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Statistical Inference for the Lomax Distribution under Progressively Type-II Censoring with Binomial Removal

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Abstract. This paper considers parameter estimations in Lomax distribution under progressive type-II censoring with random removals, assuming that the number of units removed at each failure time has a binomial distribution. The maximum likelihood estimators (MLEs) are derived using the expectation-maximization (EM) algorithm. The Bayes estimates of the parameters are obtained using both the squared error and the asymmetric loss functions based on the Lindley approximation. We compare the performance of our procedures using a simulation study and real data.

Keywords. Bayes estimator, binomial censoring scheme, EM algorithm, maximum likelihood estimator, Lomax distribution, Lindley approximation, type II progressive censoring.

1 Introduction

Lomax distribution was first introduced in the literature of Lomax (1954) for modeling of business failure data. It has also been applied in areas of

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statistical modeling such as economics and reliability theory. In the statistical literature, many works have been devoted to Lomax distribution. Several authors have estimated the parameters of Lomax distribution using both classical and Bayesian techniques. The scheme of progressive type-II censored sampling is an important scheme in lifetime experiments that allows the removal of surviving experimental units before the termination of the test. For more information on the subject of progressively censoring, see Balakrishnan (2000) and Balakrishnan (2007). Under these assumptions, n units are placed on test at time zero, and m failures are going to be observed. When the first failure is observed, r_1 surviving units is randomly selected and removed from the experiment, and so on. Finally, this process stops at the time of the mth failure and remaining $r_m = n - r_1 - r_2 - \cdots - r_{m-1} - m$ surviving units are all removed. Note that, in this scheme, $r_1, r_2, \cdots, r_m$ are all prefixed. However, in some practical situations, these numbers may be random, see Yuen (1996). In some reliability experiments, an experimenter may decide that it is inappropriate or too dangerous to carry on the testing on some of the tested units even though these units have not failed. In these cases, the removal pattern at each failure is random, and researchers choose a distribution for removing the number of unfailed units independent of the lifetime distribution. They discussed classical and Bayesian estimation problems on different distributions. Statistical inference on the parameters of some distributions under progressive type-II censoring with random removals and with binomial removals has been investigated by several authors, see for example Dey (2014), Tse et al. (2000), Wu (2001), Wu et al. (2007), and Mubarak (2011). Asgharzadeh (2011) derived estimation of the scale parameter of the Lomax distribution under progressive censoring. Zaman et al. (2020) considered a partially accelerated life test for the Lomax distribution under the progressive type-II censoring scheme and obtained the MLE of the scale parameter. Helu et al. (2015) considered estimation of the parameter of the Lomax distribution under progressive censoring using the EM algorithm. In this paper, we consider progressively type-II censored data from the Lomax distribution with binomial removals. We assume that any testing unit being dropped out from the life test is independent of the others but with the same probability p. The MLEs for the parameters using the EM are obtained and also using the sampling distribution of the MLEs, the confidence intervals for the parameters are presented in Section 2. In Section 3, Lindley's approximation is used to obtain the Bayes estimates of the parameters under the squared error, LINEX, and entropy loss functions.

Two-sided Bayes probability and credible intervals for the parameters are also used discussed. To compare MLEs and Bayes estimators, Monte Carlo simulations are conducted in Section 4, and finally, in Section 5 a real dataset is analyzed for illustrative purposes.

2 Estimation of Parameters

Suppose that the random variable X has a Lomax distribution with shape parameter α and scale parameter β, with the probability density function (pdf) and the cumulative distribution function (CDF) given by

$$\begin{aligned} f_X(x) =&\alpha\beta(1+\beta x)^{-(\alpha+1)}, \quad x>0,\ \alpha>0,\ \beta>0, \\ F_X(x) =&1-(1+\beta x)^{-\alpha}, \quad x>0. \end{aligned} \tag{1}$$

Let $X=(X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n})$ be a progressively type-II right censored sample from a life test of size m from a sample of size n, and $(R_1, R_2, ..., R_m)$ be the progressive censoring scheme where lifetimes have a Lomax distribution with pdf as given by (1). In this paper, the shape parameter is assumed to be constant. Then, the likelihood function based on this progressively type-II censored sample is

$$L(\beta) = k\prod_{i=1}^{m} f(x_{i:m:n}, \beta)[1-F(x_{i:m:n}, \beta)]^{r_i}, \tag{2}$$

where $k = n(n-1-r_1)(n-2-r_1-r_2)\cdots(n-m+1-r_1-r_2-\cdots-r_{m-1})$. Substituting (1), in (2), for a progressive type-II with predetermined number of removals $R=r$, the conditional likelihood (L) and log-likelihood function (l) are, respectively,

$$L(\beta \mid R=r) = k\alpha^m\beta^m\prod_{i=1}^{m}(1+\beta x_i)^{-((r_i+1)\alpha+1)}, \tag{3}$$

$$l(\beta \mid R=r) = \log k + m\log\alpha + m\log\beta - \sum_{i=1}^{m}((r_i+1)\alpha+1)\log(1+\beta x_i). \tag{4}$$

Suppose that an individual unit is removed from the life test. As we know that, it is independent of other units but with the same probability p. Then, the number of units removed at each failure time follows a binomial distribution such that

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1}, \; r_1 = 0, \cdots, n-m, \tag{5}$$

and

$$\begin{aligned} P(R_i = r_i \mid R_{i-1} = r_{i-1}, \cdots, R_1 = r_1) &= \binom{n-m-\sum_{k=1}^{i-1} r_k}{r_i} \\ &\quad \times p^{r_i}(1-p)^{n-m-\sum_{k=1}^{i} r_k}, \end{aligned} \tag{6}$$

where $r_i = 0, \cdots, n-m-\sum_{k=1}^{i-1} r_k, i = 2, 3, \cdots, m-1$.

Moreover, suppose that R_i is independent of X_i. Then, the joint likelihood function $X = (X_1, X_2, ..., X_m)$ and $R = (R_1, R_2, ..., R_m)$ can be expressed as

$$L(\beta, p; x, r) = L(\beta, x \mid R = r)P(R = r), \tag{7}$$

where

$$\begin{aligned} P(R = r) &= P(R_{m-1} = r_{m-1} \mid R_{m-2} = r_{m-2}, \cdots, R_1 = r_1) \\ &\cdots \; P(R_2 = r_2 \mid R_1 = r_1)P(R_1 = r_1). \end{aligned} \tag{8}$$

Substituting (5) and (6) in (8), we have

$$\begin{aligned} P(R = r) &= \frac{(n-m)!}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{i=1}^{m-1} r_i)!} \\ &\times \; p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}. \end{aligned} \tag{9}$$

Also, substituting (4) and (8) in (7), then the likelihood function can be written

$$L(\beta, p; x, r) = AL_1(\beta)L_2(p), \tag{10}$$

where

$$A = \frac{k\alpha^m (n-m)!}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{i=1}^{m-1} r_i)!},$$

$$L_1(\beta) = \beta^m \prod_{i=1}^{m} (1 + \beta x_i)^{-(r_i+1)\alpha-1}, \tag{11}$$

$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}. \tag{12}$$

In the following, the maximum likelihood estimation of the parameters β and p are obtained based on progressively type-II censored sample with binomial removals.

Due to the $L_1(\beta)$ does not depend on the parameter p, so the MLE of β can be derived from (11) directly. The log-likelihood function L_1 is

$$l_1(\beta) = l_1(\beta \mid R = r) = m \log \beta - \sum_{i=1}^{m} ((r_i + 1)\alpha + 1) \log(1 + \beta x_i). \tag{13}$$

In fact, the MLE of β can be obtained from the following equation

$$\frac{\partial l_1(\beta)}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^{m} \frac{((r_i + 1)\alpha + 1)x_i}{1 + \beta x_i} = 0. \tag{14}$$

The recent equation is nonlinear and does not have an explicit answer. Then, equation (14) must be solved by numerical methods to obtain the MLE of β.

The EM algorithm was introduced by Dempster et al. (1977) to handle any missing or incomplete data situation; readers are referred to a book by Mclachlan (1997) for a detailed discussion on the EM algorithm and its applications. Let $X = (X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n})$ be an incomplete observed data and $Z = (Z_1, Z_2, \cdots, Z_m)$ with $z_j = (z_{j1}, z_{j2}, \cdots, z_{jR_j}), j = 1, \cdots, m$, be the censored data. We consider the censored data as missing data. The combination of $(X, Z) = Y$ forms the complete data set. Log-likelihood function based on Y is

$$l_C(Y;\beta) \propto n\log(\alpha\beta) - (\alpha+1)\sum_{j=1}^{m} \log(1+\beta x_j) - (\alpha+1)\sum_{j=1}^{m}\sum_{k=1}^{R_j} \log(1+\beta z_{jk}). \tag{15}$$

Then, the MLE of β for complete sample of Y can be obtained from the following equation

$$\frac{\partial l_C(Y;\beta)}{\partial \beta} = \frac{n}{\beta} - \sum_{j=1}^{m} \frac{(\alpha+1)x_j}{(1+\beta x_j)} - \sum_{j=1}^{m}\sum_{k=1}^{R_j} \frac{(\alpha+1)z_{jk}}{(1+\beta z_{jk})} = 0. \tag{16}$$

The EM algorithm has two steps. In the E-step (expectation step), the expected value of the complete log-likelihood $l_C(Y;\beta)$ has been calculated with respect to the conditional distribution of Z given the observed data X and the current estimate of the parameter $\beta^{(k-1)}$ at the $(k-1)$th iteration.

$$\frac{n}{\beta} - \sum_{j=1}^{m} \frac{(\alpha+1)x_j}{(1+\beta x_j)} - (\alpha+1)\sum_{j=1}^{m}\sum_{k=1}^{R_j} A(\beta) = 0, \tag{17}$$

where

$$\begin{aligned} A(\beta) &= E\left(\frac{z_{jk}}{(1+\beta z_{jk})} \mid Z_{jk} > y_j\right), \\ &= \alpha(1+\beta x_j)^{\alpha}\int_{x_j}^{\infty} \beta\frac{z_{jk}}{(1+\beta z_{jk})}(\frac{1}{1+\beta z_{jk}})^{\alpha+1} dz_j, \\ &= \frac{\alpha\beta x_j + (1+\beta x_j)(1+\alpha)}{\beta(1+\alpha)(1+\beta x_j)}. \end{aligned} \tag{18}$$

In the M-step (maximization step), EM algorithm will maximize $A(\beta)$ with respect to β to give an update value $\beta^{(k)}$ until convergence with an acceptable error.

$$\frac{\partial l_C(Y;\beta)}{\partial \beta} = \frac{n}{\beta^{(k+1)}} - \sum_{j=1}^{m}\frac{(\alpha+1)x_j}{1+\beta^{(k+1)}x_j} - \sum_{j=1}^{m} R_j A(\beta^k) = 0, \tag{19}$$

$$\hat{\beta}_{EM} = \frac{n}{\sum_{j=1}^{m}\frac{(\alpha+1)x_j}{1+\beta^{(k+1)}x_j} + \sum_{j=1}^{m} R_j A(\beta^k)}. \tag{20}$$

On the other hand, $L_2(p)$ is only a function based on p, then from (12) is used to obtain MLE of p. The log-likelihood function of L_2 is

$$\log L_2(p) = \sum_{i=1}^{m-1} r_i \log p + \left((m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i\right)\log(1-p). \tag{21}$$

Thus, the MLE of p be found immediately

$$\hat{p} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i-1)r_i}. \tag{22}$$

2.1 Approximate MLE for β

Suppose that the random variable of X has Lomax distribution with parameters α and β. It is easy to show that $Y = \beta X$ also has the Lomax distribution with shape parameter α and scale parameter 1, ($Y \sim Lomax(\alpha, 1)$). Then the pdf and the cdf of Y are

$$f_Y(y) = \alpha(1+y)^{-(1+\alpha)}, \qquad y > 0,$$

and

$$F_Y(y) = 1-(1+y)^{-\alpha}, \qquad y > 0,$$

respectively. Using the transformation $x_i = \frac{y_i}{\beta}$ and $n = m+\sum_{i=1}^{m} r_i$, equation in (14) may be rewritten as

$$\begin{aligned} \frac{\partial l_1(\beta)}{\partial \beta} &= \frac{m}{\beta} - \sum_{i=1}^{m} \frac{((r_i+1)\alpha+1)\frac{y_i}{\beta}}{1+y_i}, \\ &= \frac{m}{\beta} - \frac{1}{\beta}\sum_{i=1}^{m}((r_i+1)\alpha+1)(1-(1-F(y_i)))^{\frac{1}{\alpha}}, \\ &= \frac{-n\alpha}{\beta} - \frac{1}{\beta}\sum_{i=1}^{m}((r_i+1)\alpha+1)(1-F(y_i))^{\frac{1}{\alpha}}, \\ &= 0. \end{aligned} \tag{23}$$

For estimation of β, let us consider $g(y_i) = (1-F(y_i))^{\frac{1}{\alpha}}$. Then, for approximating the term $g(y_i)$ by expanding it in a Taylor series around $E(Y_{i:m:n}) = \nu_{i:m:n}$. Balakrishnan (2000) showed that if $U_{i:m:n}$ be the ith progressively type-II censored order statistics from $U(0,1)$, then $F(Y_{i:m:n}) \stackrel{d}{=} U_{i:m:n}$. Therefore $Y_{i:m:n} = dF^{-1}(U_{i:m:n})$ and $\nu_{i:m:n} = E(Y_{i:m:n}) \approx F^{-1}(\eta_{i:m:n})$, where $\eta_{i:m:n} = E(U_{i:m:n})$.
Balakrishnan (2000) showed that

$$\eta_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} \frac{j+R_{m-j+1}+...+R_m}{j+1+R_{m-j+1}+...+R_m}, \qquad i = 1,...,m.$$

Moreover, for the $Lomax(\alpha, 1)$ distribution, we have

$$F^{-1}(u) = (1-u)^{-\frac{1}{\alpha}} - 1.$$

So $\nu_{i:m:n}$ can be approximated with $(1-\eta_{i:m:n})^{-\frac{1}{\alpha}}-1$.
We extended the function $g(y_i)$ around the point $\nu_{i:m:n}$ and keep only the first two terms,

$$\begin{aligned} g(y_i) &\approx g(\nu_{i:m:n}) + (y_i - \nu_{i:m:n}) g'(\nu_{i:m:n}), \\ &= \gamma_i + y_i \delta_i, \end{aligned} \tag{24}$$

where

$$\gamma_i = g(\nu_{i:m:n}) - \nu_{i:m:n} g'(\nu_{i:m:n}) = \frac{1+2\nu_{i:m:n}}{(1+\nu_{i:m:n})^2}, \quad i=1,...,m, \tag{25}$$

and

$$\delta_i = g'(\nu_{i:m:n}) = -\frac{1}{(1+\nu_{i:m:n})^2}, \qquad i=1,...,m. \tag{26}$$

Using the equation (25), the equation (23) can be rewritten as

$$\frac{\partial l_1(\beta)}{\partial \beta} = \frac{-n\alpha}{\beta} - \frac{1}{\beta}\sum_{i=1}^{m}((r_i+1)\alpha+1)(\gamma_i + y_i\delta_i) = 0. \tag{27}$$

Therefore

$$-n\alpha - \sum_{i=1}^{m}((r_i+1)\alpha+1)\gamma_i - \beta\sum_{i=1}^{m}\delta_i((r_i+1)\alpha+1)x_i = 0. \tag{28}$$

And finally by solving the equation (28) for β, we have an approximate of MLE of β as follows

$$\hat{\beta}_{AML} = -\frac{n\alpha + \sum_{i=1}^{m}((r_i+1)\alpha+1)\gamma_i}{\sum_{i=1}^{m}\delta_i((r_i+1)\alpha+1)x_i}. \tag{29}$$

2.2 Interval Estimation

In this subsection, we are going to derive the approximate confidence intervals for the parameters $\xi = (\beta, p)$ based on the asymptotic distributions of the MLE. It is known that the asymptotic distribution of the MLE ξ is $(\hat{\xi}-\xi) \rightarrow N_2(0, I^{-1}(\xi))$, where $I^{-1}(\xi)$, the inverse of the observed information matrix of the parameters $\xi = (\beta, p)$ may be written as (see Lawless (2011))

$$I^{-1}(\xi) = \begin{bmatrix} -\frac{\partial^2 \log L(\beta,p)}{\partial \beta^2} & -\frac{\partial^2 \log L(\beta,p)}{\partial \beta \partial p} \\ -\frac{\partial^2 \log L(\beta,p)}{\partial p \partial \beta} & -\frac{\partial^2 \log L(\beta,p)}{\partial p^2} \end{bmatrix}^{-1}_{(\beta,p)=(\hat\beta,\hat p)} = \begin{bmatrix} \mathrm{Var}(\hat\beta) & \mathrm{cov}(\hat\beta,\hat p) \\ \mathrm{cov}(\hat p,\hat\beta) & \mathrm{Var}(\hat p) \end{bmatrix}$$

where

$$\begin{aligned} \frac{\partial^2 \log L(\beta,p)}{\partial \beta^2} &= \frac{-m}{\beta^2} - \sum_{i=1}^{m} \frac{((r_i+1)\alpha+1)x_i^2}{(1+\beta x_i)^2}, \\ \frac{\partial^2 \log L(\beta,p)}{\partial p^2} &= \frac{-\sum_{i=1}^{m-1} r_i}{p^2} - \frac{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i}{(1-p)^2}, \\ \frac{\partial^2 \log L(\beta,p)}{\partial \beta \partial p} &= \frac{\partial^2 \log L(\beta,p)}{\partial p \partial \beta} = 0. \end{aligned}$$

The approximate $100(1-\tau)\%$ confidence intervals of the parameters β and p are derived, respectively,

$$\begin{aligned} \hat\beta_{AML} &\pm z_{\frac{\tau}{2}}\sqrt{\mathrm{var}(\hat\beta)}, \\ \hat p_{AML} &\pm z_{\frac{\tau}{2}}\sqrt{\mathrm{var}(\hat p)}. \end{aligned}$$

where $z_{\frac{\tau}{2}}$ is the upper $\frac{\tau}{2}$th percentile of the standard normal distribution.

3 Bayes Estimation

In this section, we want to derive the point and interval estimates of the parameters β and p based on progressively type-II censored data with binomial removals. We have assumed square error loss (SEL) function, LINEX (Linear-Exponential) loss function and entropy loss function.

$$\begin{aligned} L_s(d(\theta),\hat d(\theta)) &= (\hat d(\theta) - d(\theta))^2, \\ L_L(d(\theta),\hat d(\theta)) &= e^{c(\hat d(\theta)-d(\theta))} - c(\hat d(\theta) - d(\theta)) - 1, \quad c \neq 0, \\ L_E(d(\theta),\hat d(\theta)) &= (\frac{\hat d(\theta)}{d(\theta)})^{\nu} - \nu \log(\frac{\hat d(\theta)}{d(\theta)}) - 1, \quad \nu \neq 0, \end{aligned}$$

where $\hat d(\theta)$ is the estimation of parameter of $d(\theta)$.
It is known that the Bayes estimator of parameter under SEL is mean of

posterior distribution and it under LINEX and entropy loss functions is, respectively, (see Zellner (1986))

$$\begin{aligned}\hat{d}_L(\theta) &= -\frac{1}{c} ln\{E_\theta(e^{-cd(\theta)} \mid x)\},\\ \hat{d}_E(\theta) &= \{E_\theta(d^{-\nu}(\theta) \mid x)\}^{-\frac{1}{\nu}}.\end{aligned}$$

The prior density of the parameters β and p are considered independent priors as

$$\begin{aligned}\pi_1(\beta) &\propto \beta^{r-1}e^{-\gamma\beta}, \quad \beta, r, \gamma > 0,\\ \pi_1(p) &\propto p^{a-1}(1-p)^{b-1}, \quad 0 < p < 1, \; a, b > 0.\end{aligned} \tag{30}$$

Then the join prior distribution for β and p is given by

$$\pi(\beta, p) \propto \beta^{r-1}e^{-\gamma\beta}p^{a-1}(1-p)^{b-1}. \tag{31}$$

Using the likelihood function (10) and the join prior distribution (31), the posterior density of $\xi = (\beta, p)$ is

$$\begin{aligned}\pi(\beta, p \mid x, r) &= \frac{\beta^{m+r-1}e^{-\gamma\beta}\prod_{i=1}^{m}(1+\beta x_i)^{-(r_i+1)\alpha-1}}{\int_0^\infty \beta^{m+r-1}e^{-\gamma\beta}\prod_{i=1}^{m}(1+\beta x_i)^{-(r_i+1)\alpha-1}d\beta}\\ &\times \frac{p^{a+\sum_{i=1}^{m-1} r_i-1}(1-p)^{b+(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}}{\int_0^1 p^{a+\sum_{i=1}^{m-1} r_i-1}(1-p)^{b+(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}dp}\end{aligned} \tag{32}$$

Using the equation in (32), the marginal posterior pdfs of p and β are respectively,

$$\pi_1(p \mid x, r) = \frac{p^{a+\sum_{i=1}^{m-1} r_i-1}(1-p)^{b+(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}}{B(a+\sum_{i=1}^{m-1} r_i, b+(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i+1)}, \tag{33}$$

$$\pi_2(\beta \mid x, r) = \frac{\beta^{m+r-1}e^{-\gamma\beta}\prod_{i=1}^{m}(1+\beta x_i)^{-t}}{\prod_{i=1}^{m}\sum_{k=0}^{\infty}\binom{k+t-1}{k}x_i^k\Gamma(r+k)(\frac{m}{\gamma})^{r+k}}, \tag{34}$$

where B is beta function and $t = (r_i+1)\alpha - 1$.
The Bayes estimator of a function $U = U(\theta)$ of the parameter θ, under

squared error loss function is the posterior mean

$$\hat{U}_{BS} = E(U \mid X) = \int u\pi(\theta \mid x)d\theta = \frac{\int u\pi(\theta \mid x)d\theta}{\int \pi(\theta \mid x)d\theta}.$$

The Bayes estimator is the ratio of two integrals that cannot be obtained in a closed-form. The various methods suggested to approximate the ratio of integrals of the above form, in this paper, Lindley's approximate method is used (see Lindley (1980)). This method has been used by many authors to obtain Bayes estimators of the parameters for some lifetime distributions, see among others, Howlader (1980), Soliman (2001).
In a two-parameter case, $\theta = (\theta_1, \theta_2)$, based on Lindley's approximation, the approximate Bayes estimation of a function $U(\theta) = U(\theta_1, \theta_2)$, under the SEL function, leads to

$$I(x) = E[U(\theta) \mid X] = U(\theta) + \frac{A}{2} + \rho_1 A_{12} + \rho_2 A_{21} + \frac{1}{2}[l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{21}], \tag{35}$$

can be evaluated as

$$I(x) = U(\hat{\theta}_1, \hat{\theta}_2) + \frac{A}{2} + \rho_1 A_{12} + ..., \tag{36}$$

where

$$\begin{aligned}
l_{ks} &= \frac{\partial L^{k+s}}{\partial\theta_1^k \partial\theta_2^s}, \quad k, s = 0, 1, 2, 3, \ k + s = 3, \\
A &= \sum_{i=1}^{2}\sum_{j=1}^{2} U_{ij}\delta_{ij}, \quad i, j = 1, 2, \\
\rho_i &= \frac{\partial\rho}{\partial\theta_i}, U_i = \frac{\partial U}{\partial\theta_i}, U_{ij} = \frac{\partial^2 U}{\partial\theta_i \partial\theta_j}, \\
A_{ij} &= U_i\delta_{ii} + U_j\delta_{ji}, \\
B_{ij} &= (U_i\delta_{ii} + U_j\delta_{ij})\delta_{ii}, \\
C_{ij} &= 3U_i\delta_{ii}\delta_{ij} + U_j(\delta_{ii}\delta_{jj} + 2\delta_{ij}^2),
\end{aligned}$$

where δ_{ij} is the (i, j)th element of the inverse of the matrix $\{-l_{ij} = -\frac{\partial^2 l}{\partial\theta_i \partial\theta_j}\}$, all evaluated at the MLE of parameters.

a) Bayes estimator of p under SEL function is

$$\hat{p}_{BS} = \frac{a + \sum_1^{m-1} r_i}{a + \sum_1^{m-1} r_i + b + (m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i + 1}. \quad (37)$$

The posterior risk of the Bayes estimator and the maximum likelihood estimator of p is obtained as follows

$$R(\hat{p}_B) = \frac{(a + \sum_1^{m-1} r_i)(a + \sum_1^{m-1} r_i + 1)C - (a + \sum_1^{m-1} r_i)^2 C^2}{C^2(C+1)}, \quad (38)$$

$$\begin{aligned} R(\hat{p}_{ML}) &= \frac{(a + \sum_1^{m-1} r_i)(a + \sum_1^{m-1} r_i + 1)}{C(C+1)} \\ &- \left(\frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i-1)r_i}\right)^2, \end{aligned} \quad (39)$$

where $C = a + \sum_1^{m-1} r_i + b + (m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i + 1$.

b) Lindley approximation of Bayes estimator of β under SEL function
If $U(\theta) = \beta$ then

$$\begin{aligned}
U_1 &= \frac{\partial u}{\partial \beta} = 1, \quad U_{11} = \frac{\partial^2 U}{\partial \beta \partial \beta} = 0, \quad U_{12} = \frac{\partial^2 U}{\partial \beta \partial p} = 0, \\
U_2 &= \frac{\partial u}{\partial p} = 0, \quad U_{21} = \frac{\partial^2 U}{\partial \beta \partial p} = 0, \quad U_{22} = \frac{\partial^2 U}{\partial p \partial p} = 0 \Rightarrow A = 0, \\
A_{12} &= U_1\delta_{11} + U_2\delta_{21} = \delta_{11}, \\
A_{21} &= U_2\delta_{22} + U_1\delta_{12} = \delta_{12}, \\
B_{12} &= (U_1\delta_{11} + U_2\delta_{12})\delta_{11} = \delta_{11}^2, \\
B_{21} &= (U_2\delta_{22} + U_1\delta_{21})\delta_{22} = \delta_{21}\delta_{22}, \\
C_{12} &= 3U_1\delta_{11}\delta_{12} + U_2(\delta_{11}\delta_{22} + 2\delta_{12}^2) = 3\delta_{11}\delta_{12}, \\
C_{21} &= 3U_2\delta_{22}\delta_{12} + U_1(\delta_{22}\delta_{11} + 2\delta_{21}^2) = \delta_{11}\delta_{22} + 2\delta_{21}^2.
\end{aligned}$$

From equation (10) the log-Likelihood function can be obtained as

$$
\begin{aligned}
l(\beta, p \mid R=r) &= \log A + m \log \beta - \sum_{i=1}^{m}((r_i+1)\alpha+1)\log(1+\beta x_i) \\
&+ \sum_{i=1}^{m-1} r_i \log p + ((m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i)\log(1-p).
\end{aligned} \tag{40}
$$

Then, we get

$$
\begin{aligned}
l_{12} &= \frac{\partial l}{\partial\beta\partial p^2} = \frac{\partial l}{\partial p^2}\left(\frac{m}{\beta} - \sum_{i=1}^{m}\frac{((r_i+1)\alpha+1)x_i}{1+\beta x_i}\right) = 0, \\
l_{21} &= \frac{\partial l}{\partial p\partial\beta^2} = \frac{\partial l}{\partial\beta^2}(\frac{\sum_{i=1}^{m-1} r_i}{p} - \frac{((m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i)}{1-p}) = 0, \\
l_{30} &= \frac{\partial^3 l}{\partial\beta^3} = \frac{2m}{\beta^3} - \sum_{i=1}^{m}\frac{2((r_i+1)\alpha+1)x_i^3}{(1+\beta x_i)^3}, \\
l_{03} &= \frac{\partial^3 l}{\partial p^3} = \frac{2\sum_{i=1}^{m-1} r_i}{p^3} - \frac{2((m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i)}{(1-p)^3},
\end{aligned}
$$

$$
(-l_{ij})^{-1} = \begin{bmatrix} \frac{1}{B_1} & 0 \\ 0 & \frac{1}{B_2} \end{bmatrix}, \quad \delta_{11} = \frac{1}{B_1}, \quad \delta_{12} = \delta_{21} = 0, \quad \delta_{22} = \frac{1}{B_2},
$$

where

$$
B_1 = \frac{m}{\beta^2} - \sum_{i=1}^{m}\frac{((r_i+1)\alpha+1)x_i^2}{(1+\beta x_i)^2},
$$

and

$$
B_2 = \frac{\sum_{i=1}^{m-1} r_i}{p^2} + \frac{((m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i)}{(1-p)^2}.
$$

Substituting all the above components in (35), the Bayes estimation of β under the SEL function, becomes

$$
\hat{\beta}_{BS}^{Lindley} = \beta + R_1, \tag{41}
$$

where $R_1 = \rho_1\delta_{11} + \frac{1}{2}l_{30}\delta_{11}^2$ and $\rho_1 = \frac{r-1}{\beta} - \gamma$.
The posterior risk of the Bayes estimator and the maximum likelihood estimator of β are obtained as follows

$$\begin{aligned} R(\hat{\beta}_B) &= [1 + (2(r-1) - 2\hat{\beta}_B\gamma]\delta_{11} \\ &+ \Big(\frac{2m}{\hat{\beta}_B^2} - \hat{\beta}_B \sum_1^m ((r_i+1)\alpha + 1)\frac{2x_i^3}{(1+\hat{\beta}_B x_i)^3}\Big)\delta_{11}^2, \end{aligned} \tag{42}$$

$$\begin{aligned} R(\hat{\beta}_{ML}) &= [1 + (2(r-1) - 2\hat{\beta}_{ML}\gamma]\delta_{11} \\ &+ \Big(\frac{2m}{\hat{\beta}_{ML}^2} - \hat{\beta}_{ML} \sum_1^m ((r_i+1)\alpha + 1)\frac{2x_i^3}{(1+\hat{\beta}_B x_i)^3}\Big)\delta_{11}^2. \end{aligned} \tag{43}$$

c) Lindley approximation of Bayes estimator of β under Linex loss function
If $U(\theta) = e^{-c\beta}$ then

$U_1 = -ce^{-c\beta}$, $U_{11} = c^2e^{-c\beta}$, $U_2 = U_{21} = U_{12} = U_{22} = 0 \Rightarrow A = c^2e^{-c\beta}\delta_{11}$,
$A_{12} = -ce^{-c\beta}\delta_{11}$, $A_{21} = B_{21} = 0$, $B_{12} = -ce^{-c\beta}\delta_{11}^2$, $C_{12} = -3ce^{-c\beta}\delta_{11}$,
$C_{21} = -ce^{-c\beta}\delta_{11}\delta_{22}$.

Substituting in (35),

$$\hat{\beta}_{BL}^{Lindley} = \beta - \frac{1}{c}\log\left\{1 + \frac{1}{2}c^2\delta_{11} - cR_1\right\}. \tag{44}$$

d) Lindley approximation of Bayes estimator of p under Linex loss function.
By substituting $U(\theta) = e^{-cp}$ in (35), similar to before mentioned equations it can be obtained

$$\hat{p}_{BL}^{Lindley} = p - \frac{1}{c}\log\left\{1 + \frac{1}{2}c^2\delta_{22} - cR_2\right\}, \tag{45}$$

where $R_2 = \rho_2\delta_{22} + \frac{1}{2}l_{03}\delta_{22}^2$ and $\rho_2 = \frac{a-1}{p} - \frac{b-1}{1-p}$.

e) Lindley approximation of Bayes estimator of β under entropy loss function

If $U(\theta) = \beta^{-\nu}$ then

$$\begin{aligned}
U_1 &= -\nu\beta^{-\nu-1}, \quad U_{11} = \nu(\nu+1)\beta^{-\nu-2}, \quad U_2 = U_{21} = U_{12} = U_{22} = 0, \\
A &= \nu(\nu+1)\beta^{-\nu-2}\delta_{11}, \\
A_{12} &= -\nu\beta^{-\nu-1}\delta_{11}, \quad A_{21} = B_{21} = C_{12} = 0, \quad B_{12} = -\nu\beta^{-\nu-1}\delta_{11}^2, \\
C_{21} &= -\nu\beta^{-\nu-1}\delta_{11}\delta_{22}.
\end{aligned}$$

Substituting in (35),

$$\hat{\beta}_{BE}^{Lindley} = \beta\left\{1 + \frac{1}{2}\nu(\nu+1)\beta^{-2}\delta_{11} - \nu\beta^{-1}R_1\right\}^{-\frac{1}{\nu}}. \tag{46}$$

f) Lindley approximation of Bayes estimator of p under entropy loss function

$$\hat{p}_{BE}^{Lindley} = p\left\{1 + \frac{1}{2}\nu(\nu+1)p^{-2}\delta_{22} - \nu p^{-1}R_2\right\}^{-\frac{1}{\nu}}. \tag{47}$$

Note that (41-47) are to be evaluated at MLE's $(\hat{\beta}, \hat{p})$.

3.1 Two-sided Bayes Probability Interval

A symmetric $100(1-\tau)\%$ two-sided Bayes probability interval (TBPI) of p using the marginal posterior distributions of p and β, are denoted by $[p_L, p_U]$ and $[\beta_L, \beta_U]$ which can be obtained by solving the following equations

$$\begin{aligned}
\int_0^{p_L} \pi_1(p \mid x, r) &= \frac{\tau}{2}, \int_{p_U}^{1} \pi_1(p \mid x, r) = \frac{\tau}{2}, \\
\int_0^{\beta_L} \pi_2(\beta \mid x, r) &= \frac{\tau}{2}, \int_{\beta_U}^{\infty} \pi_2(\beta \mid x, r) = \frac{\tau}{2}.
\end{aligned} \tag{48}$$

4 Simulation Study

In this section, we present a Monte Carlo simulation study to verify how our methods work in practice. We have considered different sample sizes; $n = 30, 40, 50, 100$, and different effective sample sizes; $m = 15, 20, 25, 35, 40, 65, 85$

Table 1. Estimation of p, $\alpha = 0.5$, $\beta = 0.1$, $p = 0.4$

n	m	MLE			Lindley				
		$\hat{p}$	RMSE($\hat{p}$)	CI	$\hat{p}_{BS}$	RMSE($\hat{p}_{BS}$)	$\hat{p}_{BL}$	RMSE($\hat{p}_{BL}$)	$TBPI$
30	15	0.4216	0.1066	(0.2027, 0.6405)	0.4304	0.0920	0.4257	0.0895	(0.3333, 0.7171)
30	20	0.4159	0.0853	(0.2565, 0.5753)	0.4234	0.0789	0.4201	0.0772	(0.2632, 0.5549)
40	25	0.4153	0.0856	(0.2476, 0.5829)	0.4228	0.0791	0.4195	0.0775	(0.3131, 0.6162)
50	25	0.4167	0.0862	(0.2451, 0.5883)	0.4240	0.0796	0.4208	0.0779	(0.3242, 0.6487)
50	35	0.4112	0.0647	(0.2808, 0.5417)	0.4164	0.0623	0.4145	0.0615	(0.3530, 0.5959)
100	40	0.4175	0.0871	(0.2531, 0.5820)	0.4247	0.0805	0.4215	0.0788	(0.2862, 0.5917)
100	65	0.4075	0.0550	(0.3097, 0.5053)	0.4115	0.0537	0.4101	0.0531	(0.2590, 0.4380)
100	85	0.4045	0.0413	(0.3251, 0.4839)	0.4069	0.0408	0.4061	0.0405	(0.3336, 0.4898)

with $a = b = \gamma = 2, c = r = 1$ using progressively type-II censoring under binomial removal scheme. With no loss of generality, we set $\alpha = 0.5$, $\beta = 0.1$ and $p = 0.4$. Using binomial removal technique, for a given n and m different samples were generated. The MLEs and Bayes estimates of the unknown parameters were obtained by the methods proposed in sections 2 and 3. A comparison between the performance of estimates was done based on the root mean square error (RMSE) of the estimates under 10000 replications. In addition, the 95% confidence intervals (CIs), and the 95% two-sided Bayesian probability intervals (TBPIs) based on the same 10000 replications were computed too. We have used R-software for simulation studies. Tables 1 - 2 show the summarized results of simulation study. From these Tables, it is concluded that with increasing sample size, the RMSE decreases. The results show the better performance of Bayes estimates as a comparison to MLEs. Figure 1-a presents the results of RMSEs for parameter β for different sample sizes, and Figure 1-b shows the same results for various m at $n = 100$. In general, the lengths of TBPI are shorter than the lengths of CI (see Table 3). The posterior risk of the MLE and the Bayes estimates of β and p are computed and are reported in Table 4.

5 Analysis of Real Data

We apply the procedures developed in this paper to a real-life data set. We present data analysis of the insulating fluid data presented by Nelson (1982). The data represent the times (in minutes) to the breakdown of an insulating

Table 2. Estimation of β, $\alpha = 0.5$, $\beta = 0.1$, $p = 0.4$

n	m	MLE				Lindley					
		$\hat{\beta}_{AML}$	RMSE($\hat{\beta}_{AML}$)	$\hat{\beta}_{EM}$	RMSE($\hat{\beta}_{EM}$)	$\hat{\beta}_{BS}$	RMSE($\hat{\beta}_{BS}$)	$\hat{\beta}_{BL}$	RMSE($\hat{\beta}_{BL}$)	$\hat{\beta}_{BE}$	RMSE($\hat{\beta}_{BE}$)
30	15	0.0952	0.0575	0.0662	0.0524	0.0747	0.0515	0.0742	0.0511	0.0642	0.0437
30	20	0.0950	0.0499	0.0737	0.0464	0.0834	0.0424	0.0830	0.0461	0.0825	0.0457
40	25	0.0957	0.0438	0.0716	0.0430	0.0787	0.0414	0.0784	0.0412	0.0707	0.0319
50	25	0.0964	0.0433	0.0661	0.0401	0.0718	0.0332	0.0715	0.0328	0.0634	0.0292
50	35	0.0973	0.0432	0.0764	0.0377	0.0826	0.0324	0.0824	0.0321	0.0751	0.0288
100	40	0.0986	0.0358	0.0626	0.0314	0.0659	0.0228	0.0658	0.0227	0.0608	0.0211
100	65	0.0995	0.0282	0.0745	0.0232	0.0779	0.0223	0.0777	0.0222	0.0735	0.0209
100	85	0.0999	0.0248	0.0876	0.0217	0.0914	0.0214	0.0913	0.0212	0.0868	0.0201

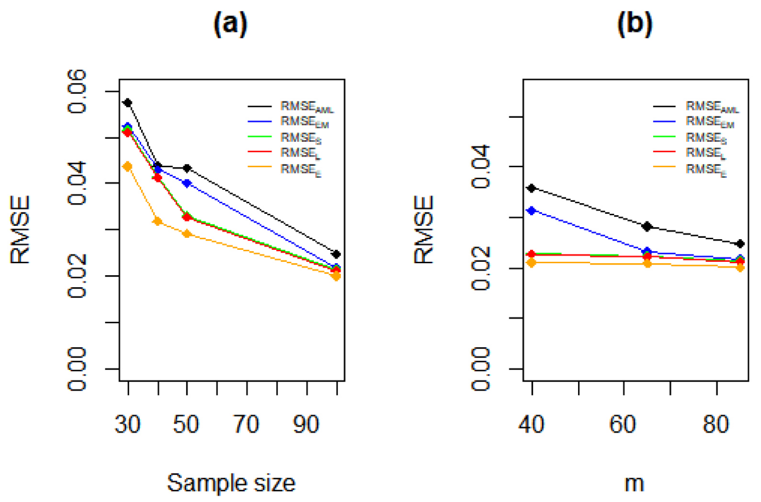

Figure 1. a) the RMSE of $\hat{\beta}$ for different sample sizes n, b) the RMSE of $\hat{\beta}$ for different effective sample sizes m at n=100.

Table 3. Interval estimation of β, $\alpha = 0.5$, $\beta = 0.1$, $p = 0.4$

n	m	CI	$TBPI$
30	15	(0.0048, 0.1427)	(0.0523, 0.1031)
30	20	(0.0142, 0.1182)	(0.0591, 0.1011)
40	25	(0.0243, 0.1188)	(0.0964, 0.2436)
50	25	(0.0286, 0.1242)	(0.0773, 0.1257)
50	35	(0.0257, 0.1063)	(0.0759, 0.1186)
100	40	(0.0257, 0.1094)	(0.0902, 0.1435)
100	65	(0.0497, 0.1254)	(0.0811, 0.1211)
100	85	(0.0401, 0.1091)	(0.0826, 0.1164)

Table 4. posterior risk of p and β, $\alpha = 0.5$, $\beta = 0.1$, $p = 0.4$

n	m	$R(\hat{p}_{ML})$	$R(\hat{p}_{BS})$	$R(\hat{\beta}_{ML})$	$R(\hat{\beta}_{BS})$	$R(\hat{\beta}_{BL})$
20	15	0.0836	0.0476	0.00548	0.0054	0.0014
30	15	0.0436	0.0071	0.0049	0.0044	0.00098
30	20	0.0291	0.0089	0.0033	0.0030	0.00073
40	25	0.0286	0.0050	0.0024	0.0021	0.00063
50	25	0.0281	0.0024	0.0020	0.0018	0.0005
50	35	0.0172	0.0022	0.0019	0.0015	0.00011

Table 5. The goodness-of-fit test results for the insulating fluid data

p	$KSstatistic$	$p-Value$
0.4	0.2308	0.8978

fluid between electrodes recorded at a Voltage of 34 kV. As indicated in Nelson (1982), the times to break down insulating fluid at these voltages are an exponential distribution. Awwad et al. (2015) analyzed the data assuming the Weibull distribution.

Here, we aim to discriminate among Lomax distribution and choose the best-preferred model for fitting this data set. We are interested to analyze this data set from the perspective of progressive type-II censoring using a binomial removal scheme. Our analysis is performed on the data generated from this data set with $p = 0.4$, $m = 40$. Figure 2 shows the plots of the QQ-plot, histogram, and the empirical CDF for real data. Table 5 summarized the goodness-of-fit test results from Kolmogorov-Smirnov (KS) test for the insulating fluid data. The results indicate that the Lomax distribution provides a quite reasonable fit for this censoring scheme.

The posterior risk of p and β and confidence intervals for $m = 40$ and $m = 50$ are presented in Table 6 and Table 7, respectively.

Table 6. Data analysis results of the insulating fluid data for $m = 40$

p	R	$R(\hat{p})$	CI_p	$R(\hat{p}_{BS})$	$R(\hat{p}_{BL})$	$TBPI$
$p = 0.4$	$R = c(8, 6, 4, 1, 0, 1, 0^{*34})$	0.0072	(0.2705, 0.5458)	0.0053	0.0029	(0.2266, 0.440)
		$R(\hat{\beta})$	CI_β	$R(\hat{\beta}_{BS})$	$R(\hat{\beta}_{BL})$	$TBPI$
		0.01007	(0.1934, 0.6329)	0.0079	0.00009	(0.4919, 0.6742)

Table 7. Data analysis result of the insulating fluid data for $m = 50$

p	R	$R(\hat{p})$	CI_p	$R(\hat{p}_{BS})$	$R(\hat{p}_{BL})$	$TBPI$
$p = 0.4$	$R = c(2, 5, 2, 0, 1, 0^{*45})$	0.0012	$(0.2321, 0.6373)$	0.0019	0.00084	$(0.2741, 0.5683)$
		$R(\hat{\beta})$	CI_β	$R(\hat{\beta}_{BS})$	$R(\hat{\beta}_{BL})$	$TBPI$
		0.0054	$(0.2832, 0.6221)$	0.0015	0.00001	$(0.5534, 0.7338)$

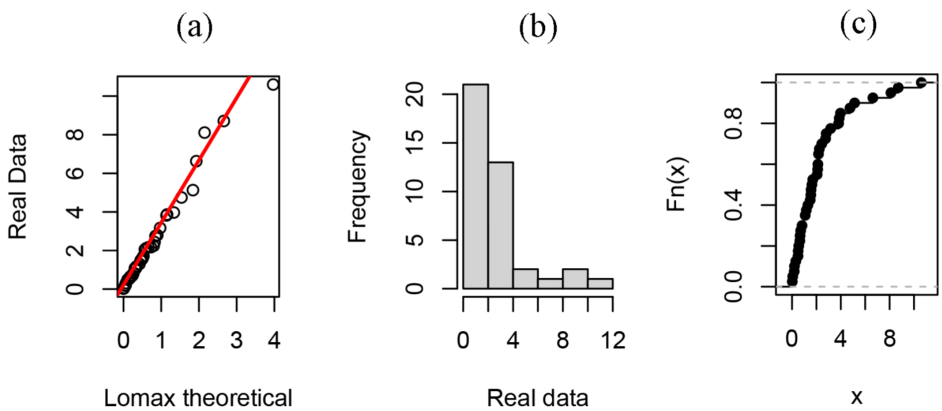

Figure 2. a) Q-Q plot, b) Histogram of real data, c) Empirical CDF

6 Conclusions

In progressive type-II censoring, during the experiment with the occurrence of each failure, numbers of test units are removed randomly until the number of failures (m) is reached. In this article, ML and Bayes estimations of the model parameters of the Lomax distribution were studied under progressive type-II censored data with binomial removals. The EM algorithm was used in estimating the scale parameter because the normal equations were non-linear. In addition, the asymptotic confidence intervals of them are obtained. The simulation results show that the RMSE of the EM method is less than the RMSE of the AML for the β parameter. Moreover, between Bayesian estimations, Bayes estimation with entropy loss function shows less RMSE. Finally, the decrease of the RMSE and the length of confidence intervals with the increase of n and $\frac{m}{n}$ is confirmed.

References

Asgharzadeh, A., and Valiollahi, R. (2011). Estimation of the scale parameter of the Lomax distribution under progressive censoring. *International journal of statistics and Economics. Series B (methodological)*, **6**, 37-48.

Awwad, R.A., Raqab, M.Z., and Al-Mudahakha, I.M. (2015). Statistical inference based on progressively type-II censored data from Weibull model, *Communications in Statistics-Simulation and Computation*, **44**, 2654-2670.

Balakrishnan, N., and Aggarwala, R. (2000). *Progressive censoring: theory, methods, and applications*, Springer Science & Business Media.

Balakrishnan, N. (2007). Progressive censoring methodology: an appraisal, *Springer*, **16**, 211-296.

Dempster, A.P., Laird, N.M., and Rubin, D.B. (1977). Maximum likelihood from incomplete data via the EM algorithm, *Journal of the royal statistical society. Series B (methodological)*, **39**, 1-38.

Dey, S. and Dey, T. (2014). Statistical inference for Rayleigh distribution under progressively Type-II censoring with binomial removal, *Applied Mathematical Modelling*, **38**, 974-982.

Helu, A., Samawib, H., and Raqaba, M.Z. (2015). Estimation on Lomax progressive censoring using the EM algorithm, *Journal of Statistical Computation and Simulation*, **85**, 1035-1052.

Howlader, H.A., and Weiss, G. (1980). Bayes estimators of the reliability of the logistic distribution, *Communications in Statistics-Theory and Methods*, **18**, 1339-1355.

Lawless, J.F. (2011). *Statistical models and methods for lifetime data*, John Wiley and Sons, 362.

Lindley, D.V. (1980). Approximate bayesian methods, *Trabajos de estad'ıstica y de Investigacian Operativa*, **31**, 223-245.

Lomax, K.S. (1954). Business failures: Another example of the analysis of failure data, *Journal of the American Statistical Association*, **49**, 847-852.

McLachlan, G.J., and Krishnan, T. (1997). *The EM Algorithm and Extensions*, New York, Wiley.

Mubarak, M. (2011). Parameter estimation based on the frechet progressive type ii censored data with binomial removals, *International Journal of Quality, Statistics, and Reliability.*

Nelson, W. (1982). *Applied Life Data Analysis*, New York, John Wiley.

Soliman, A.A.(2001). LINEX and quadratic approximate Bayes estimators applied to the Pareto model, *Communications in Statistics-Simulation and Computation*, **30**, 47-62.

Tse, S.K., Yang, C., and Yuen, H.K. (2000). Analysis of Weibull distributed lifetime data under Type-II progressive censoring with binomial removals, *Journal of Applied Statistics*, **27**, 1033-1043.

Wu, S.J., and Chang, C. (2002). Parameter estimations based on exponential progressive type-II censored data with binomial removals, *International journal of information and management sciences*, **13**, 37-46.

Wu, S.J., Chen, Y.J., and Chang, C. (2007). Statistical inference based on progressively censored samples with random removals from the Burr type XII distribution, *Journal of Statistical Computation and Simulation*, **77**, 19-27.

Yuen, H.K. and Tse, S.K. (1996). Parameters estimation for Weibull distribution lifetimes under progressive censoring with random removals, *Journal of Statistical Computation and Simulation*, **55**, 57-71.

Zaman, R., Nasiri, P., and Shadrokh, A. (2020). Statistical inference for the Lomax distribution under partially accelerated life tests with progressively type-II censoring with binomial removal. *Jordan Journal of Mathematics and Statistics*, **13**, 439-458.

Zellner, A. (1986). Bayesian estimation and prediction using asymmetric loss function. *Journal of the American Statistical Association*, **81**, 446-451.

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