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Inference about the Burr Type III Distribution under Type-II Hybrid Censored Data

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Abstract. This paper presents the statistical inference on the parameters of the Burr type III distribution, when the data are Type-II hybrid censored. The maximum likelihood estimators are developed for the unknown parameters using the EM algorithm method. We provided the observed Fisher information matrix using the missing information principle which is useful for constructing the asymptotic confidence intervals. The Bayesian estimates of the unknown parameters under the assumption of independent gamma priors are obtained using two approximations, namely Lindley's approximation and the Markov Chain Monte Carlo technique. Monte Carlo simulations are performed to observe the behavior of the proposed methods and a real dataset representing is used to illustrate the derived results.

Keywords. Bayesian estimates; Burr type III distribution; EM algorithm method; importance sampling scheme; Lindley's approximation; maximum likelihood estimators; Type-II hybrid censoring.

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1 Introduction

Burr family of distribution was introduced by Burr (1942) and is capable to approximate many well-known distributions such as Normal, Log-normal,

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Weibull, Gamma, Exponential and other type of family distributions preciously. One of the most important of them is the Burr type III distribution. The cumulative distribution function and probability density function of the Burr type III are given by,

$$F(x) = (1 + x^{-c})^{-k} \qquad x > 0 \tag{1}$$

$$f(x) = kcx^{-(c+1)}(1 + x^{-c})^{-(k+1)} \qquad x > 0 \tag{2}$$

respectively. Here $c > 0$ and $k > 0$ are the two shape parameters. Extensive work has been done on the Burr type III distribution, see for example, Al-Dayian (1999), Headrick et al. (2010) and Abd-Elfattah and Alharbey (2012).

Type-I (time) censoring, where the life testing experiment will be terminated at a prescribed time T, and Type-II (failure) censoring, where the life testing experiment will be terminated upon the R (R is pre-fixed) failure are the two most popular censoring schemes used in the reliability and experimental studies. Several authors considered different aspects of these censoring schemes, see for example Ng et al. (2006), Kundu and Howlader (2010), Sayyareh (2012) and Panahi and Sayyareh (2014). The mixture of Type-I and Type-II censoring schemes is known as hybrid censoring scheme. Epstein (1954) first introduced the Type-I hybrid censoring scheme, and it can be described as follows. Suppose n identical units are put on a life test. The test is terminated when a pre-specified number R, out of n units have failed or a pre-determined time T, has been reached, *i.e.*, hybrid life test experiment terminates at the random time $T_* = \min\{x_{R:n}, T\}$. Therefore, in Type-I hybrid censoring scheme, the experimental time and the number of failures will not exceed T and R respectively. It is clear that Type-I and Type-II censoring schemes can be obtained as special cases of hybrid censoring scheme by taking $R = n$ and $T = \infty$, respectively. As in the case of conventional Type-I censoring scheme the inferential results are obtained under the condition that the number of observed failure is at least one, and in addition there may be very few failures occurring up to the pre-fixed time T, which results in the estimator(s) of the model parameter(s) having low efficiency. For this reasons, Childs et al. (2003) introduced an alternative hybrid censoring scheme that terminated the experiment at the random time $T^* = \max\{x_{R;n}, T\}$. This hybrid censoring scheme is called Type-II hybrid censoring scheme and it has the advantage at least R failures to be observed by the end of the experiment. So, under this censoring scheme we can observe the following three types of observations

$$\begin{aligned}
I &: X_{1:n} < X_{2:n} < \cdots < X_{R:n} \text{ if } X_{R:n} > T,\\
II &: X_{1:n} < \cdots < X_{R:n} < \cdots < X_{m:n} < T < X_{(m+1):n}\\
&\qquad \text{if } R < m < n \text{ and } X_{m:n} < T < X_{(m+1):n},\\
III &: X_{1:n} < X_{2:n} < \cdots < X_{n:n} < T.
\end{aligned}$$

Note that, in case II, m failures occurred to time T while $(m+1)$-th failure occurs after T. For some of the references of this censoring scheme, the readers are referred to Kundu and Pradhan (2009), Kundu (2007), Banerjee and Kundu (2008), Rastogi and Tripathi (2011), Balakrishnan and Shafay (2011), Ling et al. (2011), Gupta and Singh (2013), Singh et al. (2013), Bhattacharya et al. (2013), Balakrishnan and Kundu (2013) and Singh et al. (2014).

In this paper we consider the statistical inference of the Burr type III distribution when the data are Type-II hybrid censored. We obtain the maximum likelihood estimators (MLEs) of the unknown parameters. It is observed that the MLEs cannot be obtained in closed forms. Thus, we propose to use the EM algorithm to compute the MLEs. We calculate the observed Fisher information matrix using the missing information principle and they have been used for constructing asymptotic confidence intervals. We also provide the Bayesian estimates of the unknown parameters under the different loss functions. It is observed that the Bayesian estimates cannot be computed explicitly, and we use the importance sampling scheme to compute the Bayesian estimates. Monte Carlo simulations are performed to compare the performances of the different methods and a real dataset is analyzed for illustrative purpose.

The rest of the paper is organized as follows. In Section 2, we describe the model, the maximum likelihood estimators and observed Fisher information matrix. The Bayesian analysis are presented in Section 3. This section, presented Lindley's approximation and importance sampling scheme. Simulation results are provided in Section 4. Real data analysis is provided in Section 5 and finally we conclude the paper in Section 6.

2 Maximum Likelihood Estimators

Suppose that $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are ordered lifetime observations of n independent units taken from model (2). Based on the observed data, the log-likelihood function for combined three different cases can be written as

$$\ell(c,k) = \ln L(c,k) = d\ln k + d\ln c - (c+1)\sum_{i=1}^{d}\ln x_{i:n} - (k+1)\sum_{i=1}^{d}\ln(1+x_{i:n}^{-c}) + (n-d)\ln\{1-(1+u^{-c})^{-k}\}, \tag{3}$$

here, d denoted the number of failures; and $u = x_{R:n}$ if $d = R$, and $u = T$ if $d > R$. d strictly positive because when it takes value zero it is difficult to evaluate the maximum likelihood estimators.
Taking derivatives with respect to c and k of (3) and equating them to zero we obtain;

$$\frac{\partial \ell(c,k)}{\partial k} = \frac{d}{k} - \sum_{i=1}^{d}\ln(1+x_{i:n}^{-c}) + (n-d)\frac{(1+u^{-c})^{-k}\ln(1+u^{-c})}{1-(1+u^{-c})^{-k}} \tag{4}$$

$$\frac{\partial \ell(c,k)}{\partial c} = \frac{d}{c} - \sum_{i=1}^{d}\ln x_{i:n} + (k+1)\sum_{i=1}^{d}\frac{x_{i:n}^{-c}\ln x_{i:n}}{1+x_{i:n}^{-c}} - (n-d)R(c,k) \tag{5}$$

where, $R(c,k) = \frac{k(1+u^{-c})^{-(k+1)}u^{-c}\ln u}{1-(1+u^{-c})^{-k}}$.
From (4) and (5), the maximum likelihood estimators of c and k is expressed as,

$$\frac{d}{k} + (n-d)\frac{(1+u^{-c})^{-k}\ln(1+u^{-c})}{1-(1+u^{-c})^{-k}} = \sum_{i=1}^{d}\ln(1+x_{i:n}^{-c})$$

$$\frac{d}{c} - (n-d)R(c,k) = \sum_{i=1}^{d}\ln x_{i:n} - (k+1)\sum_{i=1}^{d}\frac{x_{i:n}^{-c}\ln x_{i:n}}{1+x_{i:n}^{-c}},$$

respectively. The closed forms above nonlinear equations is very hard to obtained and hence, some numerical techniques are required to evaluate this estimates.
Dempster et al. (1977) introduced a general iterative approach commonly known as EM algorithm as an excellent tool for finding MLEs in cases where

observation are treated as incomplete data. Dealing with hybrid censored observation, the problem of finding MLEs of unknown parameters can be viewed as an incomplete data. Now suppose that $X = (X_{1:n}, X_{2:n}, \ldots, X_{d:n})$ and $Z = (Z_1, Z_2, \ldots, Z_{n-d})$ denote the observed and missing data, respectively. Here for a given d, $Z = (Z_1, Z_2, \ldots, Z_{n-d})$ are not observable. The censored data vector Z can be thought of as missing data and $W = (X, Z)$ represents the complete dataset.
The log-likelihood functions of the complete data are obtained as,

$$\ln L_c(c, k) = n \ln k + n \ln c - (c+1) \sum_{i=1}^{n} \ln x_i - (k+1) \sum_{i=1}^{n} \ln(1 + x_i^{-c}).$$

The E-step of the EM algorithm requires the computation of the conditional expectation $E[L_c(W; c, k)|X]$,which is equal to the pseudo log-likelihood function $L_s(c, k)$, defined as

$$E[\ln L_c(W; c, k)|X] = L_s(c, k)$$

$$\begin{aligned} L_s(c, k) = n \ln k + n \ln c - (c+1) \sum_{i=1}^{d} \ln x_{i:n} - (k+1) \sum_{i=1}^{d} \ln(1 + x_{i:n}^{-c}) \\ - (c+1) \sum_{i=1}^{n-d} E(\ln Z_i | Z_i > u) - (k+1) \sum_{i=1}^{n-d} E\{\ln(1 + Z_i^{-c}) | Z_i > u\}, \end{aligned}$$

where $E(\ln Z_i|Z_i > u) = A(u; c, k)$ and $E\{\ln(1 + Z_i^{-c})|Z_i > u\} = B(u; c, k)$ and they are obtained in Appendix A.
Now, the M-step includes the maximization of the pseudo log-likelihood function. Thus if $(c^{(s)}, k^{(s)})$ be the estimate of (c, k) at the s-th stage, the $(c^{(s+1)}, k^{(s+1)})$ can obtained by maximizing

$$\begin{aligned} \ell(c, k) = n \ln k + n \ln c - (c+1) \sum_{i=1}^{d} \ln x_{i:n} - (k+1) \sum_{i=1}^{d} \ln(1 + x_{i:n}^{-c}) \\ - (c+1)(n-d) A(u; c^{(s)}, k^{(s)}) - (k+1)(n-d) B(u; c^{(s)}, k^{(s)}). \end{aligned}$$

First, $c^{(s+1)}$ can be written by solving the fixed point type equation

$$h(c) = c$$

where

$$h(c) = \left[\frac{1}{n} \sum_{i=1}^{d} \ln x_{i:n} - \frac{\{\hat{k}(c)+1\}}{n} \sum_{i=1}^{d} \frac{x_{i:n}^{-c} \ln x_{i:n}}{(1+x_{i:n}^{-c})} + \frac{(n-d)}{n} A \right]^{-1}$$

$$\hat{k}(c) = \left\{ \frac{1}{n} \sum_{i=1}^{d} \ln(1+x_{i:n}^{-c}) + \frac{(n-d)}{n} B \right\}^{-1},$$

and $A = A(u, c^{(s)}, k^{(s)}), B = B(u, c^{(s)}, k^{(s)})$. Finally, after finding $c^{(s+1)}$, $k^{(s+1)}$ can be obtained as $k^{(s+1)} = \hat{k}(c^{(s+1)})$.

2.1 Fisher Information Matrix

In this section, we present the observed Fisher information matrix by using the missing information principles. The idea of the missing information principle of Louis (1982) can be expressed as follows:

Observed information=Complete information-Missing information.

The observed information matrix can then be inverted to obtain the asymptotic covariance matrix of the MLEs determined from the EM algorithm. Let $I_W(\theta)$, $I_X(\theta)$ and $I_{W|X}(\theta)$ denote the complete information matrix, the observed information matrix and the missing information matrix, respectively. The complete information matrix is given by

$$I_W(\theta) = -E\left\{ \frac{\partial^2 L_c(W;\theta)}{\partial \theta^2} \right\}$$

and the Fisher information matrix of the censored observations can be written as

$$I_{W|X}(\theta) = -(n-d) E\left\{ \frac{\partial^2 \ln f_{Z|X}(z|X;\theta)}{\partial \theta^2} \right\}.$$

Therefore, we obtain the observed information as

$$I_X(\theta) = I_W(\theta) - I_{W|X}(\theta)$$

and naturally, the asymptotic variance covariance matrix of $\theta = (c, k)$ can be obtained by inverting $I_X(\hat{\theta})$. Note that both the matrices $I_W(\theta)$ and $I_{W|X}(\theta)$ are of the order 2×2. Now, if the (i, j)-th, $i, j = 1, 2$, elements of

$I_W(\theta)$ is denoted by a_{ij}, they are as

$$a_{11} = \frac{n}{k^2},$$

$$a_{12} = a_{21} = -nkc \int_0^\infty x^{-2c-1}(1 + x^{-c})^{-k-2} \ln x \, dx$$

and

$$a_{22} = \frac{n}{c^2} + nkc(k+1) \int_0^\infty x^{-2c-1}(1 + x^{-c})^{-k-2} \ln^2 x \, dx.$$

Next for $I_{W|X}(\theta)$, we have

$$I_{W|X}(\theta) = (n-d) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

where

$$b_{11} = \frac{1}{k^2} - \frac{(1+u^{-c})^{-k} \ln^2(1+u^{-c})}{\{1-(1+u^{-c})^{-k}\}^2}$$

$$\begin{aligned} b_{12} = b_{21} = & \frac{(1+u^{-c})^{-(k+1)} u^{-c} \ln u \{k \ln(1+u^{-c}) - 1\}}{1-(1+u^{-c})^{-k}} \\ & + \frac{k(1+u^{-c})^{-(2k+1)} u^{-c} \ln u \ln(1+u^{-c})}{\{1-(1+u^{-c})^{-k}\}^2} \\ & + \frac{kcu^{-2c}}{1-(1+u^{-c})^{-k}} \int_0^1 w^{2c-1} \left\{1 + \left(\frac{u}{w}\right)^{-c}\right\}^{-(k+2)} \ln\left(\frac{u}{w}\right) dw \end{aligned}$$

$$\begin{aligned} b_{22} = & \frac{1}{c^2} + \frac{k(k+1)cu^{-2c}}{1-(1+u^{-c})^{-k}} \int_0^1 w^{2c-1} \left\{1 + \left(\frac{u}{w}\right)^{-c}\right\}^{-(k+3)} \ln^2\left(\frac{u}{w}\right) dw \\ & + \frac{k(k+1)(1+u^{-c})^{-(k-2)}(u^{-c}\ln u)^2}{1-(1+u^{-c})^{-k}} + \frac{\{k(1+u^{-c})^{-(k-1)} u^{-c} \ln u\}^2}{\{1-(1+u^{-c})^{-k}\}^2}. \end{aligned}$$

Since it is well known that MLEs are asymptotically Normal and consequently the pivotal quantities

$$\frac{\hat{c} - c}{\sqrt{\mathrm{var}(\hat{c})}}, \qquad \frac{\hat{k} - k}{\sqrt{\mathrm{var}(\hat{k})}}$$

are approximately distributed as standard Normal. The $(1-p)100\%$ approximate confidence intervals for the unknown parameters c and k are respectively given by $\hat{c} \pm z_{p/2}\sqrt{\text{var}(\hat{c})}$ and $\hat{k} \pm z_{p/2}\sqrt{\text{var}(\hat{k})}$ where $z_{p/2}$ is the $(p/2)$-th upper percentile of the standard Normal distribution.

3 Bayesian Estimates

The Bayesian approach allows both sample and prior information to be incorporated into analysis, which will improve the quality of the inferences. In this section Bayesian estimates of the shapes parameters are obtained in the case of Type-II hybrid censoring. Assume that c and k have the following independent gamma priors;

$$\pi_1(c) \propto c^{a-1}e^{-bc}, \quad c > 0$$

$$\pi_2(k) \propto k^{p-1}e^{-qk}, \quad k > 0$$

here a, b, p and q, are chosen to reflect the prior knowledge about the unknown parameters c and k. Based on the above prior assumptions, the posterior density function of c and k, given the data is

$$\Pi(c,k|\underline{X}) = \frac{L(c,k|\underline{X})\Pi(c,k)}{\int_0^\infty \int_0^\infty L(c,k|\underline{X})\Pi(c,k)dc\,dk}$$

where, $X = (x_{1:n}, x_{2:n}, \ldots, x_{d:n})$.
In this section, we obtain Bayesian estimates of the unknown parameters c and k against the squared error, linex and entropy loss functions. These loss functions are defined as, respectively,

$$L_s\{g(c,k), \hat{g}(c,k)\} = \{\hat{g}(c,k) - g(c,k)\}^2$$

$$L_{Ln}\{g(c,k), \hat{g}(c,k)\} = e^{h\,\{\hat{g}(c,k)-g(c,k)\}} - h\{\hat{g}(c,k) - g(c,k) - 1\}, \qquad h \neq 0$$

$$L_E(g(c,k), \hat{g}(c,k)) \propto \left\{\frac{\hat{g}(c,k)}{g(c,k)}\right\} - w\log\left\{\frac{\hat{g}(c,k)}{g(c,k)}\right\} - 1, \qquad w \neq 0$$

here $\hat{g}(c,k)$ denotes an estimate of some parametric function $g(c,k)$. Therefore, if $g(c,k)$ is any function of c and k, then the Bayesian estimate of $g(c,k)$ under square error loss function is evaluated as

$$\hat{g}_S(c,k) = E\{g(c,k)|\underline{X}\},$$

where

$$E\{g(c,k)|\underline{X}\} = \frac{1}{z}\int_0^\infty \int_0^\infty g(c,k) k^{d+p-1} c^{d+a-1} e^{-bc} e^{-kq}$$

$$\prod_{i=1}^{d} \left\{ x_{i:n}^{-(c+1)} (1+x_{i:n}^{-c})^{-(k+1)} \right\} \left\{ 1-(1+u^{-c})^{-k} \right\}^{n-d} dc\, dk.$$

Similarly, for the linex loss function we have

$$\hat{g}_{Ln}(c,k) = -\frac{1}{h} \ln E\{e^{-h\, g(c,k)}|\underline{X}\}, \qquad h \neq 0$$

where

$$E\{e^{-h\, g(c,k)}|\underline{X}\} = \frac{1}{z}\int_0^\infty \int_0^\infty e^{-h\, g(c,k)} k^{d+p-1} c^{d+a-1} e^{-bc} e^{-kq}$$

$$\prod_{i=1}^{d} \left\{ x_{i:n}^{-(c+1)} (1+x_{i:n}^{-c})^{-(k+1)} \right\} \left\{ 1-(1+u^{-c})^{-k} \right\}^{n-d} dc\, dk.$$

Proceeding in a similar manner, Bayesian estimate of $g(c,k)$ against the entropy loss function is derived as

$$\hat{g}_E(c,k) = E\{g(c,k)^{-w}|\underline{X}\}^{\frac{-1}{w}}, \qquad w \neq 0$$

where

$$E\{g(c,k)^{-w}|\underline{X}\} = \frac{1}{z}\int_0^\infty \int_0^\infty g(c,k)^{-w} k^{d+p-1} c^{d+a-1} e^{-bc} e^{-kq}$$

$$\prod_{i=1}^{d} \left\{ x_{i:n}^{-(c+1)} (1+x_{i:n}^{-c})^{-(k+1)} \right\} \left\{ 1-(1+u^{-c})^{-k} \right\}^{n-d} dc\, dk.$$

Where Z is defined as follow

$$z = \int_0^\infty \int_0^\infty k^{d+p-1} c^{d+a-1} e^{-b\,c} e^{-k\,q}$$

$$\prod_{i=1}^{d} \left\{ x_{i:n}^{-(c+1)} (1+x_{i:n}^{-c})^{-(k+1)} \right\} \left\{ 1-(1+u^{-c})^{-k} \right\}^{n-d} dc\, dk.$$

As these estimators cannot be computed explicitly, so we adopt two different procedures to approximate them, (a) Lindley's approximation, and (b) MCMC method.

3.1 Lindley's Approximation

In the previous section, we observed that the different Bayesian estimate have not explicit closed forms. For these evaluations, numerical techniques are required. One of the most numerical techniques is Lindley's method (Lindley, 1980), which approaches the ratio of the integrals as a whole and produces a single numerical result. If n is sufficiently large, according to Lindley (1980), any ratio of the integrals of the form

$$I(\underline{X}) = \frac{\int_0^\infty \int_0^\infty g(c,k) e^{\ell(c,k|\underline{x})+\rho(c,k)} dc\, dk}{\int_0^\infty \int_0^\infty e^{\ell(c,k|\underline{x})+\rho(c,k)} dc\, dk}$$

where $g(c,k)$ is function of c and k only, $\ell(c,k|\underline{x})$ is the log-likelihood and $\rho(c,k) = \log \pi(c,k)$, can be approximated as

$$\begin{aligned} I(\underline{X}) = g(\hat{c},\hat{k}) + \frac{1}{2}\big\{ & \big(\hat{g}_{kk} + 2\hat{g}_k \hat{\rho}_k\big)\hat{v}_{kk} \\ & + \big(\hat{g}_{ck} + 2\hat{g}_c\hat{\rho}_k\big)\hat{v}_{ck} + \big(\hat{g}_{kc} + 2\hat{g}_k\hat{\rho}_c\big)\hat{v}_{kc} + \big(\hat{g}_{cc} + 2\hat{g}_c\hat{\rho}_c\big)\hat{v}_{cc}\big\} \\ & + \frac{1}{2}\big\{\big(\hat{g}_k\hat{v}_{kk} + \hat{g}_c\hat{v}_{kc}\big)\big(\hat{\ell}_{kkk}\hat{v}_{kk} + \hat{\ell}_{kck}\hat{v}_{kc} + \hat{\ell}_{ckk}\hat{v}_{ck} + \hat{\ell}_{cck}\hat{v}_{cc}\big) \\ & + \big(\hat{g}_k\hat{v}_{ck} + \hat{g}_c\hat{v}_{cc}\big)\big(\hat{\ell}_{ckk}\hat{v}_{kk} + \hat{\ell}_{kcc}\hat{v}_{kc} + \hat{\ell}_{ckc}\hat{v}_{ck} + \hat{\ell}_{ccc}\hat{v}_{cc}\big)\big\} \end{aligned}$$

here, $\hat{c}$ and $\hat{k}$ are the MLEs of c and k respectively. Also, $\hat{g}_{cc}$ is the second derivative of the function $g(c,k)$ with respect to c at $(\hat{c},\hat{k})$. Also $v_{ij} = (i,j) - th$ elements of the inverse of the matrix $\big\{-\frac{\partial^2 \ell(c,k|\underline{X})}{\partial c \partial k}\big\}^{-1}$ evaluated as $(\hat{c},\hat{k})$ and

$$\hat{\rho}_c = \frac{q-1}{\hat{c}} - b$$

$$\hat{\rho}_k = \frac{p-1}{\hat{k}} - q.$$

For the detailed derivations, see Appendix B.

Now, based on the above defined expressions, we calculate the approximation Bayesian estimates under different loss functions. For the squared error loss function we get that

$$g(c,k) = c, \quad g_c = 1, \quad g_{cc} = g_k = g_{kk} = g_{ck} = g_{kc} = 0,$$

and the corresponding Bayesian estimate of c is

$$\hat{c}_S = E\big(c|\underline{X}\big) = \hat{c} + \big(\hat{u}_c\hat{\rho}_k\hat{v}_{ck} + \hat{u}_c\hat{\rho}_c\hat{v}_{cc}\big)$$
$$+ 0.5\big\{\hat{u}_c\hat{v}_{kc}(\hat{\ell}_{kkk}\hat{v}_{kk} + \hat{\ell}_{kck}\hat{v}_{kc} + \hat{\ell}_{ckk}\hat{v}_{ck} + \hat{\ell}_{cck}\hat{v}_{cc})\big\}$$
$$+ 0.5\big\{\hat{u}_c\hat{v}_{cc}(\hat{\ell}_{ckk}\hat{v}_{kk} + \hat{\ell}_{kcc}\hat{v}_{kc} + \hat{\ell}_{ckc}\hat{v}_{ck} + \hat{\ell}_{ccc}\hat{v}_{cc})\big\}.$$

Proceeding similarly, the Bayesian estimate of k under square error loss function can be obtained as (here $g(c,k) = k$, $g_k = 1$, $g_{kk} = g_c = g_{cc} = g_{ck} = g_{kc} = 0$)

$$\hat{k}_S = E\big(k|\underline{X}\big) = \hat{k} + \big(\hat{u}_k\hat{\rho}_k\hat{v}_{kk} + \hat{u}_k\hat{\rho}_c\hat{v}_{ck}\big)$$
$$+ 0.5\big\{\hat{u}_k\hat{v}_{kk}(\hat{\ell}_{kkk}\hat{v}_{kk} + \hat{\ell}_{kck}\hat{v}_{kc} + \hat{\ell}_{ckk}\hat{v}_{ck} + \hat{\ell}_{cck}\hat{v}_{cc})\big\}$$
$$+ 0.5\big\{\hat{u}_k\hat{v}_{ck}(\hat{\ell}_{ckk}\hat{v}_{kk} + \hat{\ell}_{kcc}\hat{v}_{kc} + \hat{\ell}_{ckc}\hat{v}_{ck} + \hat{\ell}_{ccc}\hat{v}_{cc})\big\}.$$

Also, the Bayesian estimate of c under linex loss function is obtained as $g(c,k) = e^{-h\,c}$, $g_c = -he^{-h\,c}$, $g_{cc} = h^2e^{-h\,c}$, $g_k = g_{kk} = g_{ck} = g_{kc} = 0$,

$$\hat{c}_{Ln} = -\frac{1}{h}\ln\big\{E\big(e^{-h\,c}|\underline{X}\big)\big\}, \qquad h \neq 0$$

where

$$E\big(e^{-h\,c}|\underline{X}\big) = e^{-h\hat{c}} + 0.5\big\{2\hat{u}_c\hat{\rho}_k\hat{v}_{ck} + (\hat{u}_{cc} + 2\hat{u}_c\hat{\rho}_c)\hat{v}_{cc}\big\}$$
$$+ 0.5\big\{\hat{u}_c\hat{v}_{kc}(\hat{\ell}_{kkk}\hat{v}_{kk} + \hat{\ell}_{kck}\hat{v}_{kc} + \hat{\ell}_{ckk}\hat{v}_{ck} + \hat{\ell}_{cck}\hat{v}_{cc})\big\}$$
$$+ 0.5\big\{\hat{u}_c\hat{v}_{cc}(\hat{\ell}_{ckk}\hat{v}_{kk} + \hat{\ell}_{kcc}\hat{v}_{kc} + \hat{\ell}_{ckc}\hat{v}_{ck} + \hat{\ell}_{ccc}\hat{v}_{cc})\big\}.$$

Similarly, for k, we have $g(c,k) = e^{-h\,k}$, $g_k = -he^{-h\,k}$, $g_{kk} = h^2e^{-h\,k}$, $g_c = g_{cc} = g_{kc} = g_{ck} = 0$ and

$$\hat{k}_{Ln} = -\frac{1}{h}\ln\big\{E\big(e^{-h\,k}|\underline{X}\big)\big\}, \qquad h \neq 0$$

where

$$E\big(e^{-h\,k}|\underline{X}\big) = e^{-h\hat{k}} + 0.5\big\{2\hat{u}_k\hat{\rho}_c\hat{v}_{ck} + (\hat{u}_{kk} + 2\hat{u}_k\hat{\rho}_k)\hat{v}_{kk}\big\}$$
$$+ 0.5\big\{\hat{u}_k\hat{v}_{kk}(\hat{\ell}_{kkk}\hat{v}_{kk} + \hat{\ell}_{kck}\hat{v}_{kc} + \hat{\ell}_{ckk}\hat{v}_{ck} + \hat{\ell}_{cck}\hat{v}_{cc})\big\}$$
$$+ 0.5\big\{\hat{u}_k\hat{v}_{ck}(\hat{\ell}_{ckk}\hat{v}_{kk} + \hat{\ell}_{kcc}\hat{v}_{kc} + \hat{\ell}_{ckc}\hat{v}_{ck} + \hat{\ell}_{ccc}\hat{v}_{cc})\big\}.$$

Finally, we obtain the Bayesian estimate under the entropy loss function. For the parameter c, we observe that $g(c,k) = c^{-w}$, $g_c = -wc^{-(w+1)}$, $g_{cc} = w\,(w+1)\,c^{-(w+2)}$, $g_k = g_{kk} = g_{ck} = g_{kc} = 0$,

$$
\begin{aligned}
E\big(c^{-w}|\underline{X}\big) = \hat{c}^{-w} &+ 0.5\big\{2\hat{u}_c\hat{\rho}_k\hat{v}_{ck} + (\hat{u}_{cc} + 2\hat{u}_c\hat{\rho}_c)\hat{v}_{cc}\big\} \\
&+ 0.5\big\{\hat{u}_c\hat{v}_{kc}(\hat{\ell}_{kkk}\hat{v}_{kk} + \hat{\ell}_{kck}\hat{v}_{kc} + \hat{\ell}_{ckk}\hat{v}_{ck} + \hat{\ell}_{cck}\hat{v}_{cc})\big\} \\
&+ 0.5\big\{\hat{u}_c\hat{v}_{cc}(\hat{\ell}_{ckk}\hat{v}_{kk} + \hat{\ell}_{kcc}\hat{v}_{kc} + \hat{\ell}_{ckc}\hat{v}_{ck} + \hat{\ell}_{ccc}\hat{v}_{cc})\big\}
\end{aligned}
$$

where

$$
\hat{c}_E = E\left(c^{-w}|\underline{X}\right)^{-\frac{1}{w}}, \qquad w \neq 0.
$$

Also, for the parameter k, $g(c,k) = k^{-w}$, $g_k = -w\,k^{-(w+1)}$, $g_{kk} = w(w+1)k^{-(w+2)}$, $g_c = g_{cc} = g_{ck} = g_{kc} = 0$,

$$
\begin{aligned}
E\big(k^{-w}|\underline{X}\big) = \hat{k}^{-w} &+ 0.5\big\{2\hat{u}_k\hat{\rho}_c\hat{v}_{ck} + (\hat{u}_{kk} + 2\hat{u}_k\hat{\rho}_k)\hat{v}_{kk}\big\} \\
&+ 0.5\big\{\hat{u}_k\hat{v}_{kk}(\hat{\ell}_{kkk}\hat{v}_{kk} + \hat{\ell}_{kck}\hat{v}_{kc} + \hat{\ell}_{ckk}\hat{v}_{ck} + \hat{\ell}_{cck}\hat{v}_{cc})\big\} \\
&+ 0.5\big\{\hat{u}_k\hat{v}_{ck}(\hat{\ell}_{ckk}\hat{v}_{kk} + \hat{\ell}_{kcc}\hat{v}_{kc} + \hat{\ell}_{ckc}\hat{v}_{ck} + \hat{\ell}_{ccc}\hat{v}_{cc})\big\}
\end{aligned}
$$

consequently, the approximate Bayesian estimate of k is given by

$$
\hat{k}_E = \left\{E\left(k^{-w}|\underline{X}\right)\right\}^{-\frac{1}{w}}, \qquad w \neq 0.
$$

3.2 Importance Sampling

In this section we would like to provide the importance sampling scheme to draw samples from the posterior density function and then compute the Bayesian estimates. Therefore, under the stated prior distribution of c and k, the corresponding posterior distribution can be rewritten as

$$
\begin{aligned}
\Pi(c,k|\underline{X}) \propto\ & k^{d+p-1}c^{d+a-1}e^{-bc}e^{-kq} \\
& \prod_{i=1}^{d}\left\{x_{i:n}^{-(c+1)}(1+x_{i:n}^{-c})^{-(k+1)}\right\}\left\{1-(1+u^{-c})^{-k}\right\}^{n-d}
\end{aligned}
$$

on the other word

$$\Pi(c,k|\underline{X}) \propto \Gamma_c\left(d+a, b+\sum_{i=1}^{d}\ln x_{i:n}\right) \times \Gamma_{k|c}\left\{d+p, q+\sum_{i=1}^{d}\ln(1+x_{i:n}^{-c})\right\} \times s(c,k)$$

where

$$s(c,k) = \frac{\left\{1-(1+u^{-c})^{-k}\right\}^{(n-d)} e^{\{-\sum_{i=1}^{d}\ln(1+x_{i:n}^{-c})\}}}{\left\{q+\sum_{i=1}^{d}\ln(1+x_{i:n}^{-c})\right\}^{(d+p)}}.$$

Now, using Kundu and Pradhan (2009), we propose the following algorithm to compute the Bayesian estimates of $g(c,k)$.

Step 1: Generate $c_1 \sim \Gamma_c(d+a, b+\sum_{i=1}^{d}\ln x_{i:n})$, $k_1 \sim \Gamma_{k|c}\{d+p, q+\sum_{i=1}^{d}\ln(1+x_{i:n}^{-c})\}$.

Step 2: Repeat Step 1, M times to obtain $(c_1,k_1),(c_2,k_2),\ldots,(c_M,k_M)$.

Step 3: Now, Bayesian estimates of $g(c,k)$ under the square error, linex and entropy loss functions can be obtained as,

$$\hat{g}_S(c,k) = \frac{\sum_{i=1}^{M} g(c_i,k_i)s(c_i,k_i)}{\sum_{i=1}^{M} s(c_i,k_i)}$$

$$\hat{g}_{Ln}(c,k) = -\frac{1}{h}\ln\left\{\frac{\sum_{i=1}^{M} e^{-h\,g(c_i,k_i)}s(c_i,k_i)}{\sum_{i=1}^{M} s(c_i,k_i)}\right\}$$

$$\hat{g}_E(c,k) = \left\{\frac{\sum_{i=1}^{M} g(c_i,k_i)^{-w}s(c_i,k_i)}{\sum_{i=1}^{M} s(c_i,k_i)}\right\}^{\frac{-1}{w}},$$

respectively.

4 Simulation Results

Since the performance of the different methods cannot be compared theoretically, Monte Carlo simulations to compare the performances of the different estimators of the unknown parameters of a Burr type III distribution are performed. We mainly compare the performances of the EM and Bayesian estimates of the unknown parameters for different choices of n, R and T, in terms of their average biases and mean squared errors (MSEs) for different censoring schemes. We also compare the average lengths of the asymptotic confidence intervals. We have taken in all cases, $c = 1.5$ and $k = 1$ and Bayesian estimates are obtained under (prior 1: $a = p = 1$, $b = 18$, $q = 1$), $w = 0.5$ and $h = 0.5$. We replicate the process 10000 times and report the average estimates, average biases, the MSEs and length of the asymptotic confidence intervals. All the computations are performed by using R software. The average estimates, the average biases and the mean squared errors of c and k, denoted by Es, B and MSE, respectively. The results based on MLEs (EM), different Bayesian estimates under Lindley's approximation against square error (BLS), linex (BLL) and entropy (BLE) loss functions and Bayesian estimates based on MCMC method under square error (BMS), linex (BML) and entropy (BME) loss functions are reported in Tables 1 and 2. The average length of asymptotic confidence intervals of c and k are reported in Tables 4 and 5. Furthermore, we want to observe the effect of the hyper parameters on the Bayesian estimates. So, we have considered other informative prior (prior 2: $a = 4$, $b = 25$, $p = 3$, $q = 4$). The results for this prior are reported in Table 3. Note that we report the result only for $n = 40$, but for $n = 60$ can be obtained similarly.

From the Tables 1-5 the following general observations for three methods and two parameters can be made:
(i) for fixed n and R when T increases from 1 to 3, the average biases, the MSEs and the length of asymptotic confidence intervals decrease, (ii) for fixed R and T as n increases from 40 to 60 the average biases, the MSEs and the length of asymptotic confidence intervals, decrease, (iii) for fixed n and T as R increases, the average biases, the MSEs and the length of asymptotic confidence intervals, decrease.

It is observe that from the tabulated estimates and mean square error values that the performance of all Bayesian estimates of c and k are satisfactory compared to the respective maximum likelihood estimates. This holds true for almost all tabulated choices of n, R and T. The Bayesian estimates under entropy loss function based on MCMC method are better choice among all its rivals and for all values n, R and T. Further, the Bayesin estimates based on prior 2 perform better than the Bayesian estimates based on prior 1, in terms of MSEs and the average biases.

Table 1. The EM and Baseyan average estimates, the mean squared errors and the average biases of c and k, for $a = 1,\ b = 18,\ p = q = 1$ and $n = 40$.

			$R = 20$			$R = 30$		
			Es	MSE	B	Es	MSE	B
EM	$T = 1$	c	1.681141,	0.032812,	0.152006	1.608066,	0.018640,	0.092946
		k	0.805478,	0.172731,	0.092677	1.013913,	0.001951,	0.053349
	$T = 3$	c	1.355317,	0.020938,	0.126347	1.573292,	0.005371,	0.049366
		k	1.168241,	0.028305,	0.164014	1.053081,	0.002817,	0.017552
BLS	$T = 1$	c	1.681138,	0.032804,	0.152003	1.608061,	0.018633,	0.092941
		k	0.805497,	0.172728,	0.092673	1.013910,	0.001946,	0.053344
	$T = 3$	c	1.355323,	0.020934,	0.126343	1.573288,	0.005368,	0.049361
		k	1.168233,	0.028302,	0.164012	1.053062,	0.002810,	0.017541
BLL	$T = 1$	c	1.681139,	0.032805,	0.152004	1.608063,	0.018637,	0.092941
		k	0.804598,	0.172729,	0.092674	1.013911,	0.001948,	0.053346
	$T = 3$	c	1.355322,	0.020937,	0.126346	1.573287,	0.005370,	0.049364
		k	1.168234,	0.028304,	0.164014	1.053067,	0.002806,	0.017546
BLE	$T = 1$	c	1.681136,	0.032804,	0.152002	1.608061,	0.018631,	0.092941
		k	0.805498,	0.172729,	0.292674	1.013910,	0.001943,	0.053343
	$T = 3$	c	1.355324,	0.020934,	0.126346	1.573285,	0.005364,	0.049356
		k	1.168234,	0.028301,	0.164011	1.053060,	0.002807,	0.017540
BMS	$T = 1$	c	1.305156,	0.031320,	0.106000	1.388051,	0.023663,	0.056972
		k	0.842447,	0.127642,	0.082536	1.007921,	0.000876,	0.035636
	$T = 3$	c	1.475351,	0.000984,	0.086352	1.537684,	0.000361,	0.059461
		k	1.095437,	0.007926,	0.097601	1.077579,	0.006835,	0.007891
BML	$T = 1$	c	1.305143,	0.031322,	0.106001	1.388048,	0.023665,	0.056976
		k	0.842438,	0.127651,	0.082547	1.007933,	0.008772,	0.035647
	$T = 3$	c	1.475352,	0.000983,	0.086357	1.537690,	0.000365,	0.059468
		k	1.095441,	0.007934,	0.097604	1.077582,	0.006837,	0.007892
BME	$T = 1$	c	1.305157,	0.031319,	0.106000	1.388052,	0.023661,	0.056970
		k	0.842452,	0.127603,	0.082531	1.007914,	0.008755,	0.035634
	$T = 3$	c	1.475357,	0.000982,	0.086351	1.537683,	0.000355,	0.059456
		k	1.095436,	0.007925,	0.097593	1.077578,	0.006833,	0.007888

Table 2. The EM and Baseyan average estimates, the mean squared errors and the average biases of c and k, for $a = 1,\ b = 18,\ p = q = 1$ and $n = 60$.

			$R = 30$			$R = 40$		
			Es	MSE	B	Es	MSE	B
EM	$T = 1$	c	1.390912,	0.015425,	0.090513	1.424567,	0.006590,	0.055406
		k	0.862237,	0.018978,	0.123166	0.915284,	0.007176,	0.073879
	$T = 3$	c	1.463211,	0.005131,	0.040932	1.482829,	0.004502,	0.037250
		k	0.964260,	0.002203,	0.015291	0.969311,	0.003196,	0.034509
BLS	$T = 1$	c	1.390934,	0.015421,	0.090510	1.424572,	0.006588,	0.055403
		k	0.862242,	0.018975,	0.123162	0.915287,	0.007171,	0.073875
	$T = 3$	c	1.463216,	0.005127,	0.040926	1.482832,	0.004500,	0.037247
		k	0.964261,	0.002201,	0.015289	0.969313,	0.003194,	0.034508
BLL	$T = 1$	c	1.390921,	0.015424,	0.090512	1.424571,	0.006589,	0.055404
		k	0.862239,	0.018977,	0.123164	0.915286,	0.007173,	0.073877
	$T = 3$	c	1.463212,	0.005128,	0.040927	1.482830,	0.004501,	0.037248
		k	0.964260,	0.002202,	0.015290	0.969313,	0.003193,	0.034507
BLE	$T = 1$	c	1.390937,	0.015419,	0.090509	1.424574,	0.006586,	0.055401
		k	0.862245,	0.018970,	0.123161	0.915291,	0.007172,	0.073875
	$T = 3$	c	1.460220,	0.005122,	0.040925	1.482833,	0.004501,	0.037247
		k	0.964263,	0.002498,	0.015284	0.949313,	0.003193,	0.034507
BMS	$T = 1$	c	1.420029,	0.006597,	0.085330	1.464729,	0.001244,	0.092778
		k	0.986949,	0.005689,	0.018339	0.858455,	0.020035,	0.127382
	$T = 3$	c	1.491276,	0.000448,	0.018263	1.499505,	0.000401,	0.051185
		k	0.956320,	0.004708,	0.052689	0.964708,	0.004263,	0.051545
BML	$T = 1$	c	1.420022,	0.006599,	0.085332	1.464726,	0.001246,	0.092783
		k	0.986943,	0.005682,	0.018331	0.858450,	0.020031,	0.127388
	$T = 3$	c	1.491268,	0.000453,	0.018266	1.499503,	0.000406,	0.051186
		k	0.956317,	0.004709,	0.052691	0.964701,	0.004257,	0.051549
BME	$T = 1$	c	1.420036,	0.006594,	0.085327	1.464734,	0.001241,	0.092774
		k	0.986952,	0.005692,	0.018329	0.858463,	0.020031,	0.127376
	$T = 3$	c	1.491283,	0.000441,	0.018257	1.499512,	0.000400,	0.051181
		k	0.956327,	0.004704,	0.052683	0.964703,	0.004257,	0.051540

Table 3. The Baseyan average estimates, the mean squared errors and the average biases of c and k, for $a = 4,\ b = 25,\ p = 3,\ q = 4$ and $n = 40$.

			$R = 20$			$R = 30$		
			Es	MSE	B	Es	MSE	B
BLS	$T = 1$	c	1.338034,	0.026233,	0.156945	1.365562,	0.018090,	0.128252
		k	0.868453,	0.015781,	0.031119	0.898162,	0.010370,	0.088473
	$T = 3$	c	1.363914,	0.018519,	0.115533	1.448396,	0.002663,	0.046366
		k	0.911498,	0.007832,	0.081760	0.890461,	0.011998,	0.097919
BLL	$T = 1$	c	1.338031,	0.026239,	0.156952	1.365561,	0.018093,	0.128251
		k	0.868451,	0.015782,	0.031121	0.898162,	0.010370,	0.088474
	$T = 3$	c	1.363911,	0.018515,	0.115532	1.448393,	0.002667,	0.046368
		k	0.911492,	0.007842,	0.081769	0.890461,	0.012006,	0.097926
BLE	$T = 1$	c	1.338041,	0.026226,	0.156941	1.365570,	0.018081,	0.128243
		k	0.868458,	0.015774,	0.031111	0.898166,	0.010361,	0.088473
	$T = 3$	c	1.363952,	0.018510,	0.115521	1.448412,	0.002652,	0.046360
		k	0.911499,	0.007831,	0.081757	0.890471,	0.011991,	0.097912
BMS	$T = 1$	c	1.344493,	0.024045,	0.150488	1.431906,	0.008550,	0.082406
		k	0.902989,	0.049410,	0.085116	0.912943,	0.007578,	0.071281
	$T = 3$	c	1.384414,	0.022409,	0.108378	1.378439,	0.020512,	0.046243
		k	0.909580,	0.018175,	0.084456	0.915418,	0.007154,	0.075136
BML	$T = 1$	c	1.334439,	0.027410,	0.150062	1.351528,	0.022044,	0.130305
		k	0.929698,	0.015781,	0.063123	0.930893,	0.004775,	0.048252
	$T = 3$	c	1.350691,	0.022293,	0.143705	1.364234,	0.018432,	0.130296
		k	0.922804,	0.005959,	0.060539	0.933939,	0.004364,	0.055995
BME	$T = 1$	c	1.351707,	0.021991,	0.131067	1.365593,	0.018067,	0.128154
		k	0.931881,	0.004640,	0.047219	0.942311,	0.003328,	0.046693
	$T = 3$	c	1.365249,	0.018157,	0.113957	1.379837,	0.014439,	0.113304
		k	0.928922,	0.003730,	0.034455	0.978368,	0.000798,	0.053175

Table 4. The average estimates of asymptotic confidence intervals of c and k, for $n = 40$.

			$R = 20$	$R = 30$
$n = 40$	$T = 1$	c	[1.366992, 1.995289]	[1.305119, 1.911012]
		k	[0.297636, 1.103113]	[0.720045, 1.307781]
	$T = 3$	c	[1.040740, 1.669893]	[1.288341, 1.883174]
		k	[0.887526, 1.448941]	[0.768438, 1.337685]

Table 5. The average estimates of asymptotic confidence intervals of c and k, for $n = 60$.

			$R = 30$	$R = 40$
$n = 60$	$T = 1$	c	[0.988537, 1.593173]	[1.114872, 1.734272]
		k	[0.596534, 1.127940]	[0.630652, 1.199906]
	$T = 3$	c	[1.154201, 1.772220]	[1.178845, 1.790669]
		k	[0.676023, 1.252496]	[0.682785, 1.255836]

5 Real Data Analysis

This section, involves analysis of the failure data from a sample of 23 ball bearings which have been used for illustration purpose in many articles (see Lawless, 1982, and Kundu and Pradhan, 2009). The data are given as below

$$17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80,$$

$$68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.$$

We consider the following two sampling schemes:
Scheme 1 is $R = 18$ and $T = 100$
and Scheme 2 is $R = 16$ and $T = 70$.
In both the cases we have estimated the unknown parameters using the MLEs and the Bayesian estimates. Also the 95%, approximate confidence intervals are obtained. For computing the Bayesian estimates we have mainly considered the squared error loss function in both the cases. Before progressing further, we have first fitted the Burr III distribution to the complete data and it is observed that $\hat{c} = 1.8343$ and $\hat{k} = 11.4186 \times 10^2$. The Kolmogorov-Smirnov statistics and the associated p-values of MLE are 0.1328 and 0.812, respectively. Therefore, the high p-value clearly indicates that the Burr III distribution can be used to analyze this dataset. Also, the fitted probability distribution function (pdf) and the relative histogram of this data is presented in Figure 1. For Scheme 1, the MLEs, Bayesian estimates under Lindley's approximation and Bayesian estimates under MCMC method of the unknown parameters (c, k) are $(1.6755, 7.0889 \times 10^2)$, $(1.4302, 10.4450 \times 10^2)$ and $(1.4112, 10.4041 \times 10^2)$ respectively. Also, the approximate confidence intervals of c and k are $(1.35785, 1.99316)$, $(0.00, 15.04504 \times 10^2)$ respectively. Similarly, for Scheme 2, the MLEs, Bayesian estimates under Lindley's approximation and Bayesian estimates under MCMC method of the unknown parameters (c, k) are $(1.6433, 6.3331 \times 10^2)$, $(1.4563, 8.4370 \times 10^2)$ and

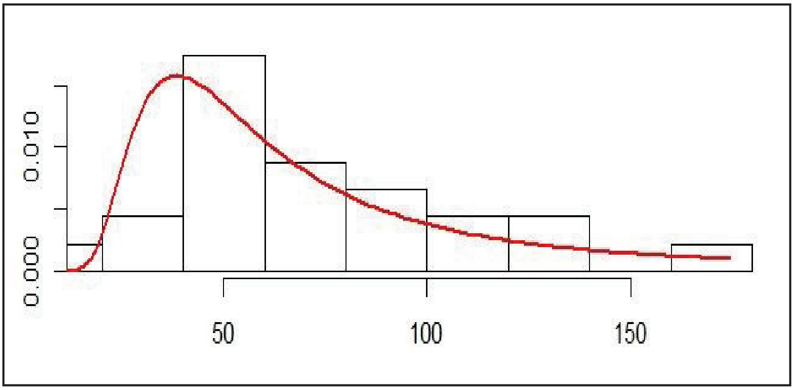

Figure 1. The fitted pdf and the relative histogram of data.

$(1.4197, 10.4464 \times 10^2)$, respectively and the approximate confidence intervals of c and k are, $(1.3821, 1.9045)$ and $(82.6483, 11.8398 \times 10^2)$ respectively.

6 Conclusions

In this paper, the statistical inference based on Type-II hybrid censored data from a Burr type III distribution performed. It is observed that the MLE method cannot be obtained in closed form. The EM estimators of the considered parameters obtained. Observed Fisher information matrix is obtained by applying missing value principle and is further utilized to constructing the approximate confidence intervals. Also, Bayesian estimates of the unknown parameters are obtained using different approximation under square error, linex and entropy loss functions. A simulation study has been constructed to examine the performance of the different schemes for the Burr type III distribution. Finally one dataset analyzed for illustrative purposes.

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Appendix A

Theorem 1. *Given* $\boldsymbol{X} = (X_{1:n} = x_{1:n}, X_{2:n} = x_{2:n}, \ldots, X_{d:n} = x_{d:n})$ *the conditional probability distribution function of* Z_i *for* $i = 1, 2, \ldots, n-d$ *is*

$$f_{\boldsymbol{Z}|\boldsymbol{X}}(z_i|X_{1:n} = x_{1:n}, X_{2:n} = x_{2:n}, \ldots, X_{d:n} = x_{d:n})\frac{f_{BIII}(z_i)}{1 - F_{BIII}(u)}; \qquad z_i > u$$

where, $F_{BIII}(x)$ and $f_{BIII}(x)$ are defined in (1) and (2) respectively and Z_i and Z_j for $i \neq j$ are conditionally independent. Note that, for case I, $d = R$ and $u = x_{R:n}$ and for case II, $d > R$ and $u = T$.

Proof. The joint probability of the complete sample is

$$\prod_{j=1}^{d} f_{BIII}(x_{j:n}) \prod_{i=1}^{n-d} f_{BIII}(z_i)$$

and the joint probability of the observed

$$\prod_{j=1}^{d} f_{BIII}(x_{j:n})\{1 - F_{BIII}(u)\}^{(n-d)}.$$

Then the conditional probability Z_i given $X_{1:n} = x_{1:n}, X_{2:n} = x_{2:n}, \ldots, X_{d:n} = x_{d:n}$ is obtained by

$$f_{\mathbf{Z}|\mathbf{X}}(z|x) = \frac{\prod_{j=1}^{d} f_{BIII}(x_{j:n}) \prod_{i=1}^{n-d} f_{BIII}(z_i)}{\prod_{j=1}^{d} f_{BIII}(x_{j:n})(1 - F_{BIII}(u))^{(n-d)}} = \prod_{i=1}^{n-d} \frac{f_{BIII}(z_i)}{1 - F_{BIII}(u)}.$$

So, based on the factorization theorem; Z_i are conditionally independent and follow the left truncated distribution at u. □

Now, using Theorem 1, we can write

$$\begin{aligned} E(\log Z_i | Z_i > u) &= \frac{kc}{1 - F(u; c, k)} \int_u^{\infty} \log(x) x^{-(c+1)} (1 + x^{-c})^{-(k+1)} dx \\ &= A(u; c, k), say, \end{aligned}$$

and

$$\begin{aligned} E\{\log(1 + Z_i^{-c}) | Z_i > u\} &= \frac{kc}{1 - F(u; c, k)} \\ &\quad \times \int_u^{\infty} \log(1 + x^{-c}) x^{-(c+1)} (1 + x^{-c})^{-(k+1)} dx \\ &= B(u; c, k), say. \end{aligned}$$

Appendix B

For the two parameters Burr type III Distribution under Type-II hybrid censored data, we have

$$\hat{\ell}_{kk} = \frac{\partial^2 \ell}{\partial k^2}|_{k=\hat{k},c=\hat{c}} = -\frac{d}{\hat{k}^2} - (n-d)\left[\frac{\ln^2(1+u^{-\hat{c}})(1+u^{-\hat{c}})^{-\hat{k}}}{\left\{1-(1+u^{-\hat{c}})^{-\hat{k}}\right\}^2}\right]$$

$$\hat{\ell}_{kc} = \frac{\partial^2 \ell}{\partial k \partial c}|_{k=\hat{k},c=\hat{c}} = \sum_{i=1}^{d} \frac{x_{i:n}^{-\hat{c}} \ln x_{i:n}}{1+x_{i:n}^{-\hat{c}}}$$
$$+ (n-d)\left[\frac{(1+u^{-\hat{c}})^{-(\hat{k}+1)} u^{-\hat{c}} \ln u \left\{1-(1+u^{-\hat{c}})^{-\hat{k}} - \hat{k}\ln(1+u^{-\hat{c}})\right\}}{\left\{1-(1+u^{-\hat{c}})^{-\hat{k}}\right\}^2}\right]$$

$$\hat{\ell}_{cc} = \frac{\partial^2 \ell}{\partial c^2}|_{k=\hat{k},c=\hat{c}} = -\frac{d}{\hat{c}^2} + (\hat{k}+1)\sum_{i=1}^{d} \frac{x_{i:n}^{-\hat{c}} \ln^2 x_{i:n}}{(1+x_{i:n}^{-\hat{c}})^2}$$
$$+(n-d)\left[\frac{\hat{k}(\hat{k}+1)(1+u^{-\hat{c}})^{-(\hat{k}+2)}(u^{-\hat{c}}\ln u)^2 + \left\{\hat{k}(1+u^{-\hat{c}})^{-(\hat{k}+2)} u^{-\hat{c}} \ln u\right\}^2}{\left\{1-(1+u^{-\hat{c}})^{-\hat{k}}\right\}^2}\right]$$

$$\hat{\ell}_{kkk} = \frac{\partial^3 \ell}{\partial k^3}|_{k=\hat{k},c=\hat{c}} = -\frac{2d}{\hat{k}^3} - (n-d)\left[\frac{\ln^3(1+u^{-\hat{c}})(1+u^{-\hat{c}})^{-\hat{k}}}{\left\{1-(1+u^{-\hat{c}})^{-\hat{k}}\right\}^2}\right]$$
$$- (n-d)\left[\frac{\ln^3(1+u^{-\hat{c}})(1+u^{-\hat{c}})^{-2\hat{k}}}{\left\{1-(1+u^{-\hat{c}})^{-\hat{k}}\right\}^3}\right]$$

$$\hat{\ell}_{ccc} = \frac{\partial^3 \ell}{\partial^3 c}|_{k=\hat{k},c=\hat{c}} = \frac{2d}{\hat{c}^3} + (\hat{k}+1)\sum_{i=1}^{d} \frac{x_{i:n}^{-\hat{c}} \ln^3 x_{i:n}(1-x_{i:n}^{-\hat{c}})}{(1+x_{i:n}^{-\hat{c}})^3}$$
$$- (n-d)\left[\frac{\left\{2\hat{k}^2(\hat{k}+1)(1+u^{-\hat{c}})^{-2(\hat{k}+2)} u^{-\hat{c}} \ln^2 u\right\}\left\{(\hat{k}+1)u^{-\hat{c}} - 1\right\}}{\left\{1-(1+u^{-\hat{c}})^{-\hat{k}}\right\}^2}\right]$$
$$+ (n-d)\left[\frac{\left\{2\hat{k}(1+u^{-\hat{c}})^{-(\hat{k}+1)} u^{-\hat{c}} \ln u\right\}\left\{\hat{k}(1+u^{-\hat{c}})^{-(\hat{k}+2)} u^{-\hat{c}} \ln u\right\}^2}{\left\{1-(1+u^{-\hat{c}})^{-\hat{k}}\right\}^3}\right]$$
$$+(n-d)\left[\frac{\left\{2\hat{k}(1+u^{-\hat{c}})^{-(\hat{k}+1)} u^{-\hat{c}} \ln u\right\}\left\{\hat{k}(\hat{k}+1)(1+u^{-\hat{c}})^{-(\hat{k}+2)}(u^{-\hat{c}}\ln u)^2\right\}}{\left\{1-(1+u^{-\hat{c}})^{-\hat{k}}\right\}^3}\right]$$

$$-(n-d)\left[\frac{\{\hat{k}(\hat{k}+1)(1+u^{-\hat{c}})^{-(\hat{k}+2)}(u^{-\hat{c}}\ln u)^3\}\{(\hat{k}+2)(1+u^{-\hat{c}})-2u^{-2\hat{c}}\ln^2 u\}}{\{1-(1+u^{-\hat{c}})^{-\hat{k}}\}^2}\right]$$

$$\hat{\ell}_{ckc}=\hat{\ell}_{kcc}=\frac{\partial^3\ell}{\partial k\partial^2 c}|_{k=\hat{k},c=\hat{c}}=-\sum_{i=1}^{d}\frac{x_{i:n}^{-\hat{c}}\ln^2 x_{i:n}}{1+x_{i:n}^{-\hat{c}}}$$
$$+(n-d)\left[\frac{\hat{k}(\hat{k}+1)(u^{-\hat{c}}\ln u)^2(1+u^{-\hat{c}})^{-(\hat{k}+2)}\ln(1+u^{-\hat{c}})}{\{1-(1+u^{-\hat{c}})^{-\hat{k}}\}^2}\right]$$
$$-(n-d)\left[\frac{\hat{k}\{(1+u^{-\hat{c}})^{-(\hat{k}+1)}u^{-\hat{c}}\ln u\}^2\{1-(1+u^{-\hat{c}})^{-\hat{k}}-\hat{k}\ln(1+u^{-\hat{c}})\}}{\{1-(1+u^{-\hat{c}})^{-\hat{k}}\}^3}\right]$$
$$+(n-d)\left[\frac{(2k+1)(u^{-\hat{c}}\ln u)^2(1+u^{-\hat{c}})^{-(2k+2)}+u^{-\hat{c}}\ln^2 u(1+u^{-\hat{c}})^{-(2k1)}}{\{1-(1+u^{-\hat{c}})^{-\hat{k}}\}^2}\right]$$
$$+(n-d)\left[\frac{(\hat{k}+1)(u^{-\hat{c}}\ln u)^2(1+u^{-\hat{c}})^{-(k+2)}+u^{-\hat{c}}\ln^2 u(1+u^{-\hat{c}})^{-(k+1)}}{\{1-(1+u^{-\hat{c}})^{-\hat{k}}\}^2}\right]$$
$$-(n-d)\left[\frac{\hat{k}(u^{-\hat{c}}\ln u)^2(1+u^{-\hat{c}})^{-(\hat{k}+2)}-u^{-\hat{c}}\ln^2 u\,(1+u^{-\hat{c}})^{-(\hat{k}+1)}\ln(1+u^{-\hat{c}})}{\{1-(1+u^{-\hat{c}})^{-\hat{k}}\}^2}\right]$$

$$\hat{\ell}_{cck}=\frac{\partial^3\ell}{\partial^2 c\partial k}|_{k=\hat{k},c=\hat{c}}=\sum_{i=1}^{d}\frac{x_{i:n}^{-\hat{c}}\ln^2 x_{i:n}}{(1+x_{i:n}^{-\hat{c}})^2}$$
$$+(n-d)\left[\frac{\{(1+u^{-\hat{c}})^{-(\hat{k}+2)}(u^{-\hat{c}}\ln u)^2\}\{(2\hat{k}+1)-\hat{k}(\hat{k}+1)\ln(1+u^{-\hat{c}})\}}{\{1-(1+u^{-\hat{c}})^{-\hat{k}}\}^2}\right]$$
$$-(n-d)\left[\frac{2(1+u^{-\hat{c}})^{-\hat{k}}\ln(1+u^{-\hat{c}})\{\hat{k}(\hat{k}+1)(1+u^{-\hat{c}})^{-(\hat{k}+2)}(u^{-\hat{c}}\ln u)^2\}}{\{1-(1+u^{-\hat{c}})^{-\hat{k}}\}^3}\right]$$
$$-(n-d)\left[\frac{\{2(1+u^{-\hat{c}})^{-\hat{k}}\ln(1+u^{-\hat{c}})\}\{(\hat{k}(1+u^{-\hat{c}})^{-(\hat{k}+2)}u^{-\hat{c}}\ln u)^2\}}{\{1-(1+u^{-\hat{c}})^{-\hat{k}}\}^3}\right]$$
$$+(n-d)\left[\frac{2\hat{k}\{(1+u^{-\hat{c}})^{-(\hat{k}+2)}u^{-\hat{c}}\ln u\}^2\{1-\hat{k}\ln(1+u^{-\hat{c}})\}}{\{1-(1+u^{-\hat{c}})^{-\hat{k}}\}^2}\right]$$

$$\hat{\ell}_{kck} = \hat{\ell}_{ckk} = \frac{\partial^3 \ell}{\partial c \partial^2 k}|_{k=\hat{k},c=\hat{c}} = -(n-d)\left[\frac{2(1+u^{-\hat{c}})^{-2(\hat{k}+1)}\ln(1+u^{-\hat{c}})u^{-\hat{c}}\ln u}{\left\{1-(1+u^{-\hat{c}})^{-\hat{k}}\right\}^3}\right]$$
$$-(n-d)\left[\frac{\left\{2(1+u^{-\hat{c}})^{-2(\hat{k}+1)}\ln(1+u^{-\hat{c}})u^{-\hat{c}}\ln u\right\}\left\{(1+u^{-\hat{c}})^{-\hat{k}} - \hat{k}\ln(1+u^{-\hat{c}})\right\}}{\left\{1-(1+u^{-\hat{c}})^{-\hat{k}}\right\}^3}\right]$$
$$+(n-d)\left[\frac{\left\{(1+u^{-\hat{c}})^{-(\hat{k}+1)}u^{-\hat{c}}\ln u\ln(1+u^{-\hat{c}})\right\}\left\{1-2(1+u^{-\hat{c}})-k\right\}}{1-(1+u^{-\hat{c}})^{-\hat{k}}}\right].$$

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