mu}{\sigma})\right]^{\alpha}}. \qquad (2) \end{aligned}$$

Upon differentiating the expression of the cdf of X in 2, we obtain the pdf of X as

$$f(x;\mu,\sigma,\alpha,\theta) = \frac{(1-\theta)\alpha\phi(\frac{x-\mu}{\sigma})\left[\Phi(\frac{x-\mu}{\sigma})\right]^{\alpha-1}}{\sigma(1-\theta\left[\Phi(\frac{x-\mu}{\sigma})\right]^{\alpha})^2}. \qquad (3)$$

From 2, we obtain

$$\lim_{\theta \to 0} F(x;\mu,\sigma,\alpha,\theta) = \left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}.$$

The Pdf of the PNG distribution is drawn in Figure 1 for some values of the parameters. It is observed that the PNG pdf is an unimodal and increase of α and θ shift the pdfs to the right with increases in the heights of the pdfs.

The normal–geometric distribution introduced by Mahmoudi and mahmoodian (2017) is a special case of the PNG distribution with $\alpha = 1$.

Proposition 1. The densities of PNG class of distributions can be written as infinite number of linear combination of the density of the PN distribution with parameters μ, σ and $n\alpha$.

Proof. We have

$$f(x;\mu,\sigma,\alpha,\theta) = \sum_{n=1}^{\infty} P(N=n) g_{X_{(n)}}(x;n),$$

where $g_{X_{(n)}}(x;n)$ denotes the density function of $X_{(n)} = \max(X_1,..,X_n)$. Note that, applying the fact that $X_{(n)} \sim PN(\mu,\sigma,n\alpha)$, the PNG pdf can be written as follows:

$$f(x;\mu,\sigma,\alpha,\theta) = \sum_{n=1}^{\infty} P(N=n) f_{PN}(x;\mu,\sigma,n\alpha).$$

$\square$

The failure rate function of the PNG distribution is given by

$$h(x;\mu,\sigma,\alpha,\theta) = \frac{(1-\theta)\alpha\phi(\frac{y-\mu}{\sigma})\left[\Phi(\frac{x-\mu}{\sigma})\right]^{\alpha-1}}{\sigma(1-\left[\Phi(\frac{x-\mu}{\sigma})\right]^{\alpha})(1-\theta\left[\Phi(\frac{x-\mu}{\sigma})\right]^{\alpha})}.$$

Plots of the PNG failure rate function for selected parameter values are given in Figure 2.

***Theorem* 1.** *Suppose that $X_1 \sim PNG(0,1,\alpha_1,\theta_1)$ and $X_2 \sim PNG(0,1,\alpha_2,\theta_2)$. If $\theta_1 > \theta_2$ and $\alpha_1 = \alpha_2$, then $X_2 <_{LR} X_1$.*

Proof. we have

$$\frac{d}{dx}\log\left(\frac{f(x;0,1,\alpha_1,\theta_1)}{f(x;0,1,\alpha_2,\theta_2)}\right) = \frac{(\alpha_1-\alpha_2)\,\phi(x)}{\Phi(x)}$$
$$+\frac{2\phi(x)\left\{\theta_1\alpha_1\,[\Phi(x)]^{\alpha_1-1} - \theta_2\alpha_2\,[\Phi(x)]^{\alpha_2-1} + \theta_1\theta_2\,(\alpha_2-\alpha_1)\,[\Phi(x)]^{\alpha_1+\alpha_2-1}\right\}}{(1-\theta_2\,[\Phi(x)]^{\alpha_2})\,(1-\theta_1\,[\Phi(x)]^{\alpha_1})}. \tag{4}$$

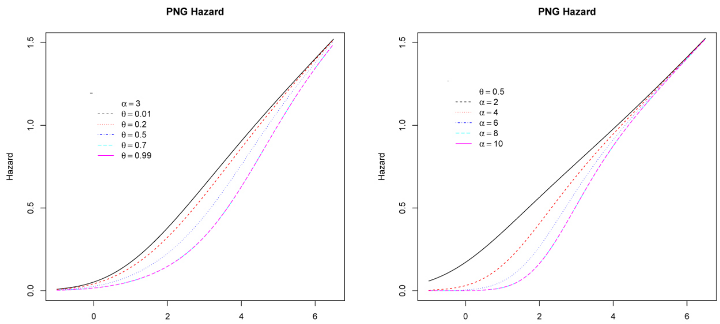

Figure 2. The hazard rate function of PNG distribution for different values α, θ, $\mu = 0$ and $\sigma = 1$.

Clearly if $\theta_1 > \theta_2$ and $\alpha_1 = \alpha_2$, then 4 is positive. Therefore, the PNG has the likelihood ratio ordering, which implies it has the failure rate ordering as well as the stochastic ordering and the mean residual life ordering. $\square$

The qth quantile of the PNG distributions is given by

$$x_q = \mu + \sigma\Phi^{-1}\left(\left[\frac{q}{1-\theta\,(1-q)}\right]^{\frac{1}{\alpha}}\right).$$

Using the above expression, generating random data from the PNG distribution is simple.

3 Moment Generating and Moments

In the following, using Proposition ??, without loss of generality, we provide the moment generating function, kth moment and the first moment of a random variable $X \sim PNG(0, 1, \alpha, \theta)$. Here we use, $\Phi_n(\cdot\ ; \mathbf{\Sigma})$ for the cdf of $N_n(\mathbf{0}, \mathbf{\Sigma})$. Furthermore, for $r, k \in \mathbb{N}$, let $\mathbf{1}_r$, $\mathbf{0}_r$ and $\mathbf{I}_r$ denote the vector of ones, the vector of zeros and the identity matrix of dimension r, respectively. If $X \sim PNG(0, 1, \alpha, \theta)$ and α is integer, then the moment generating function, kth moment and mean of X are given by

$$M_X(t) = \sum_{n=1}^{\infty}(1-\theta)\theta^{n-1}M_{X_{(n)}}(t)$$

Note that when α is integer, applying the fact that distribution of maximum of a size n from the PN distribution is the same as distribution of maximum of a size $n\alpha$ from the normal distribution and based on Jamalizadeh and Balakrishnan (2010)

$$M_X(t) = \exp\left(\frac{1}{2}t^2\right)\sum_{n=1}^{\infty}(1-\theta)\theta^{n-1}\times n\alpha\Phi_{n\alpha-1}(\mathbf{1}_{n\alpha-1}t;\mathbf{I}_{n\alpha-1}+\mathbf{1}_{n\alpha-1}^T\mathbf{1}_{n\alpha-1}),$$

$$\begin{aligned} E(X^{k+1}) &= \sum_{n=1}^{\infty}(1-\theta)\theta^{n-1}kE(X^{k-1}) + \sum_{n=1}^{\infty}(1-\theta)\theta^{n-1} \\ &\times \frac{(n\alpha-1)\Phi_{n\alpha-2}\left(\mathbf{0}_{n\alpha-2};\mathbf{I}_{n\alpha-2}+\frac{1}{2}\mathbf{1}_{n\alpha-2}^T\mathbf{1}_{n\alpha-2}\right)}{2\sqrt{\pi}\Phi_{n\alpha-1}\left(\mathbf{0}_{n\alpha-1};\mathbf{I}_{n\alpha-1}+\mathbf{1}_{n\alpha-1}^T\mathbf{1}_{n\alpha-1}\right)} \\ &\times E\left(Z^k_{n\alpha-2,\frac{1}{\sqrt{2}}\mathbf{1}_{n\alpha-2},\mathbf{I}_{n\alpha-2}}\right), \end{aligned}$$

and

$$E(X) = \frac{1}{2\sqrt{\pi}}\sum_{n=1}^{\infty}(1-\theta)\theta^{n-1}\times n\alpha(n\alpha-1)\Phi_{n\alpha-2}\left(\mathbf{0}_{n\alpha-2};\mathbf{I}_{n\alpha-2}+\frac{1}{2}\mathbf{1}_{n\alpha-2}^T\mathbf{1}_{n\alpha-2}\right),$$

respectively. One can derive the second moment of PNG distribution as

$$\begin{aligned} E(X^2) &= 1+\frac{\alpha(1-\theta)}{4\sqrt{3}\pi}\sum_{n=1}^{\infty}\theta^{n-1}n(n\alpha-1)(n\alpha-3) \\ &\times \Phi_{n\alpha-3}\left(\mathbf{0}_{n\alpha-3};\mathbf{I}_{n\alpha-3}+\frac{1}{3}\mathbf{1}_{n\alpha-3}^T\mathbf{1}_{n\alpha-3}\right). \end{aligned}$$

Table 1 lists the first four moments, variance, skewness and kurtosis of the $PNG(0,1,\alpha,\theta)$ for different values α and θ.

Table 1. The first four moments, variance, skewness and kurtusis of PNG distribution for $\mu = 0, \sigma = 1$.

	$\alpha = 2$			$\alpha = 4$			$\alpha = 6$		
	$\theta = 0.1$	$\theta = 0.5$	$\theta = 0.9$	$\theta = 0.1$	$\theta = 0.5$	$\theta = 0.9$	$\theta = 0.1$	$\theta = 0.5$	$\theta = 0.9$
$E(X)$	0.6133	0.8878	1.6179	1.0710	1.3056	1.9439	1.3055	1.5219	2.1176
$E(X^2)$	1.0602	1.4749	3.2261	1.6424	2.2125	4.2558	2.1241	2.7513	4.9051
$E(X^3)$	1.5505	2.4995	6.9808	2.9022	4.2427	10.0280	3.9515	5.5531	12.1061
$E(X^4)$	3.2632	5.2069	16.3990	5.8471	9.0400	25.1023	8.1930	12.2525	31.4899
VAR	0.6842	0.6866	0.6085	0.4954	0.5078	0.4770	0.4198	0.4352	0.4211
SK	0.1080	-0.0513	-0.4375	0.2356	0.0773	-0.3023	0.3033	0.1443	-0.2351
KUR	3.0507	3.0585	3.6037	3.1391	3.0795	3.4361	3.2039	3.1082	3.3726

4 Estimation and Inference

Let $x_1, \cdots, x_n$ be n observations from $PNG(\mu, \sigma, \alpha, \theta)$ and $\mathbf{\Theta} = (\mu, \sigma, \alpha, \theta)^T$ be the parameter vector. Then the log-likelihood function can be written as

$$\begin{aligned} l_n(\mathbf{\Theta}; \mathbf{x}) &= n\log(\alpha\theta) - n\log(\sigma) - \frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n}\log(C^{'}\left([\theta\Phi\left(u_i\right)]^{\alpha}\right) \\ &+ (\alpha - 1)\sum_{i=1}^{n}\log\left(\Phi\left(u_i\right)\right) - n\log\left(\frac{\theta}{1-\theta}\right), \end{aligned}$$

where $u_i = \frac{x_i - \mu}{\sigma}$ for $i = 1, \cdots, n$. The maximum likelihood estimators (MLEs) of $\mathbf{\Theta} = (\mu, \sigma, \alpha, \theta)^T$ can be obtained by maximizing the log-likelihood function with respect to the PNG parameters. The normal equations can be obtained by taking derivatives of l_n with respect to μ, σ, α and θ, respectively, and equating them to 0. Clearly, MLEs cannot be obtained in closed forms. We propose to use EM algorithm to compute the MLEs. Let, $(X_i, Z_i), i = 1, 2, \cdot, n$ be the complete data, where X_i is the observed data and Z_i is considered as missing data. We define a hypothetical complete-data distribution with a joint probability density function in the form

$$g(z, x; \mathbf{\Theta}) = \frac{\alpha z(1-\theta)\theta^{z-1}}{\sigma}\phi\left(\frac{x_i - \mu}{\sigma}\right)\Phi^{\alpha z - 1}\left(\frac{x_i - \mu}{\sigma}\right),$$

where $\theta, \sigma > 0,\ \alpha > 1, x \in \mathbb{R}$ and $z \in N$. Suppose $\mathbf{\Theta}^{(h)} = (\mu^{(h)}, \sigma^{(h)}, \alpha^{(h)}, \theta^{(h)})$ is the current estimate (in the hth iteration) of $\mathbf{\Theta}$. Based on the EM algorithm principle, in the E-step, we should first present the expectation of $(Z \mid X; \mathbf{\Theta}^{(h)})$. The conditional probability mass function of Z given $X = x$ is

$$g(z \mid x) = \frac{g(z, x; \mathbf{\Theta})}{f(x)} = \frac{\theta^{z-1} z \Phi^{\alpha(z-1)}\left(\frac{x_i-\mu}{\sigma}\right)}{\left(1-\theta\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}\right)^{-2}}.$$

The conditional expectation become

$$E(Z \mid X = x) = 1 + \frac{2\theta\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}}{1-\theta\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}}. \tag{5}$$

The M-step of EM cycle is completed by using the maximum likelihood estimation over $\mathbf{\Theta}$, with the missing Z's replaced by their conditional expectations given in 5. The complete-data log likelihood, ignoring the values that do not depend on the parameters, is obtained

$$\begin{aligned} l_n = (\mathbf{x}, \mathbf{z}; \mathbf{\Theta}) \quad \propto \quad & \sum_{i=1}^{n} z_i \log\theta + n\log\alpha - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \\ + \quad & \sum_{i=1}^{n}(\alpha z_i - 1)\log\left(\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right) - n\log\left(\frac{\theta}{1-\theta}\right). \end{aligned}$$

The components of the score function $U_c(\mathbf{x}, \mathbf{z}; \mathbf{\Theta}) = \left(\frac{\partial l_n^*}{\partial\mu}, \frac{\partial l_n^*}{\partial\sigma}, \frac{\partial l_n^*}{\partial\alpha}, \frac{\partial l_n^*}{\partial\theta}\right)$, are

$$\begin{aligned} \frac{\partial l_n^*}{\partial\mu} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) - \frac{1}{\sigma}\sum_{i=1}^{n}(\alpha z_i - 1)\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}, \\ \frac{\partial l_n^*}{\partial\sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{1}{\sigma^2}\sum_{i=1}^{n}(\alpha z_i - 1)\frac{(x_i-\mu)\,\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}, \\ \frac{\partial l_n^*}{\partial\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} z_i \log\left(\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right), \\ \frac{\partial l_n^*}{\partial\theta} &= \frac{1}{\theta}\sum_{i=1}^{n} z_i - \frac{n}{\theta\,(1-\theta)}. \end{aligned}$$

The maximum likelihood estimates can be obtained from the iterative algorithm given by

$$\frac{1}{\widehat{\sigma}^{(h)}}\sum_{i=1}^{n}\left(x_i-\widehat{\mu}^{(h+1)}\right)-\sum_{i=1}^{n}\left(\widehat{\alpha}^{(h)}\widehat{z}_i^{(h)}-1\right)\frac{\phi\left(\frac{x_i-\widehat{\mu}^{(h+1)}}{\widehat{\sigma}^{(h)}}\right)}{\Phi\left(\frac{x_i-\widehat{\mu}^{(h+1)}}{\widehat{\sigma}^{(h)}}\right)}=0,$$

$$\frac{1}{\left(\widehat{\sigma}^{(h+1)}\right)^2}\sum_{i=1}^{n}\left(x_i-\widehat{\mu}^{(h)}\right)^2-\frac{1}{\widehat{\sigma}^{(h+1)}}\sum_{i=1}^{n}\left(\widehat{\alpha}^{(h)}\widehat{z}_i^{(h)}-1\right)\frac{\left(x_i-\widehat{\mu}^{(h)}\right)\phi\left(\frac{x_i-\widehat{\mu}^{(h)}}{\widehat{\sigma}^{(h+1)}}\right)}{\Phi\left(\frac{x_i-\widehat{\mu}^{(h)}}{\widehat{\sigma}^{(h+1)}}\right)}=n,$$

$$\sum_{i=1}^{n}\widehat{z}_i^{(h)}\log\left(\Phi\left(\frac{x_i-\widehat{\mu}^{(h)}}{\widehat{\sigma}^{(h)}}\right)\right)=0, \qquad (1-\widehat{\theta}^{(h+1)})\sum_{i=1}^{n}\widehat{z}_i^{(h)}=n,$$

where $\widehat{\mu}^{(h+1)}$, $\widehat{\sigma}^{(h+1)}$, $\widehat{\alpha}^{(h+1)}$ and $\widehat{\theta}^{(h+1)}$ are found numerically. Here, for $i=1,...,n$, we have than

$$\widehat{z}_i^{(h)}=1+\frac{2\widehat{\theta}^{(h)}\left[\Phi\left(\frac{x_i-\widehat{\mu}^{(h)}}{\widehat{\sigma}^{(h)}}\right)\right]^{\widehat{\alpha}^{(h)}}}{1-\widehat{\theta}^{(h)}\left[\Phi\left(\frac{x_i-\widehat{\mu}^{(h)}}{\widehat{\sigma}^{(h)}}\right)\right]^{\widehat{\alpha}^{(h)}}}.$$

5 Simulation

Now, we present some Monte Carlo simulation results to show how the proposed EM algorithm performs PNG parameters. The simulations was performed using the R software. The simulations are performed 1000 times under the PNG distribution with different sets of parameters and sample sizes $n = 50, 100, 300$ and 500. We compute we computed the biases and root of mean-squared errors (RMSE) of MLEs. The results are reported in Table 2. Note that in the simulation example, the biases generally decrease to zero as sample size increases. Also, the RMSE of the estimators becomes smaller when the sample size increases and this illustrates the consistency of the estimators.

Table 2. The biases of the 1000 MLE's and RMSE of EM estimators for the PNG distribution.

		Bias				RMSE			
n	$(\mu,\sigma,\alpha,\theta)$	$\widehat{\mu}$	$\widehat{\sigma}$	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\mu}$	$\widehat{\sigma}$	$\widehat{\alpha}$	$\widehat{\theta}$
red25	(0.0, 1.0,2,0.2)	-0.0029	-0.423	-0.1201	0.0801	0.0071	0.1201	0.5927	0.3121
	(0.0, 1.0,2,0.5)	-0.0015	-0.0163	0.0776	-0.0553	0.0058	0.1148	0.8002	0.3052
	(0.0, 1.0,2,0.9)	0.0010	0.0171	1.2800	-0.1191	0.0690	0.1182	5.000	0.2701
	(0.0, 1.0,3,0.2)	-0.0023	-0.0461	0.8301	0.3015	0.0053	0.1171	0.9059	0.3164
	(0.0, 1.0,3,0.5)	-0.0008	-0.0154	0.1401	-0.0587	0.0040	0.1084	1.1203	0.3199
	(0.0, 1.0,3,0.9)	-0.0111	0.0228	1.7693	-0.1227	0.1706	0.1200	7.1906	0.2671
50	(0.0, 1.0,2,0.2)	-0.0025	-0.411	-0.1096	0.0773	0.0060	0.1181	0.5767	0.2977
	(0.0, 1.0,2,0.5)	-0.0010	-0.0153	0.0764	-0.0547	0.0050	0.1126	0.7903	0.3052
	(0.0, 1.0,2,0.9)	0.0010	0.0161	1.2847	-0.1171	0.0690	0.1121	4.7024	0.2618
	(0.0, 1.0,3,0.2)	-0.0019	-0.0438	0.8230	0.3005	0.0047	0.1121	0.9050	0.3137
	(0.0, 1.0,3,0.5)	-0.0008	-0.0142	0.1351	-0.0587	0.0035	0.1057	1.1934	0.3188
	(0.0, 1.0,3,0.9)	-0.0101	0.0224	1.7679	-0.1218	0.1694	0.1182	7.1701	0.2638
100	(0.0, 1.0,2,0.2)	-0.0016	-0.0267	-0.1095	0.0690	0.0042	0.0878	0.4604	0.2597
	(0.0, 1.0,2,0.5)	-0.0007	-0.0013	0.0707	-0.0618	0.0034	0.0852	0.6503	0.2734
	(0.0, 1.0,2,0.9)	0.0000	0.0062	0.4764	-0.0482	0.0030	0.0692	1.5248	0.1403
	(0.0, 1.0,3,0.2)	-0.0015	-0.022	0.8547	0.0667	0.0036	0.0837	0.6776	0.2633
	(0.0, 1.0,3,0.5)	-0.0005	-0.0105	0.0788	-0.0466	0.0024	0.0840	0.9509	0.2764
	(0.0, 1.0,3,0.9)	0.0000	0.0049	0.6673	-0.048	0.0022	0.0638	2.1000	0.1300
300	(0.0, 1.0,2,0.2)	0.0007	-0.0106	-0.0577	0.0286	0.0020	0.0543	0.2920	0.1858
	(0.0, 1.0,2,0.5)	0.0000	0.0007	0.0417	-0.0288	0.0012	0.0544	0.4070	0.1802
	(0.0, 1.0,2,0.9)	-0.0002	0.0045	0.1028	-0.0144	0.0021	0.0410	0.8081	0.0623
	(0.0, 1.0,3,0.2)	-0.0006	-0.0092	-0.0825	0.0356	0.0016	0.0516	0.4099	0.1911
	(0.0, 1.0,3,0.5)	-0.0001	-0.0014	0.0402	-0.0313	0.0009	0.0546	0.5735	0.1933
	(0.0, 1.0,3,0.9)	-0.0001	0.0009	0.1992	-0.0146	0.0014	0.0391	1.1400	0.0632
500	(0.0, 1.0,2,0.2)	-0.0005	-0.0068	-0.0338	0.0211	0.0013	0.0419	0.2307	0.1545
	(0.0, 1.0,2,0.5)	0.0001	0.0020	0.0282	-0.0223	0.0009	0.0416	0.3105	0.1430
	(0.0, 1.0,2,0.9)	-0.0001	0.0022	0.0883	-0.0095	0.0015	0.0312	0.5989	0.0427
	(0.0, 1.0,3,0.2)	-0.0004	-0.0036	-0.0223	0.0049	0.0012	0.0382	0.3207	0.1556
	(0.0, 1.0,3,0.5)	0.0000	-0.0011	0.0275	-0.0173	0.0007	0.427	0.4399	0.1480
	(0.0, 1.0,3,0.9)	0.0001	0.0002	0.1149	-0.0083	0.0011	0.0312	0.9184	0.0457

Table 3. Parameter estimates, AIC and BIC for velocities of 82 distant galaxies.

Dist.	Parameter estimates	K-S	P-value	$-\log(L)$	AIC	BIC
PNG	$\widehat{\mu}=-3.82$, $\widehat{\sigma}$=7.64, $\widehat{\alpha}$=11.87, $\widehat{\theta}$=0.99	red0.088	red0.418	232.45	472.89	482.51
PN	$\widehat{\mu}$=22.34, $\widehat{\sigma}$=4.01, $\widehat{\alpha}$=0.67	red0.092	red0.367	240.28	486.56	493.78
Normal	$\widehat{\mu}$= 20.83, $\widehat{\sigma}$=4.54	red0.092	red0.367	240.42	484.83	489.65
SN	$\widehat{\mu}$=24.61, $\widehat{\sigma}$=5.90, $\widehat{\alpha}=-1.39$	red0.091	red0.363	239.21	484.42	491.64

6 Application

Two real data are used to illustrate to show that the PNG distribution can provide a better fit with the data sets than the PN, the SN and normal distributions.

The first data set concerning the velocities of 82 distant galaxies, diverging from our own galaxy and are available at *http://www.stats.bris.ac.uk/∼peter /mixdata*. For the PNG, the PN, the SN and the normal distributions the MLEs of the parameters, red the Kolmogorov–Smirnov statistic with its respective p-value, the log-likelihood, the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) were derived and displayed in Table 3. Based on the AIC and the BIC, one can see that the PNG distribution is the best among the fitted models. Also the estimated density functions of the data including the respected histograms, plotted in Figure 3 confirm this conclusion.

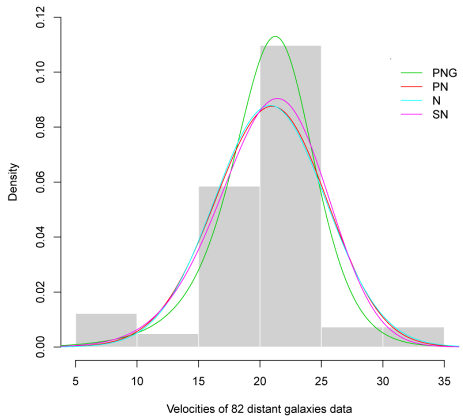

Figure 3. Velocities of 82 distant galaxies data histogram and fitted pdf of distributions.

Table 4. Parameter estimates, AIC and BIC for AIS data.

Dist.	Parameter estimates	K-S	P-value	$-\log(L)$	AIC	BIC
PNG	$\widehat{\mu}$= 140.83, $\widehat{\sigma}$=13.01,$\widehat{\alpha}$= 4.72, $\widehat{\theta}$=0.98	0.0492	0.9689	348.11	704.22	714.64
PN	$\widehat{\mu}$=182.08, $\widehat{\sigma}$= 5.37, $\widehat{\alpha}$=0.29	0.0811	0.5257	351.30	708.61	716.42
Normal	$\widehat{\mu}$=174.59, $\widehat{\sigma}$=8.21	0.0817	0.5174	352.32	708.63	713.85
SN	$\widehat{\mu}$=170.32, $\widehat{\sigma}$=8.002, $\widehat{\theta}$=0.0016	0.0831	0.5091	352.03	710.64	718.45

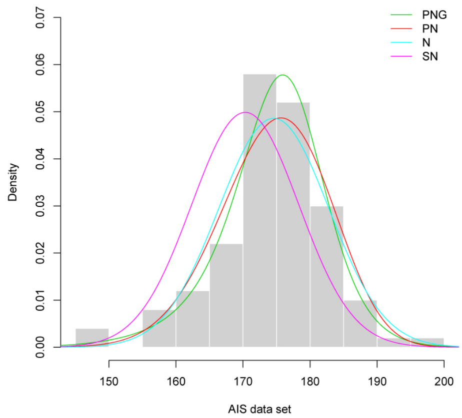

Figure 4. AIs data histogram and fitted pdf of distributions.

The second data represents the heights (in centimeters) of 100 Australian athletes studied by Cook and Weisberg (1994). Table 4 gives the MLEs of the parameters, red the Kolmogorov–Smirnov statistic with its respective p-value, the log-likelihood, the AIC and BIC for the PNG, PN, SN and Normal models for the second data set. According to the AIC and the BIC, the PNG distribution is the best among considered models. Also the plots of the densities in Figure 4 confirmed this conclusion.

7 Bivariate

In this section we introduce the bivariate power normal geometric distribution (BPNG) distribution, whose marginals are power normal geometric distributions. Suppose $\{X_1, \cdots, X_n\}$ and $\{Y_1, \cdots, Y_n\}$ are two sequence of mutually independent and identically distributed (i.i.d.) from $PN(\mu_1, \sigma_1, \alpha_1)$ and $PN(\mu_2, \sigma_2, \alpha_2)$, respectively. Let N be a geometric variable independent of X_i's and Y_i's. Consider the following bivariate random variable (X, Y), where

$$U = \max(X_1, \cdots, X_N) \qquad \text{and} \qquad V = \max(Y_1, \cdots, Y_N).$$

The joint cdf of (U, V) is

$$\begin{aligned} F_{U,V}(u, v) &= P(U \leq u, V \leq v) \\ &= \sum_{n=1}^{\infty}(1-\theta)\theta^n \left(\left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1}\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2}\right)^n \\ &= \frac{(1-\theta)\left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1}\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2}}{1-\theta\left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1}\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2}}. \end{aligned} \tag{6}$$

The bivariate random vector (U, V) is said to have a bivariate normal geometric distributions, denoted by $BPNPS(\mu_1, \mu_2, \sigma_1, \sigma_2, \alpha_1, \alpha_2, \theta)$; if (U, V) has the joint cdf 6. The joint cdf of (U, V) can be obtained as

$$f_{U,V}(u, v) = (1-\theta)\left[\frac{f_0(u, v)(1-\theta F_0(u, v)) + 2\theta f_1(u, v) f_2(u, v)}{(1-\theta F_0(u, v))^3}\right],$$

where

$$\begin{aligned} F_0(u, v) &= \left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1}\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2}, \\ f_0(u, v) &= \frac{\alpha_1\alpha_2}{\sigma_1\sigma_2}\phi\left(\frac{u-\mu_1}{\sigma_1}\right)\phi\left(\frac{v-\mu_2}{\sigma_2}\right)\left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1-1}\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2-1}, \\ f_1(u, v) &= \frac{\alpha_1}{\sigma_1}\phi\left(\frac{u-\mu_1}{\sigma_1}\right)\left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1-1}\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2}, \\ f_2(u, v) &= \frac{\alpha_2}{\sigma_2}\phi\left(\frac{v-\mu_2}{\sigma_2}\right)\left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1}\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2-1}. \end{aligned}$$

Theorem 2. *If* $(U, V) \sim BPNG(\mu_1, \mu_2, \sigma_1, \sigma_2, \alpha_1, \alpha_2, \theta)$*, then*
(i) $U \sim PNG(\mu_1, \sigma_1, \alpha_1, \theta)$ *and* $V \sim PNG(\mu_2, \sigma_2, \alpha_2, \theta)$.
(ii) The conditional distributions have the following forms.

$$P(V \leq v \mid U \leq u) = \frac{\left\{1-\theta\left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1}\right\}\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2}}{1-\theta\left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1}\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2}}.$$

$$P(U \leq u \mid V \leq v) = \frac{\left\{1-\theta\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2}\right\}\left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1}}{1-\theta\left[\Phi\left(\frac{u-\mu_1}{\sigma_1}\right)\right]^{\alpha_1}\left[\Phi\left(\frac{v-\mu_2}{\sigma_2}\right)\right]^{\alpha_2}}.$$

8 Conclusion

In this paper we introduce a new four-parameter distribution called the power normal-geometric distributions, which is an alternative to the Azzalini skew-normal distribution. We have derived different mathematical properties of this distribution. The parameters estimation via EM algorithm is proposed, and their performances are evaluated by the Monte Carlo method. We fitted the PNG distribution to two real data sets to show the potentially of the new proposed distribution. Finally the model has been generalized to the bivariate case. More work is needed to study this proposed bivariate distribution in details. red

Acknowledgement

The author would like to thank the Referees for their valuable comments and suggestions which have contributed to substantially improving the manuscript.

References

Adamidis, K. and Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics and Probability Letters,* **39**, 35-42.

Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics,* **12**, 171-178.

Bahrami, E. and Yarmoghaddam, N. (2019). A New Lifetime Distribution: The Beta Modified Weibull Power Series Distribution. *Journal of Statistical Research of Iran,* **16**, 1-31.

Barreto-Souza, W., Morais, A.L., and Cordeiro, G.M. (2011). The Weibull-geometric distribution. *Journal of Statistical Computation and Simulation*, **81**, 645-657.

Chahkandi, M. and Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate. *Computational Statistics and Data Analysis*, **53**, 4433-4440.

Cook, R.D. and Weisberg, S. (1994). *An introduction to regression graphics.* Wiley, New York.

Gupta, R.C. and Gupta, R.D. (2008). Analyzing skewed data by power normal model. *Test,* **17**, 197-210.

Jamalizadeh, A. and Balakrishnan, N. (2008). Distributions of order statistics and linear combinations of order statistics from an elliptical distribution as mixtures of unified skew-elliptical distributions. *Journal of Multivariate Analysis,* **101**, 1412-1427.

Mahmoudi, E. and Mahmoodian, H. (2017). Normal-power series class of distributions: Model, properties and applications. *Journal of Statistical Theory and Applications,* **10**, 512–541.

Mahmoudi, E. and Jafari, A.A. (2012). Generalized exponential-power series distributions. *Computational Statistics and Data Analysis*, **56**, 4047-4066. red

Mahmoudi, E. and Jafari, A. A. (2017). The compound class of linear failure rate-power series distributions: Model, properties and applications. *Communication in Statistics-Simulation and Computation*, **46**, 1414–1440

Morais, A.L. and Barreto-Souza, W. (2011). A compound class of Weibull and power series distributions. *Computational Statistics and Data Analysis*, **55**, 1410-1425.

Marshall, A.W. and Olkin, I. (1997). A new method of adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika,* **84**, 641-652.

Muhammad, M. (2017). The complementary exponentiated Burr XII Poisson distribution: Model, properties and application. *Journal of Statistics Applications and Probability*, **6**, 33-48.

Roozegar, R. and Nadarajah, S. (2017). The quadratic hazard rate power series distribution. *Journal of Testing and Evaluation,* **45**, 1058-1072.

Roozegar, R., Hamedani, G.G., Amiri, L. and Esfandiyari, F. (2020). A New Family of Lifetime Distributions: Theory, Application and Characterizations. *Annals of Data Science,* **7**, 109-138. red

Tahmasebi, S. and Jafari A.A. (2015). Generalized Gompertz-power series distributions. *Hacettepe Journal of Mathematics and Statistics,* **45**, 1579-1604.

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