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The Lifetime Behavior of a New Discrete Time Mixed δ-shock Model

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Abstract. In this study, a mixed δ-shock model with discrete-time is defined by combining δ-shock and extreme shock models. In this model, a system with multiple states fails in two ways: first, when k interarrival times between two consecutive shocks with magnitude larger than the critical threshold γ are in $[\delta_1, \delta_2], \delta_1 < \delta_2$; and second, when the interarrival time between two consecutive shocks is less than δ_1. The lifetime of the system and the Markov chain of the system's lifetime under the proposed mixed δ-shock model is obtained. Also, the mean lifetime of the system is calculated and a numerical example for validating the analytical results is established here.

Keywords. Discrete time, extreme shocks, interarrival times, lifetime, markov chain, mixed δ-shock model.

1 Introduction

A shock model is introduced to represent the operating system failure process. In reliability, four major shock models are studied: i) Shanthikumar and Sumita (1983) and Gut (1990) introduced the cumulative shock model; ii) Gut and Hüsler (1999) studied an extreme shock model that results in the system failure if the magnitude of a shock is more than a threshold γ; iii) the

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run shock model proposed by Mallor and Omey (2001), and iv) the δ-shock model, a special type of shock model, in which the system fails if the interarrival time between two consecutive shocks is less than a critical threshold δ, and it has been studied in detail by Li et al. (1999), Wang and Zhang (2001), Bai and Xiao (2008) and Eryilmaz (2013). Li and Zhao (2007) have also stated different applications of the δ-shock model including in engineering reliability, electrical systems, inventory theory, insurance mathematics, and customer relation management (CRM). Eryilmaz (2015a) presented an extension of the extreme shock model in which if the magnitude of a shock alternates between two critical levels, then the system switches to a lower partial function state with a reduced capacity. Eryilmaz and Tekin (2019) studied a new mixed shock model that combines run and extreme shock models. Lorvand et al. (2020) investigated an extended discrete-time mixed δ-shock model. Eryilmaz and Kan (2021) studied a mixed shock model for the case when the times between successive shocks and the magnitudes of shocks have discrete phase-type distributions.

Li et al. (1999) presented failure time distribution under a δ-shock model according to a Poisson distribution with a mean λ in per unit time. Eryilmaz (2012) studied the life behavior of a system by assuming the arrival shocks are a type of mixed shock model under the discrete probability distribution. Eryilmaz and Bayramoglu (2014) considered a δ-shock model with arrival processes from a renewal process. Parvardeh and Balakrishnan (2015) verified a new δ-shock model, as an extension of the model of Eryilmaz and Bayramoglu (2014), in which the system fails if the interarrival time between two consecutive shocks is less than a threshold δ, or the magnitude of the shock is more than a threshold γ. Tuncel and Eryilmaz (2018) investigated the δ-shock model in which the interarrival time among two successive shocks is independent but not identically. The shock arrival process and the optimal replacement policy for the δ-shock model are described by a Polya process studied by Eryilmaz (2017). Lorvand et al. (2019) studied the distributional properties of a new mixed δ-shock model in which a system fails in three different ways.

Eryilmaz (2015b) surveyed three different discrete-time shock models in two ways: (i) shocks are independent, and (ii) shocks are Markov dependent. Eryilmaz (2016) studied a system under two shock models over Markovian by assuming the system fails if the cumulative shock magnitude overpasses a threshold or when the cumulative effect of the shocks in consecutive periods is upper than a threshold. Nair et al. (2018) provided the theoretical concepts and results required to model and analyze the discrete reliability systems.

The purpose of this paper is to discuss the discrete-time version of the introduced mixed δ-shock model by Roozegar and Entezari (2022). In this proposed mixed δ-shock model is assumed that i) the magnitudes of arrival shocks are random, and ii) interarrival times between two consecutive shocks are independent and identically distributed (i.i.d.) sequence of geometric distribution with parameter p. According to the definition, the multi-state system would fail in two ways: first, when k interarrival times between two consecutive shocks with a magnitude larger than the critical threshold γ are in $[\delta_1, \delta_2], \delta_1 < \delta_2$; and second, when the interarrival time between two consecutive shocks is less than δ_1.

As a practical example, an earthquake is a natural phenomenon that causes damage in an area. After the earthquake, it will have aftershocks that if the specific number (k) of these aftershocks occurs in a determined time $[\delta_1, \delta_2]$ and the intensity aftershock (Z) is bigger than threshold γ, it will cause fatal damage, or if the aftershock occurs in a determined time less than δ_1, it will cause fatal damage.

As another application, the economic, social and political occurrences are one of significant factors in stock market that can be considered as shocks. Usually, the stock market requires time to recover after each occurrence, denoted by δ_1. If the next occurrence happens before the recovery time, this market will bankrupt. Also, if the stock market faces a specific number of occurrences in the time interval $[\delta_1, \delta_2]$ with predetermined intensity γ, this market will bankrupt. This can be considered as an application of our proposed mixed δ shock model.

The rest of this paper is the following. We investigate the lifetime of the system and the mean of lifetime under this discrete-time mixed δ-shock model in section 2. Also, the Markov chain of the lifetime of the system is derived in section 3. In section 4, an example of this study is carried out to evaluate the accuracy of the analytical results established here. Finally, the concluding remarks of this paper are presented in 5.

We will use the following terms to examine the properties of the behavior of the lifetime of this mixed δ-shock model:

N	Number of interarrival times between two consecutive shocks until the system fails completely
Z_i	The magnitude of the ith shock
X_i	Interarrival time between the $(i-1)$th and ith shocks, for $i = 1, 2, ...$
δ_j	The critical threshold for δ-shock, $j = 1, 2$
γ	The critical threshold for shock magnitude
k	Number of interarrival times between two consecutive shocks with our considered condition $\delta_1 < X_i < \delta_2, Z_i > \gamma$
T	Lifetime of the system
F	Cumulative distribution function

2 The Behavior of the System's Lifetime

To obtain the lifetime of this discrete-time mixed δ-shock model, let N denote the number of interarrival times between two successive shocks that cause the system to fail. So, $N = n$ means that n shocks arrived at the system. Then, it can be enumerated as follows, for $j = 0, 1, \ldots, k-1$ and $l = 0, 1, \ldots, n-j-1$:

$$
\begin{aligned}
(N = n) = &\Big[k-1 \; of \; (n-1) \; (X_i, Z_i) \; are \; (\delta_1 < X_i < \delta_2, Z_i > \gamma) \\
&\quad and \; \Big\{ j \; of \; n-k \; (X_i, Z_i) \; are \; (\delta_1 < X_i < \delta_2, Z_i < \gamma) \\
&\quad and \; n-k-j \; of \; n-k \; (X_i, Z_i) \; are \; X_i > \delta_2) \Big\} \\
&\quad and \; \delta_1 < X_n < \delta_2, Z_n > \gamma \Big] \\
&\cup \Big[k-1 \; of \; (n-1)(X_i, Z_i) \; are \; (\delta_1 < X_i < \delta_2, Z_i > \gamma) \\
&\quad and \; \Big\{ j \; of \; n-k \; (X_i, Z_i) \; are \; (\delta_1 < X_i < \delta_2, Z_i < \gamma) \\
&\quad and \; n-k-j \; of \; n-k \; (X_i, Z_i) \; are \; X_i > \delta_2) \Big\} \; and \; X_n < \delta_1 \Big] \\
&\quad \cup \Big[j \; of \; (n-1) \; (X_i, Z_i) \; are \; (\delta_1 < X_i < \delta_2, Z_i > \gamma) \\
&\quad and \; \Big\{ l \; of \; n-j-1 \; (X_i, Z_i) \; are \; (\delta_1 < X_i < \delta_2, Z_i < \gamma) \\
&\quad and \; (n-j-1-l) \; of \; n-j-1 \; (X_i, Z_i) \; are \; X_i > \delta_2) \Big\} \; and \; X_n < \delta_1 \Big].
\end{aligned}
$$

Before obtaining the pmf of the system's lifetime under this discrete-time mixed model, the following Lemma is useful.

***Lemma* 1.** *(Makri et al. (2007)) The number of allocations of α indistinguishable balls into r distinguishable cells, in such a way that each of $m\,(0 \leq m \leq r)$ specified cells is occupied by at most k balls, is given by*

$$H_m(\alpha, r, k) = \sum_{j=0}^{\min(m,[\frac{\alpha}{k+1}])} (-1)^j \binom{m}{j} \binom{\alpha-(k+1)j+r-1}{\alpha-(k+1)j},$$

for $\alpha \geq 0, r > 0$ and $H_m(\alpha, r, k) = 0$ otherwise, where $[x]$ denotes the integer part of x.

The following Theorem derived the pmf of lifetime of this discrete-time mixed δ-shock model according to the definition of N.

***Theorem* 1.** *Suppose X_i are the interarrival times between two consecutive shocks and Z_i are the magnitudes of shocks and these are mutually independent, for $i = 1, 2, \cdots$. Let $T = \sum_{i=1}^{N} X_i$ be the lifetime of the system. Then, the pmf of the system's lifetime is as follows:*

$$\begin{aligned}
P(T=n) = &\sum_{i=k+1}^{[\frac{n+(k+1)(\delta_2-\delta_1)}{(\delta_2-\delta_1)+1}]} \binom{i-2}{k-1} \Bigg[\sum_{l=0}^{\min(k,[\frac{n-i-(i-(k+1))(\delta_2-\delta_1)}{(\delta_2-\delta_1)}])} (-1)^l \binom{k}{l} \\
&\times \binom{n-(i-(k-l)-1)(\delta_2-\delta_1)-1}{i-1} p^i (1-p)^{n-i} F^{i-k}(\gamma) \bar{F}^k(\gamma) \\
&+ \sum_{j=0}^{k-1} \sum_{i=j+1}^{[\frac{n+(j+1)(\delta_2-\delta_1)}{(\delta_2-\delta_1)+1}]} \binom{i-1}{j} \Bigg[\sum_{l=0}^{\min(j,[\frac{n-i-(i-(j+1))(\delta_2-\delta_1)}{(\delta_2-\delta_1)}])} (-1)^l \binom{j}{l} \\
&\times \binom{n-(i-(j-1)-1)(\delta_2-\delta_1)-1}{i-1} p^i (1-p)^{n-i} F^{i-j-1}(\gamma) \bar{F}^j(\gamma).
\end{aligned} \tag{1}$$

Proof. We presume W_k is the waiting time until $k+1$ out of 1s are separated by at most $[\delta_1, \delta_2]$, $Z > \gamma$ failures in $I_1, I_2, \ldots$ or is the waiting time until an interarrival between successive shocks are less than δ_1. For $n = 1, 2, \ldots,$

$$I_n = \begin{cases} 1 & \text{a shock occures in the period } n, \\ 0 & \text{otherwise.} \end{cases}$$

We have two patterns:

(1) $\underbrace{0\ldots0}_{y_1\geq 0}\ \underbrace{1}_{Z_1<\gamma}\ \underbrace{0\ldots0}_{y_2}\ \underbrace{1}_{Z_2<\gamma}\ \underbrace{0\ldots0}_{y_3}\ \underbrace{1}_{Z_3<\gamma}\cdots\ \underbrace{1}_{Z_{i-2}<\gamma}\ \underbrace{0\ldots0}_{y_{i-1}}\ \underbrace{1}_{Z_{i-1}<\gamma}\ \underbrace{0\ldots0}_{y_i}\ \underbrace{1}_{Z_i>\gamma}$

where y_1 is the number of 0s up to getting the first 1s and y_s is the number of 0s among $(s-1)$th and sth 1, so that k of y_s's are in $[\delta_1, \delta_2]$ and the magnitude of these shocks is bigger than threshold γ for $s = 2, 3, \ldots,$ and the other shocks are bigger than δ_2. Therefore by using Lemma 1, the number of integer solutions to the equation $y_1 + \ldots + y_i = n - i$ under these conditions is

$$\sum_{l=0}^{\min(k,[\frac{n-i-(i-(k+1))(\delta_2-\delta_1)}{(\delta_2-\delta_1)}])} (-1)^l \binom{k}{l} \binom{n-(i-(k-l)-1)(\delta_2-\delta_1)-1}{i-1}.$$

(2) $\underbrace{0\ldots0}_{y_1\geq 0}\ \underbrace{1}_{Z_1<\gamma}\ \underbrace{0\ldots0}_{y_2}\ \underbrace{1}_{Z_2<\gamma}\ \underbrace{0\ldots0}_{y_3}\ \underbrace{1}_{Z_3<\gamma}\cdots\ \underbrace{1}_{Z_{i-2}<\gamma}\ \underbrace{0\ldots0}_{y_{i-1}}\ \underbrace{1}_{Z_{i-1}<\gamma}\ \underbrace{0\ldots0}_{y_i}\ \underbrace{1}_{Z_i>0}$

where j of y_s's are in $[\delta_1, \delta_2]$ and $Z_s > \gamma$ for $s = 2, 3, \ldots, i,\quad j = 0, 1, \ldots, k-1$ and the rest of y_s's are bigger than δ_2, when the last is less than δ_1. Therefore by using Lemma 1, the number of integer solutions to the equation $y_1 + \ldots + y_i = n - i$ under these conditions is

$$\sum_{l=0}^{\min(j,[\frac{n-i-(i-(j+1))(\delta_2-\delta_1)}{(\delta_2-\delta_1)}])} (-1)^l \binom{j}{l} \binom{n-(i-(j-1)-1)(\delta_2-\delta_1)-1}{i-1}.$$

Finally, we have

$$
\begin{aligned}
P(T=n) &= P(W_k=n) = \sum_{i\geq k+1} \sum_{l=0}^{\min(k,[\frac{n-i-(i-(k+1))(\delta_2-\delta_1)}{(\delta_2-\delta_1)}])} (-1)^l \binom{k}{l} \\
&\quad \times \binom{n-(i-(k-l)-1)(\delta_2-\delta_1)-1}{i-1} p^i (1-p)^{n-i} F^{i-k}(\gamma)\bar{F}^k(\gamma) \\
&\quad + \sum_{j=0}^{k-1} \sum_{i\geq j+1} \sum_{l=0}^{\min(j,[\frac{n-i-(i-(j+1))(\delta_2-\delta_1)}{(\delta_2-\delta_1)}])} (-1)^l \binom{j}{l} \\
&\quad \times \binom{n-(i-(j-1)-1)(\delta_2-\delta_1)-1}{i-1} p^i (1-p)^{n-i} F^{i-j-1}(\gamma)\bar{F}^j(\gamma) \\
&= \sum_{i=k+1}^{[\frac{n+(k+1)(\delta_2-\delta_1)}{(\delta_2-\delta_1)+1}]} \binom{i-2}{k-1} \sum_{l=0}^{\min(k,[\frac{n-i-(i-(k+1))(\delta_2-\delta_1)}{(\delta_2-\delta_1)}])} (-1)^l \binom{k}{l} \\
&\quad \times \binom{n-(i-(k-l)-1)(\delta_2-\delta_1)-1}{i-1} p^i (1-p)^{n-i} F^{i-k}(\gamma)\bar{F}^k(\gamma) \\
&\quad + \sum_{j=0}^{k-1} \sum_{i=j+1}^{[\frac{n+(j+1)(\delta_2-\delta_1)}{(\delta_2-\delta_1)+1}]} \binom{i-1}{j} \sum_{l=0}^{\min(j,[\frac{n-i-(i-(j+1))(\delta_2-\delta_1)}{(\delta_2-\delta_1)}])} (-1)^l \binom{j}{l} \\
&\quad \times \binom{n-(i-(j-1)-1)(\delta_2-\delta_1)-1}{i-1} p^i (1-p)^{n-i} F^{i-j-1}(\gamma)\bar{F}^j(\gamma).
\end{aligned}
$$

Note that the continuation of proof is similar to Lorvand et al. (2020). □

Now, suppose M denotes the number of interarrival times between two consecutive shocks till the system receives the first shock with interarrival time between δ_1 and δ_2 with its magnitude larger than γ, or with interarrival time being less than δ_1 that causes the system to get out of the complete working condition. Let $S=\sum_{i=1}^{M} X_i$ be the time that the system remains in complete working condition. Then, to obtain the lifetime of S, we first enumerate M as follows:

$$
\begin{aligned}
(M=m) &= (X_1>\delta_2,\cdots,X_{m-1}>\delta_2,\delta_1<X_m<\delta_2,Z_m>\gamma) \\
&\quad \cup(X_1>\delta_2,\cdots,X_{m-1}>\delta_2,X_m<\delta_1).
\end{aligned}
$$

***Theorem* 2.** *The system's lifetime of $S=\sum_{i=1}^{M} X_i$ is given by*

$$\begin{aligned}P(S=m)=&\sum_{i=2}^{[\frac{m+2\delta_1}{\delta_1+1}]}\left(\binom{m-(i-2)\delta_1-1}{i-1}-\binom{m-(i-1)\delta_1-1}{i-1}\right)p^i(1-p)^{m-i}F^{i-1}(\gamma)\\&+\sum_{i=2}^{[\frac{m+(\delta_2-\delta_1)}{(\delta_2-\delta_1)+1}]}\left(\binom{m-(i-2)(\delta_2-\delta_1)-1}{i-1}-\binom{m-(i-1)(\delta_2-\delta_1)-1}{i-1}\right)\\&\times p^i(1-p)^{m-i}F^{i-1}(\gamma)\bar{F}(\gamma).\end{aligned} \tag{2}$$

Proof. The proof of this Theorem is similar to Lorvand et al. (2020). Note the difference between this Theorem and Lorvand et al. (2020)'s Theorem is in the definition of conditions in our shock model. □

Now, in the next Theorem, the lifetime of system in partially working condition $(T-S)$ is derived by using the definitions of N and M.

***Theorem* 3.** *With the assumptions mentioned in Theorem 1, the pmf of $T-S$ is presented by*

$$P(T-S=m)=\begin{cases}\sum\limits_{n=1}^{\infty}\sum\limits_{i=1}^{[\frac{n+(\delta_2-\delta_1)}{(\delta_2-\delta_1)+1}]}\binom{n-(i-2)\delta_1-1}{i-1}p^i(1-p)^{n-i}F^{i-1}(\gamma)\bar{F}(\gamma), & m=0,\\ (1-P(T-S=0))P(\sum\limits_{s=0}^{N^*}X_s=m), & m>0,\end{cases} \tag{3}$$

where $P(\sum_{s=0}^{N^*}X_s=m)$ is given by replacing k by $k-1$ in Theorem 1.

Proof. The proof of this Theorem is the same as Lorvand et al. (2020). Note the difference among this Theorem and Lorvand et al. (2020)'s Theorem is the definition of conditions in the mixed δ-shock model. □

***Theorem* 4.** *The probability generating function of a system's lifetime is*

$$\begin{aligned}\varphi(t)=E(t^{\sum_{t=0}^{N}X_i}) &= \frac{pt}{1-qt}\left\{\left(\frac{\bar{F}(\gamma)\frac{pt((qt)^{\delta_1}-(qt)^{\delta_2})}{(1-qt)}}{1-\frac{pt(qt)^{\delta_2}}{1-qt}-F(\gamma)\frac{pt((qt)^{\delta_1}-(qt)^{\delta_2})}{1-qt}}\right)^k\right.\\&\left.+\frac{\left[1-\left(\frac{\bar{F}(\gamma)\frac{pt((qt)^{\delta_1}-(qt)^{\delta_2})}{(1-qt)}}{1-\frac{pt(qt)^{\delta_2}}{1-qt}}\right)^k\right]\frac{pt(1-(qt)^{\delta_1})}{1-qt}}{1-\frac{pt(qt)^{\delta_2}}{1-qt}-\bar{F}(\gamma)\frac{pt((qt)^{\delta_1}-(qt)^{\delta_2})}{1-qt}}\right\}.\end{aligned}$$

Proof. We know

$$E(t^{\sum_{i=0}^{N} X_i}) = E(t^{X_0})E(t^{\sum_{i=1}^{N} X_i}).$$

Since X_0 has the geometric distribution, $E(t^{X_0}) = \frac{pt}{1-qt}$ and

$$\begin{aligned}
E(t^{\sum_{i=1}^{N} X_i}) &= \sum_{n=1}^{\infty} E(t^{\sum_{i=1}^{N} X_i}|N=n)P(N=n) \\
&= \sum_{n=k}^{\infty}\sum_{j=0}^{n-k}\binom{n-1}{k-1}\binom{n-k}{j}\Big[E(t^{X_1}|\delta_1 < X_1 < \delta_2)P(\delta_1 < X_1 < \delta_2)F(\gamma)\Big]^j \\
&\quad \times \Big[E(t^{X_1}|X_1 > \delta_2)P(X > \delta_2)\Big]^{n-k-j} \\
&\quad \times \Big[E(t^{X_1}|\delta_1 < X_1 < \delta_2)P(\delta_1 < X_1 < \delta_2)\bar{F}(\gamma)\Big]^k \\
&\quad + \sum_{j=0}^{k-1}\sum_{n=j+1}^{\infty}\binom{n-1}{j}\Big[E(t^{X_1}|\delta_1 < X_1 < \delta_2)P(\delta_1 < X_1 < \delta_2)\bar{F}(\gamma)\Big]^j \\
&\quad \times \Big[E(t^{X_1}|X_1 > \delta_2)P(X > \delta_2)\Big]^{n-j-1} E(t^{X_1}|X_1 \le \delta_1)P(X_1 \le \delta_1) \\
&= \left(\frac{E(t^{X_1}|\delta_1 < X_1 < \delta_2)P(\delta_1 < X_1 < \delta_2)\bar{F}(\gamma)}{1-E(t^{X_1}|X_1 > \delta_2)P(X_1 > \delta_2)-E(t^{X_1}|\delta_1 < X_1 < \delta_2)P(\delta_1 < X_1 < \delta_2)F(\gamma)}\right)^k \\
&\quad + \frac{E(t^{X_1}|X_1 \le \delta_1)P(X_1 \le \delta_1)\left[1-\left(\frac{E(t^{X_1}|\delta_1<X_1<\delta_2)P(\delta_1<X_1<\delta_2)}{(1-E(t^{X_1}|X_1>\delta_2)P(X_1>\delta_2)}\right)^k\right]}{1-E(t^{X_1}|X_1 > \delta_2)P(X_1 > \delta_2)-E(t^{X_1}|\delta_1 < X_1 < \delta_2)P(\delta_1 < X_1 < \delta_2)\bar{F}(\gamma)} \\
&= \frac{pt}{1-qt}\left\{\left(\frac{\bar{F}(\gamma)\frac{pt((qt)^{\delta_1}-(qt)^{\delta_2})}{(1-qt)}}{1-\frac{pt(qt)^{\delta_2}}{1-qt}-F(\gamma)\frac{pt((qt)^{\delta_1}-(qt)^{\delta_2})}{1-qt}}\right)^k\right. \\
&\quad \left. + \frac{\left[1-\left(\frac{\bar{F}(\gamma)\frac{pt((qt)^{\delta_1}-(qt)^{\delta_2})}{(1-qt)}}{1-\frac{pt(qt)^{\delta_2}}{1-qt}}\right)^k\right]\frac{pt(1-(qt)^{\delta_1})}{1-qt}}{1-\frac{pt(qt)^{\delta_2}}{1-qt}-\bar{F}(\gamma)\frac{pt((qt)^{\delta_1}-(qt)^{\delta_2})}{1-qt}}\right\}.
\end{aligned}$$

□

Remark 1. By using the derivation of Theorem 4 when $t = 1$, we can obtain the mean lifetime of system as following

$$
\begin{aligned}
E(T) \quad = \quad & \frac{p}{(q-1)^2}\Bigg\{\frac{2e^{\gamma\lambda}p(q-1)(q^{\delta_1}-1)\Bigg((\frac{e^{-\gamma\lambda}p(-q^{\delta_1}+q^{\delta_2})-1)}{-1+q+pq^{\delta_2}})^k-1\Bigg)}{p(q^{\delta_1}-q^{\delta_2})+e^{\gamma\lambda}(-1+q+pq^{\delta_2})} \\
& -\frac{2e^{\gamma\lambda}pq(-1+q^{\delta_1})\Bigg(-1+(\frac{e^{-\gamma\lambda}p(-q^{\delta_1}+q^{\delta_2})}{-1+q+pq^{\delta_2}})^k\Bigg)}{p(q^{\delta_1}-q^{\delta_2})+e^{\gamma\lambda}(-1+q+pq^{\delta_2})} \\
& +\frac{e^{\gamma\lambda}p(-1+q)q^{\delta_1}\Bigg(-1+(\frac{e^{-\gamma\lambda}p(-q^{\delta_1}+q^{\delta_2})}{-1+q+pq^{\delta_2}})^k\Bigg)\delta_1}{p(q^{\delta_1}-q^{\delta_2})+e^{\gamma\lambda}(-1+q+pq^{\delta_2})}
\end{aligned}
$$

$$
\begin{aligned}
& -e^{\gamma\lambda}p^2(-1+q^{\delta_1})(-1+(\frac{e^{-\gamma\lambda}p(-q^{\delta_1}+q^{\delta_2})}{-1+q+pq^{\delta_2}})^k) \\
& \times\frac{q^{\delta_1}(-1+(-1+q)\delta_1)+(-1+e^{\gamma\lambda})q^{\delta_2}(-1+(-1+q)\delta_2)}{(p(q^{\delta_1}-q^{\delta_2})+e^{\gamma\lambda}(-1+q+pq^{\delta_2}))^2} \\
& +e^{\gamma\lambda}kp(-1+q)(-1+q^{\delta_1})(\frac{e^{-\gamma\lambda}p(-q^{\delta_1}+q^{\delta_2})}{-1+q+pq^{\delta_2}})^k \\
& \times\frac{q^{\delta_1}(-1+(-1+q)\delta_1)+q^{\delta_2}(1+pq^{\delta_1}(\delta_1-\delta_2)+\delta_2-q\delta_2)}{(q^{\delta_1}-q^{\delta_2})(-1+q+pq^{\delta_2})(p(q^{\delta_1}-q^{\delta_2})+e^{\gamma\lambda}(-1+q+pq^{\delta_2}))} \\
& +\Bigg[-p(q^{\delta_1}-q^{\delta_2})^2+e^{\gamma\lambda}q^{\delta_1}(pq^{\delta_1}+(-1+q)(1+k(1+\delta_1-q\delta_1))) \\
& +q^{\delta_2}(1-q-pq^{\delta_1}+k(-1+q)(-1-pq^{\delta_1}(\delta_1-\delta_2)+(-1+q)\delta_2))\Bigg] \\
& \times\frac{(1+\frac{e^{\gamma\lambda}(-1+q+pq^{\delta_1})}{p(-q^{\delta_1}+q^{\delta_2})})^{-k}}{(q^{\delta_1}-q^{\delta_2})(e^{\gamma\lambda}(-1+q+pq^{\delta_1})+p(-q^{\delta_1}+q^{\delta_2}))}\Bigg\}.
\end{aligned}
$$

3 Markov Chain

In this section, we expanded the mixed δ-shock model to a dependent case. Suppose that

$$p_{ij}=P(I_n=j|I_n=i), \qquad i,j=0,1,$$

and also, $p_0=P(I_1=0)$ and $p_1=P(I_1=1)=1-P(I_1=0)$.

***Theorem* 5.** *The system's lifetime is presented by*

$$
\begin{aligned}
P(T=n) = {} & p_1 p_{11}^{n-1} I[n=k+1] \\
& + \sum_{j=0}^{k} \binom{k}{j} \sum_{i=k+1}^{[\frac{n+(\delta_2-\delta_1)(k+1)-(j+1)}{(\delta_2-\delta_1)+1}]} \binom{i-2}{k-1} \sum_{l=0}^{\min(j,[\frac{n-(i-k-1)(\delta_2-\delta_1)-(i+j+1)}{(\delta_2-\delta_1)-1}])} \\
& \times (-1)^l \binom{j}{l} \binom{n-(i-k-1)(\delta_2-\delta_1)-l(\delta_2-\delta_1-1)-(k+2)}{i-k+j-1} \\
& \times p_0 p_{00}^{n-2i} p_{01}^{i} p_{10}^{i-1} F^{j-k}(\gamma) \bar{F}^k(\gamma) \\
& + \sum_{h=0}^{k-1} \sum_{i=h+1}^{[\frac{n+\delta_1(h+1)}{\delta_1+1}]} \binom{i-1}{h} \sum_{l=0}^{\min(h,[\frac{n-(i-h-1)\delta_1-(i+1)}{\delta_1-1}])} (-1)^l \binom{h}{l} \\
& \times \binom{n-(i-1)\delta_1-l(\delta_1-1)-(h+2)}{i-h-1} \\
& \times p_0 p_{00}^{n-2i} p_{01}^{i-2} p_{10}^{i-2} F^{i-h}(\gamma) \bar{F}^h(\gamma) [p_{00}p_{11}+p_{01}p_{10}] \\
& + \sum_{j=0}^{k} \binom{k}{j} \sum_{i=k+1}^{[\frac{n+(\delta_2-\delta_1)(k+1)-j}{(\delta_2-\delta_1)+1}]} \binom{i-2}{k-1} \sum_{l=0}^{\min(j,[\frac{n-(i-k-1)(\delta_2-\delta_1)-(i+j)}{(\delta_2-\delta_1)-1}])} \\
& \times (-1)^l \binom{j}{l} \binom{n-(i-k-1)(\delta_2-\delta_1)-l(\delta_2-\delta_1-1)-(k+2)}{i-k+j-2} \\
& \times p_1 p_{00}^{n-2i+1} p_{01}^{i-1} p_{10}^{i-1} F^{j-k}(\gamma) \bar{F}^k(\gamma) \\
& + \sum_{h=0}^{k-1} \sum_{i=h+1}^{[\frac{n+(\delta_1)(h+1)}{\delta_1+1}]} \binom{i-1}{h} \sum_{l=0}^{\min(h,[\frac{n-(i-h-1)\delta_1}{\delta_1-1}])} (-1)^l \binom{h}{l} \\
& \times \binom{n-(i-h-1)\delta_1-l(\delta_1-1)-(h+2)}{i-h-2} \\
& \times p_1 p_{00}^{n-2i} p_{01}^{i-2} p_{10}^{i-2} F^{i-h}(\gamma) \bar{F}^h(\gamma) [p_{00}p_{11}+p_{01}p_{10}].
\end{aligned}
$$

Proof. To prove this Theorem, we can obtain five ways for the mixed δ-shock model as follows

(A)

$$
\underbrace{1 \quad 1}_{y_1=0}\underbrace{1 \quad 1}_{y_2=0} \cdots \underbrace{1 \quad 1}_{y_{k-1}=0}\underbrace{1 \quad 1}_{y_k=0},
$$

(B)

$$\underbrace{0\ldots0}_{y_1>0}\underbrace{1}_{Z_1}\underbrace{0\ldots0}_{y_2>0}\underbrace{1}_{Z_2}\underbrace{0\ldots0}_{y_3>0}\underbrace{1}_{Z_3}\cdots\underbrace{1}_{Z_{i-2}}\underbrace{0\ldots0}_{y_{i-1}>0}\underbrace{1}_{Z_{i-1}}\underbrace{0\ldots0}_{y_i>0}\underbrace{1}_{Z_i},$$

(C)

$$\begin{cases}\underbrace{0\ldots0}_{y_1>0}\underbrace{1}_{Z_1<\gamma}\underbrace{0\ldots0}_{y_2>\delta_1}\underbrace{1}_{Z_2<\gamma}\underbrace{0\ldots0}_{y_3>\delta_1}\underbrace{1}_{Z_3<\gamma}\cdots\underbrace{1}_{Z_{i-2}<\gamma}\underbrace{0\ldots0}_{y_{i-1}>\delta_1}\underbrace{1}_{Z_{i-1}<\gamma}\underbrace{0\ldots0}_{0<y_i<\delta_1},\\ \underbrace{0\ldots0}_{y_1>0}\underbrace{1}_{Z_1<\gamma}\underbrace{0\ldots0}_{y_2>\delta_1}\underbrace{1}_{Z_2<\gamma}\underbrace{0\ldots0}_{y_3>\delta_1}\underbrace{1}_{Z_3<\gamma}\cdots\underbrace{1}_{Z_{i-2}<\gamma}\underbrace{0\ldots0}_{y_{i-1}>\delta_1}\underbrace{1}_{Z_{i-1}<\gamma}\underbrace{0\ldots0}_{y_i=0},\end{cases}$$

(D)

$$\underbrace{1}_{Z_1}\underbrace{0\ldots0}_{y_1>0}\underbrace{1}_{Z_2}\underbrace{0\ldots0}_{y_2>0}\underbrace{1}_{Z_3}\cdots\underbrace{1}_{Z_{i-2}}\underbrace{0\ldots0}_{y_{i-2}>0}\underbrace{1}_{Z_{i-1}}\underbrace{0\ldots0}_{y_{i-1}>0}\underbrace{1}_{Z_i},$$

(E)

$$\begin{cases}\underbrace{1}_{Z_1<\gamma}\underbrace{0\ldots0}_{y_1>0}\underbrace{1}_{Z_2<\gamma}\underbrace{0\ldots0}_{y_2>\delta_1}\underbrace{1}_{Z_3<\gamma}\underbrace{0\ldots0}_{y_3>\delta_1}\cdots\underbrace{1}_{Z_{i-2}<\gamma}\underbrace{0\ldots0}_{y_{i-2}>0}\underbrace{1}_{0<y_{i-1}<\delta_1},\\ \underbrace{1}_{Z_1<\gamma}\underbrace{0\ldots0}_{y_1>0}\underbrace{1}_{Z_1<\gamma}\underbrace{0\ldots0}_{y_2>\delta_1}\underbrace{1}_{Z_2<\gamma}\underbrace{0\ldots0}_{y_3>\delta_1}\underbrace{1}_{Z_3<\gamma}\cdots\underbrace{1}_{y_{i-1}=0}.\end{cases}$$

Form (A):

(1) $y_1 = y_2 = \cdots = y_k = 0$,

(2) Each sequence of (A) has the probability p_1p_{11}.

Form (B):

(1) $y_1 + y_2 + \cdots + y_i = n - i$,

(2) j out of shocks are in $\delta_1 < y_i < \delta_2$ so that the magnitude of k shocks is greater than γ and $j-k$ shocks is less than γ. Also, $i-j$ of shocks are in $y_i > \delta_2$.

Hence, the probability of (B) is $p_0p_{00}^{n-2i}p_{01}^{i}p_{10}^{i-1}F^{j-k}(\gamma)\bar{F}^{k}(\gamma)$.

Form (C):

(1) $y_1 + y_2 + \cdots + y_i = n - i$,

(2) If $y_i = 0$, then the probability is $p_0p_{00}^{n-2i+1}p_{01}^{i-2}p_{10}^{i-2}p_{11}F^{i-h}(\gamma)\bar{F}^{h}(\gamma)$,

(3) If $0 < y_i < \delta_1$, the probability is $p_0p_{00}^{n-2i}p_{01}^{i-1}p_{10}^{i-1}F^{i-h}(\gamma)\bar{F}^{h}(\gamma)$.

Hence, by simplifying the two probabilities above, the probability of (C) is

$$p_0p_{00}^{n-2i}p_{01}^{i-2}p_{10}^{i-2}F^{i-h}(\gamma)\bar{F}^{h}(\gamma)[p_{00}p_{11} + p_{10}p_{01}].$$

Form (D):

(1) $y_1 + y_2 + \cdots + y_i = n - i$,

(2) k out of our shocks are in $\delta_1 < y_i < \delta_2, \quad Z > \gamma$ and $j - k$ of shocks are in $\delta_1 < y_i < \delta_2, \quad Z < \gamma$.

Hence, the probability of (D) is

$$p_1 p_{00}^{n-2i+1} p_{01}^{i-1} p_{10}^{i-1} F^{j-k}(\gamma) \bar{F}^k(\gamma).$$

Form (E):

(1) $y_1 + y_2 + \cdots + y_i = n - i$,

(2) If $y_{i-1} = 0$, the probability is $p_1 p_{00}^{n-2i+1} p_{01}^{i-2} p_{10}^{i-2} p_{11} F^{i-h}(\gamma) \bar{F}^h(\gamma)$,

(3) If $0 < y_{i-1} < \delta_1$, the probability is $p_1 p_{00}^{n-2i} p_{01}^{i-1} p_{10}^{i-1} F^{i-h}(\gamma) \bar{F}^h(\gamma)$.

Hence, by simplifying the two probabilities above, the probability of (E) is

$$p_1 p_{00}^{n-2i} p_{01}^{i-2} p_{10}^{i-2} F^{i-h}(\gamma) \bar{F}^h(\gamma) [p_{00} p_{11} + p_{10} p_{01}].$$

Finally, we can obtain our system's lifetime with these Markov chain patterns. □

4 Computational Results

In this section, an example of this study is carried out to validate the analytical results obtained here. It is assumed that the interarrival times $X_1, X_2, \cdots$ and the magnitudes of shocks $Z_1, Z_2, \cdots$ are i.i.d. random variables having the geometric and the exponential distribution with the probability $p = 0.8$ and mean 0.5, respectively, and that they are also mutually independent.

Figure 1 presents the pmf of system lifetime $P(T = n)$ for $\delta_1 = 2$, $\delta_2 = 4$, $\gamma = 0.2$, $\lambda = 2$, $p = 0.8$ with respect to $k = 1, 2, 3, 4$. As be observed, the system's lifetime decreases when k increasing. Figure 2 displays the plot of $P(T = n)$ with respect to $\gamma = 0.2, 0.5, 0.8, 1$. As shown, the system's lifetime decreases when γ increasing. Also, the plot of $P(T = n)$ is shown in Figure 3 with respect to $k = 2, 3, 4$ which the system's lifetime decreases when k increasing. In addition, with increasing interarrival time $[\delta_1, \delta_2]$ the lifetime of the system increases. Figure 4 presents a plot of $E(T)$ for $\delta_1 = 1$, $\delta_2 = 3$, $\gamma = 1$, $\lambda = 2$ with respect to $k = 1, 2, 3, 4$ and shows that the mean of the system decreases when k increasing. Also in Figure 5, with increasing values $[\delta_1, \delta_2]$ and fixed width interarrival time, the mean of the system first decreases and then increases.

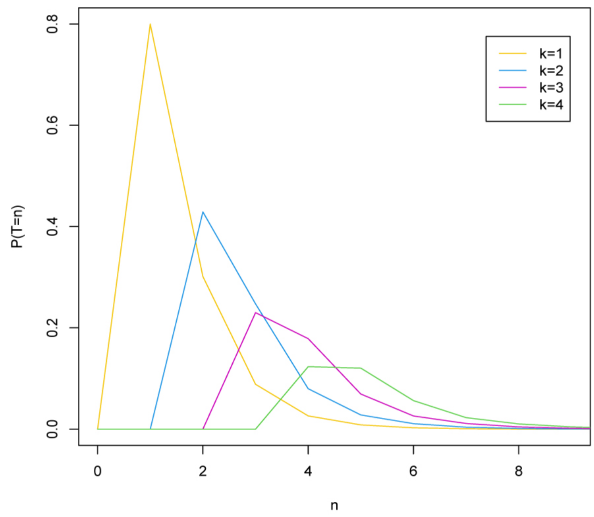

Figure 1. The $P(T = n)$ when $\delta_1 = 2, \delta_2 = 4, \gamma = 0.2, \lambda = 2, p = 0.8$ and for different values of k.

Table 1 gives the values of pmf $P(T = n)$ for $\delta_1 = 2$, $\delta_2 = 4$, $p = 0.8$ and $\lambda = 2$ with respect to different values of γ, k and n. As observed, the system's lifetime decreases when γ and n increase. When k increases, the system's lifetime is sensitive with respect to γ that for $\gamma \leq 0.5$, the system's lifetime increases and for $\gamma > 0.5$, the system's lifetime decreases. Table 2 gives the values of $E(T)$ for $\lambda = 2$ with respect to different values of parameters. As observed, the mean of system's lifetime decreases when γ and p increase. When k increases, the mean of system's lifetime is sensitive with respect to γ that for $\gamma \geq 0.5$, the mean of system's lifetime decreases and for $\gamma < 0.5$, the mean of system's lifetime increases. Also with increasing values $[\delta_1, \delta_2]$ and fixed width interarrival time, the mean of the system first decreases and then increases.

5 Conclusion

In this study, a mixed δ-shock model with discrete-time is defined by combining δ-shock and extreme shock models, such that it causes the failure of a multi-state system in two ways: first, when k interarrival times between

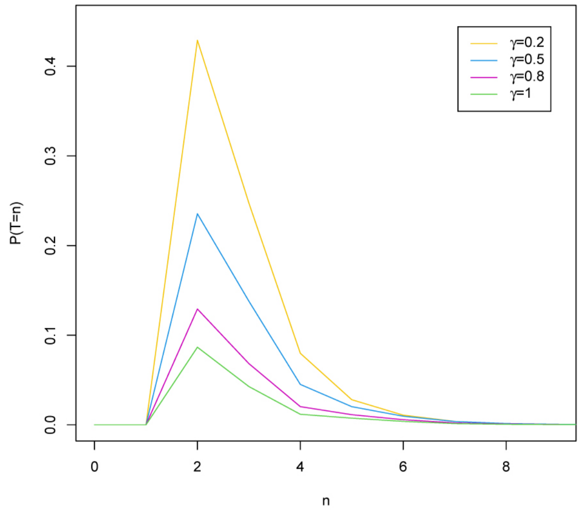

Figure 2. The $P(T=n)$ when $\delta_1=2, \delta_2=4, \lambda=2, p=0.8$ and for different values of γ.

Table 1. The pmf of system lifetime $P(T=n)$ with $\delta_1=2, \delta_2=4$ and different values of γ ana p.

		$P(T=n)$		
k	γ	$n=3$	$n=5$	$n=6$
1	0.2	0.0886	0.0084	0.0026
	0.5	0.0915	0.0131	0.0045
	0.8	0.0733	0.0137	0.0046
	1	0.0700	0.0134	0.0044
2	0.2	0.2474	0.0281	0.0108
	0.5	0.1380	0.0203	0.0096
	0.8	0.0683	0.0113	0.0058
	1	0.0428	0.0075	0.0039
3	0.2	0.2301	0.0693	0.0260
	0.5	0.0693	0.0214	0.0100
	0.8	0.0209	0.0055	0.0030
	1	0.0094	0.0022	0.0013

Table 2. The mean of system lifetime $E(T)$ for different values of parameters.

			$E(T)$		
k	$[\delta_1, \delta_2]$	γ	$p = 0.1$	$p = 0.2$	$p = 0.3$
1	$[1, 3]$	0.2	47.555	14.964	8.059
		0.5	44.878	13.617	7.155
		0.8	38.888	11.640	6.090
		1	34.329	10.324	5.447
	$[2, 4]$	0.2	37.074	12.576	7.153
		0.5	33.054	11.141	6.353
		0.8	29.101	9.919	5.739
		1	26.948	9.301	5.445
	$[5, 7]$	0.2	27.509	10.735	6.674
		0.5	25.527	10.091	6.376
		0.8	24.194	9.688	6.199
		1	23.600	9.516	6.125
2	$[1, 3]$	0.2	59.806	17.688	9.110
		0.5	44.176	12.890	6.632
		0.8	31.480	9.464	5.041
		1	25.766	8.035	4.416
	$[2, 4]$	0.2	42.434	13.761	7.622
		0.5	32.278	10.811	6.193
		0.8	26.561	9.246	5.463
		1	24.442	8.675	5.200
	$[5, 7]$	0.2	29.364	11.157	6.836
		0.5	25.918	10.192	6.428
		0.8	24.166	9.697	6.215
		1	23.508	9.508	6.132
3	$[1, 3]$	0.2	61.091	17.679	8.983
		0.5	37.847	11.202	5.888
		0.8	25.603	8.1001	4.507
		1	21.624	7.129	4.079
	$[2, 4]$	0.2	42.390	13.682	7.578
		0.5	30.439	10.400	6.049
		0.8	25.372	9.004	5.384
		1	23.735	8.537	5.153
	$[5, 7]$	0.2	29.466	11.175	6.842
		0.5	25.828	10.180	6.427
		0.8	24.113	9.690	6.214
		1	23.478	9.504	6.132

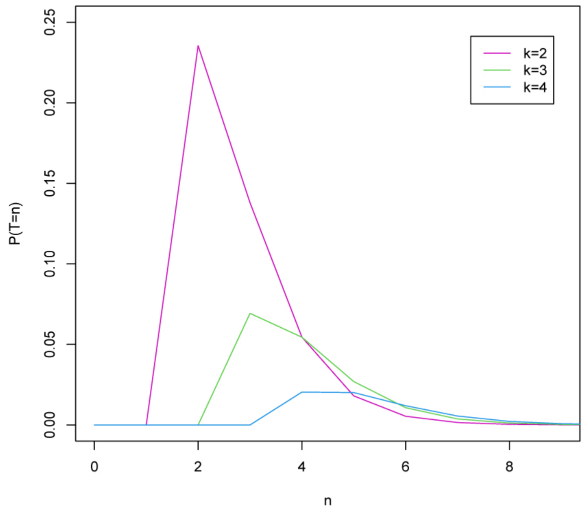

Figure 3. The $P(T = n)$ when $\gamma = 0.2, \lambda = 2, p = 0.8$ and for different values of k.

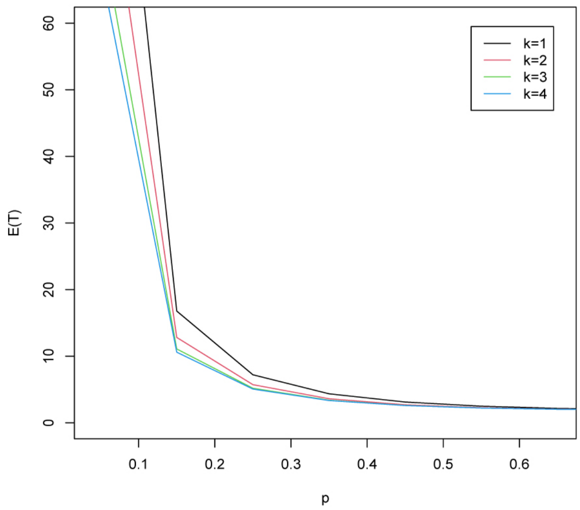

Figure 4. The $E(T)$ when $\delta_1 = 1, \delta_2 = 3, \gamma = 1, \lambda = 2$ and for different values of k.

two consecutive shocks with a magnitude of shock larger than threshold γ is

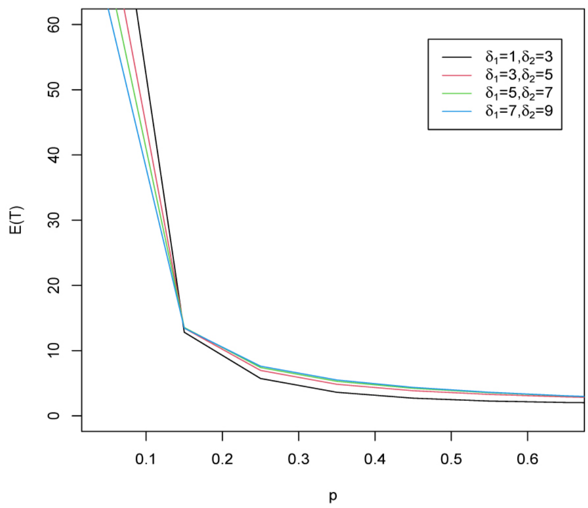

Figure 5. The $E(T)$ when $\gamma = 1, \lambda = 2$ and for different values in $[\delta_1, \delta_2]$.

in $[\delta_1, \delta_2]$, and second, when the time among two consecutive shocks is less than δ_1.

By assuming that the shocks occur independently and randomly with the magnitude Z_i and the interarrival times among two consecutive shocks X_i are i.i.d. random variables, we have derived explicit expressions for the lifetime of the proposed mixed δ-shock model for three cases: (i) system's lifetime, (ii) when the system performs completely, and (iii) when the system performs partially. Moreover, the generating function and mean of the system's lifetime have also been derived. Also, a Markovian chain has been computed for this mixed δ-shock model. Finally, a computational result for the lifetime and mean of the system has been presented.

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