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Recurrence Relations for Quotient Moment of Generalized Pareto Distribution Based on Generalized Order Statistics and Characterization

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Abstract. Generalized Pareto distribution play an important role in reliability, extreme value theory, and other branches of applied probability and statistics. This family of distributions includes exponential distribution, Pareto distribution, and Power distribution. In this paper, we established exact expressions and recurrence relations satisfied by the quotient moments of generalized order statistics for a generalized Pareto distribution. Further the results for quotient moments of order statistics and records are deduced from the relations obtained and a theorem for characterizing this distribution is presented.

Keywords. Generalized order statistics; order statistics; record values; generalized Pareto distribution; recurrence relations; conditional expectation and characterization.

MSC 2010: 62G30, 62E10.

1 Introduction

Kamps (1995) introduced the concept of generalized order statistics (*gos*) as follows: Let $X_1, X_2 \ldots$ be a sequence of independent and identically distributed (*iid*) random variables (*rv*) with absolutely continuous distribution function (*df*) $F(x)$ and probability density function (*pdf*), $f(x)$, $x \in (\alpha, \beta)$.

Let $n \in N$, $n \geqslant 2$, $k > 0$, $m \in \Re$, be the parameters such that

$$\gamma_r = k + (n-r)(m+1) > 0, \qquad \text{for all} \qquad r \in 1, 2, \ldots, n-1.$$

Then $X(1,n,m,k), \ldots, X(n,n,m,k)$, $r = 1, 2, \ldots, n$ are called *gos* if their joint *pdf* is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left\{\prod_{i=1}^{n-1}[1-F(x_i)]^m f(x_i)\right\}\{1-F(x_n)\}^{k-1}f(x_n) \tag{1}$$

on the cone $F^{-1}(0) \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant F^{-1}(1)$.
The model of *gos* contains as special cases, order statistics, record values, sequential order statistics. Choosing the parameters appropriately (Cramer, 2002), we get the variant of the *gos* given in Table 1.

Table 1. Variants of the generalized order statistics

	$\boldsymbol{\gamma_n = k}$	$\boldsymbol{\gamma_r}$	$\boldsymbol{m_r}$
i) Sequential order statistics	α_n	$(n-r+1)\alpha_r$	$\gamma_r - \gamma_{r+1} - 1$
ii) Ordinary order statistics	1	$n-r+1$	0
iii) Record values	1	1	-1
iv) Progressively type II censored order statistics	$R_n + 1$	$n-r+1+\sum_{j=r}^{n} R_j$	R_r
v) Pfeifer's record values	β_n	β_r	$\beta_r - \beta_{r+1} - 1$

For simplicity we shall assume $m_1 = m_2 = \cdots = m_{n-1} = m$.
The *pdf* of the rth *gos*, $X(r,n,m,k)$, $1 \leqslant r \leqslant n$, is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\{\bar{F}(x)\}^{\gamma_r - 1} f(x) g_m^{r-1}\{F(x)\} \tag{2}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \leqslant r < s \leqslant n$, is

$$\begin{aligned} f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = {} & \frac{C_{s-1}}{(r-1)!(s-r-1)!}\{\bar{F}(x)\}^m f(x) g_m^{r-1} \\ & \times \{F(x)\}[h_m\{F(y)\} - h_m\{F(x)\}]^{s-r-1} \\ & \times \{\bar{F}(y)\}^{\gamma_s - 1} f(y), \qquad x < y, \end{aligned} \tag{3}$$

where

$$\bar{F}(x) = 1 - F(x), \qquad C_{r-1} = \prod_{i=1}^{r} \gamma_i, \qquad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \qquad x \in [0,1).$$

Let $X(r,n,m,k)$, $r = 1,2,\ldots,n$ be *gos* from a continuous population with *df* $F(x)$ and *pdf* $f(x)$. Then the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \leqslant r < s \leqslant n$, in view of (2) and (3), is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \times \frac{[h_m\{F(y)\} - h_m\{F(x)\}]^{s-r-1}\{\bar{F}(y)\}^{\gamma_s - 1}}{\{\bar{F}(x)\}^{\gamma_{r+1}}} f(y), \quad x < y. \tag{4}$$

Kamps (1995) investigated recurrence relations for moments of *gos* based on non-identically distributed random variables, which contains order statistics and record values as special cases. Cramer and Kamps (2000) derived relations for expectations of functions of generalized order statistics within a class of distributions including a variety of identities for single and product moments of ordinary order statistics and record values as particular cases. Various developments on *gos* and related topics have been studied by Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Pawlas and Szynal (2001), Ahmad and Fawzy (2003), Ahmad (2007), Khan, et al. (2007, 2010) among others. Khan and Kumar (2010, 2011a, b) have established recurrence relations for moments of lower generalized order statistics from exponentiated Pareto, gamma and generalized exponential distributions. Kamps (1998) investigated the importance of recurrence relations for moments of order statistics in characterization.

The aim of the present study is to give exact expression and some recurrence relations for quotient moments of *gos* from generalized Pareto distribution. In Section 2 we give exact expression and recurrence relations for quotient moments of generalized Pareto distribution. Then we show that results for order statistics and record values are deduced as special cases. In

Section 3 we give exact expression and recurrence relations for conditional quotient moments of generalized Pareto distribution and we show that results for order statistics and record values are deduced. In the last section of the paper we prove a characterization result on this distribution based on recurrence relation for conditional quotient moment of the *gos*.

A random variable X is said to have generalized Pareto distribution (Hall and Wellner, 1981) if its *pdf* is of the form

$$f(x) = \frac{\beta(1+\alpha)}{(\alpha x+\beta)^2}\Big(\frac{\beta}{\alpha x+\beta}\Big)^{\frac{1}{\alpha}}, \qquad x>0, \quad \alpha,\beta>0 \tag{5}$$

and the corresponding survival function is

$$\bar{F}(x) = \Big(\frac{\beta}{\alpha x+\beta}\Big)^{(\frac{1}{\alpha})+1}, \qquad x>0, \ \alpha,\beta>0. \tag{6}$$

For more details and some applications of this distribution, one may refer to Hall and Wellner (1981) and Johnson et al. (1994).

2 Relations for Quotient Moments

In view of (5) and (6), we have

$$(1+\alpha)\bar{F}(x) = (\alpha x+\beta)f(x). \tag{7}$$

The relation in (7) will be used to derive some simple recurrence relations for quotient moment of *gos* from the generalized Pareto distribution. These recurrence relations will be enable one to obtain all the quotient moments in a simple recursive manner.

***Theorem* 1.** *For generalized Pareto distribution as given in (6) and for* $1 \leqslant r \leqslant s-2$, $k=1,2,\dots$, $i=0,1,2,\dots$ *and* $j=1,2,\dots$, *if* $m \neq -1$

$$\begin{aligned} E\Big\{\frac{X^i(r,n,m,k)}{X^{j+1}(s,n,m,k)}\Big\} &= \frac{(1+\alpha)^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\Big(\frac{\alpha}{\beta}\Big)^{j-i+1} \\ &\times \sum_{p=0}^{\infty}\sum_{q=0}^{i}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v+q}\binom{r-1}{u}\binom{s-r-1}{v} \\ &\times \binom{i}{q}\frac{(j+1)_{(p)}}{p!\{(1+\alpha)\gamma_{s-v}+\alpha(p+j+1)\}} \\ &\times \frac{1}{\{(1+\alpha)\gamma_{r-u}+\alpha(p+q+j-i+1)\}}, \end{aligned} \tag{8}$$

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and if $m=-1$

$$E\Big\{\frac{X^i(r,n,-1,k)}{X^{j+1}(s,n,-1,k)}\Big\}=\{(1+\alpha)k\}^s\Big(\frac{\alpha}{\beta}\Big)^{j-i+1}\sum_{p=0}^{\infty}\sum_{q=0}^{i}(-1)^q\binom{i}{q}$$
$$\times\frac{(j+1)_{(p)}}{p!\{(1+\alpha)k+\alpha(p+j+1)\}^{s-r}\{(1+\alpha)k+\alpha(p+q+j-i+1)\}^r}, \tag{9}$$

where

$$(j)_p=\begin{cases} j(j+1)\dots(j+p-1), & p>0\\ 1, & p=0\end{cases}$$

Proof. From (3), we have

$$E\Big\{\frac{X^i(r,n,m,k)}{X^{j+1}(s,n,m,k)}\Big\}=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty\frac{x^i}{y^{j+1}}\{\bar F(x)\}^m f(x)$$
$$\times g_m^{r-1}\{F(x)\}[h_m\{F(y)\}-h_m\{F(x)\}]^{s-r-1}$$
$$\times\{\bar F(y)\}^{\gamma_s-1}f(y)dydx. \tag{10}$$

On using binomial expansion, (10), can be obtained when $m\neq-1$ as

$$E\Big\{\frac{X^i(r,n,m,k)}{X^{j+1}(s,n,m,k)}\Big\}=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times\int_0^\infty x^i\{\bar F(x)\}^{(s-r+u-v)(m+1)-1}f(x)G(x)dx, \tag{11}$$

where

$$G(x)=\int_x^\infty y^{-(j+1)}\{\bar F(y)\}^{\gamma_{s-v}-1}f(y)dy. \tag{12}$$

By setting $t = \{\bar{F}(y)\}^{\frac{\alpha}{(1+\alpha)}}$ in (12), we get

$$G(x) = \frac{(1+\alpha)\alpha^j}{\beta^{j+1}} \int_x^{\{\bar{F}(x)\}^{\frac{\alpha}{(1+\alpha)}}} (1-t)^{-(j+1)}\ t^{\frac{\alpha(j+1)+(\alpha+1)(\gamma_{s-v}-1)+1}{\alpha}} dt$$

$$= \frac{(1+\alpha)\alpha^j}{\beta^{j+1}} \sum_{p=0}^{\infty} \frac{(j+1)_p}{p!} \int_x^{\{\bar{F}(x)\}^{\frac{\alpha}{(1+\alpha)}}} t^{\frac{\alpha(j+1+p)+(\alpha+1)(\gamma_{s-v}-1)+1}{\alpha}} dt$$

$$= (1+\alpha)\Big(\frac{\alpha}{\beta}\Big)^{j+1} \sum_{p=0}^{\infty} \frac{(j+1)_p}{p!} \cdot \frac{\{\bar{F}(x)\}^{\gamma_{s-v}+\frac{\alpha(p+j+1)}{(\alpha+1)}}}{\{(\alpha+1)\gamma_{s-v}+\alpha(p+j+1)\}}.$$

On substituting the above expression of $G(x)$ in (11), we find that

$$E\Big\{\frac{X^i(r,n,m,k)}{X^{j+1}(s,n,m,k)}\Big\} = \frac{(1+\alpha)C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\Big(\frac{\alpha}{\beta}\Big)^{j+1}$$
$$\times \sum_{p=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times \frac{(j+1)_p}{p!\{(\alpha+1)\gamma_{s-v}+\alpha(p+j+1)\}}$$
$$\times \int_0^{\infty} x^i\{\bar{F}(x)\}^{\gamma_{r-u}+\frac{\alpha(p+j+1)}{(\alpha+1)}-1} f(x)dx. \tag{13}$$

Again by setting $z = \{\bar{F}(x)\}^{\alpha/(1+\alpha)}$ in (13) and simplifying the resulting equation, we get the result given in (8).
When $m=-1$, we have that

$$E\Big\{\frac{X^i(r,n,-1,k)}{X^{j+1}(s,n,-1,k)}\Big\} = \frac{k^s}{(r-1)!(s-r-1)!}$$
$$\times \int_0^{\infty} x^i\{-\ln\bar{F}(x)\}^{r-1}\frac{f(x)}{\bar{F}(x)} I(x)dx, \tag{14}$$

where

$$I(x) = \int_x^{\infty} y^{-(j+1)}[\ln\{\bar{F}(x)\} - \ln\{\bar{F}(y)\}]^{s-r-1}\{\bar{F}(y)\}^{k-1} f(y)dy.$$

Setting $w = \ln\{\bar{F}(x)\} - \ln\{\bar{F}(y)\}$, we find that

$$I(x) = (1+\alpha)^{s-r}\Big(\frac{\alpha}{\beta}\Big)^{j+1}\sum_{p=0}^{\infty}\frac{(j+1)_p}{p!}\cdot\frac{\{\bar{F}(x)\}^{k+\frac{\alpha(p+j+1)}{(\alpha+1)}}\ \Gamma(s-r)}{\{(\alpha+1)k+\alpha(p+j+1)\}^{s-r}}.$$

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On substituting the above expression of $I(x)$ in (14), we obtain

$$E\Big\{\frac{X^i(r,n,-1,k)}{X^{j+1}(s,n,-1,k)}\Big\} = \frac{(1+\alpha)^{s-r}k^s}{(r-1)!}\Big(\frac{\alpha}{\beta}\Big)^{j+1}$$

$$\times \sum_{p=0}^{\infty} \frac{(j+1)_p}{p!\{(\alpha+1)k+\alpha(p+j+1)\}^{s-r}}$$

$$\times \int_0^{\infty} x^i\{-\ln \bar{F}(x)\}^{r-1}\{\bar{F}(x)\}^{k+\frac{\alpha(p+j+1)}{(\alpha+1)}-1} f(x)dx. \tag{15}$$

Again by setting $z = -\ln\{\bar{F}(x)\}$ in (15) and simplifying the resulting expression, we get the result given in (9). □

Special Cases

i) Putting $m = 0, \quad k = 1$ in (8), the exact expression for the quotient moments of order statistics of the generalized Pareto distribution is obtained as

$$E\Big(\frac{X^i_{r:n}}{X^{j+1}_{s:n}}\Big) = (1+\alpha)^2 C_{r,s:n}\Big(\frac{\alpha}{\beta}\Big)^{j-i+1}$$

$$\times \sum_{p=0}^{\infty}\sum_{q=0}^{i}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v+q}\binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{q}$$

$$\times \frac{(j+1)_{(p)}}{p!\{(1+\alpha)(n-s+1+v)+\alpha(p+j+1)\}}$$

$$\times \frac{1}{\{(1+\alpha)(n-r+1+u)+\alpha(p+q+j-i+1)\}},$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

ii) Putting $k = 1$ in (9), we deduce the explicit expression for the quotient moments of upper record values for generalized Pareto distribution in the form

$$E\Big(\frac{X^i_{U(r)}}{X^{j+1}_{U(s)}}\Big) = (1+\alpha)^s\Big(\frac{\alpha}{\beta}\Big)^{j-i+1}\sum_{p=0}^{\infty}\sum_{q=0}^{i}(-1)^q\binom{i}{q}$$
$$\times\frac{(j+1)_{(p)}}{p!\{\alpha(p+j+2)+1\}^{s-r}\{\alpha(p+q+j-i+2)+1\}^r}.$$

Making use of (7), we can derive recurrence relations for the quotient moments of *gos* from (4).

Theorem 2. *For $1\leqslant r\leqslant s-2$, $k\geqslant 1$, $i=0,1,2,\dots$ and $j=1,2,\dots,$*

$$\Big\{1+\frac{\alpha(j+1)}{(1+\alpha)\gamma_s}\Big\}E\Big\{\frac{X^i(r,n,m,k)}{X^{j+1}(s,n,m,k)}\Big\} = E\Big\{\frac{X^i(r,n,m,k)}{X^{j+1}(s-1,n,m,k)}\Big\} - \frac{\beta(j+1)}{(1+\alpha)\gamma_s}E\Big\{\frac{X^i(r,n,m,k)}{X^{j+2}(s,n,m,k)}\Big\}. \quad (16)$$

Proof. We have from (3)

$$E\Big\{\frac{X^i(r,n,m,k)}{X^{j+1}(s,n,m,k)}\Big\} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\times\int_0^\infty x^i\{\bar F(x)\}^m f(x)g_m^{r-1}\{F(x)\}I(x)dx \quad (17)$$

where

$$I(x) = \int_x^\infty \frac{1}{y^{j+1}}[h_m\{F(y)\}-h_m\{F(x)\}]^{s-r-1}\{\bar F(y)\}^{\gamma_s-1}f(y)dy. \quad (18)$$

Integrating $I(x)$ by parts treating $\{\bar F(y)\}^{\gamma_s-1}f(y)$ for integration and the rest of the integrand for differentiation, we get

$$I(x) = -\frac{(j+1)}{\gamma_s}\int_x^\infty \frac{1}{y^{j+2}}[h_m\{F(y)\}-h_m\{F(x)\}]^{s-r-1}\{\bar F(y)\}^{\gamma_s}dy + \frac{(s-r-1)}{\gamma_s}\int_x^\infty \frac{1}{y^{j+1}}[h_m\{F(y)\}-h_m\{F(x)\}]^{s-r-2}\{\bar F(y)\}^{\gamma_s+m}f(y)dy. \quad (19)$$

Substituting the value of $I(x)$ in (17) and simplifying the resulting expression we get the result given in (16). □

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Remark 1. Putting $m = 0$ and $k = 1$ in (16), we obtain a recurrence relation for quotient moment of order statistics as

$$\Big\{1 + \frac{\alpha(j+1)}{(1+\alpha)(n-s+1)}\Big\} E\Big(\frac{X_{r:n}^{i}}{X_{s:n}^{j+1}}\Big) = E\Big(\frac{X_{r:n}^{i}}{X_{s-1:n}^{j+1}}\Big) - \frac{\beta(j+1)}{(1+\alpha)(n-s+1)} E\Big(\frac{X_{r:n}^{i}}{X_{s:n}^{j+2}}\Big).$$

Remark 2. Setting $m = -1$ and $k \geqslant 1$ in Theorem 2, we get a recurrence relation for quotient moment of upper k record as

$$\Big\{1 + \frac{\alpha(j+1)}{(1+\alpha)k}\Big\} E\Big(\frac{X_{U(r):k}^{i}}{X_{U(s):k}^{j+1}}\Big) = E\Big(\frac{X_{U(r):k}^{i}}{X_{U(s-1):k}^{j+1}}\Big) - \frac{\beta(j+1)}{(1+\alpha)k} E\Big(\frac{X_{U(r):k}^{i}}{X_{U(s):k}^{j+2}}\Big).$$

3 Relations for Quotient Conditional Expectation

Theorem 3. *For the distribution as given in (6) and for $1 \leqslant r < s \leqslant n-2$, $j = 1, 2, \dots,$ and $k = 1, 2, \dots,$ if $m \neq -1$*

$$E\Big\{\frac{1}{X^{j}(s,n,m,k)} | X(r,n,m,k) = x\Big\} = \Big(-\frac{\alpha}{\beta}\Big)^{j} \sum_{p=0}^{\infty} \frac{(j)_p}{p!} \Big(\frac{\alpha x + \beta}{\beta}\Big)^{p} \times \prod_{w=1}^{s-r} \left\{ \frac{\gamma_{r+w}}{\frac{\gamma_{r+w} - \alpha p}{(1+\alpha)}} \right\}, \tag{20}$$

and if $m = -1$

$$E\Big\{\frac{1}{X^{j}(s,n,-1,k)} | X(r,n,-1,k) = x\Big\} = \Big(\frac{\alpha}{\beta}\Big)^{j} \sum_{p=0}^{\infty} \frac{(j)_p}{p!} \Big\{\frac{(1+\alpha)k}{\alpha(p+j)}\Big\}^{s-r} \times \Big(\frac{\beta}{\alpha x + \beta}\Big)^{p+j}. \tag{21}$$

Proof. From (4), we have

$$E\Big\{\frac{1}{X^j(s,n,m,k)}|X(r,n,m,k)=x\Big\} = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$
$$\times \int_x^\infty y^{-j}\Big[1-\Big\{\frac{\bar{F}(y)}{\bar{F}(x)}\Big\}^{m+1}\Big]^{s-r-1}\Big\{\frac{\bar{F}(y)}{\bar{F}(x)}\Big\}^{\gamma_s-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{22}$$

By setting $u=\frac{\bar{F}(y)}{\bar{F}(x)}=\left(\frac{\alpha x+\beta}{\alpha y+\beta}\right)^{(1+\alpha)/\alpha}$ from (6) in (22), we obtain

$$E\Big\{\frac{1}{X^j(s,n,m,k)}|X(r,n,m,k)=x\Big\} = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$
$$\times \int_0^1 \left[-\frac{\beta}{\alpha}\left\{1-\frac{(\alpha x+\beta)u^{-\frac{\alpha}{(1+\alpha)}}}{\beta}\right\}\right]^{-j}$$
$$\times u^{\gamma_s-1}(1-u^{m+1})^{s-r-1}du$$
$$= \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$
$$\times \Big(-\frac{\alpha}{\beta}\Big)^j\sum_{p=0}^{\infty}\frac{(j)_p}{p!}\Big(\frac{\alpha x+\beta}{\beta}\Big)^p$$
$$\times \int_0^1 u^{\gamma_s-(\frac{\alpha p}{(1+\alpha)})-1}(1-u^{m+1})^{s-r-1}du, \tag{23}$$

Again by setting $t=u^{m+1}$ in (23), we get

$$E\Big\{\frac{1}{X^j(s,n,m,k)}|X(r,n,m,k)=x\Big\} = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\Big(-\frac{\alpha}{\beta}\Big)^j$$
$$\times \sum_{p=0}^{\infty}\frac{(j)_p}{p!}\Big(\frac{\alpha x+\beta}{\beta}\Big)^p$$
$$\times \int_0^1 t^{\frac{k-\frac{\alpha p}{(1+\alpha)}}{(m+1)}+n-s-1}(1-t)^{s-r-1}dt$$

$$= \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\Big(-\frac{\alpha}{\beta}\Big)^j \sum_{p=0}^{\infty}\frac{(j)_p}{p!}\Big(\frac{\alpha x+\beta}{\beta}\Big)^p$$

$$\times \frac{\Big\{\frac{\frac{k-\alpha p}{(1+\alpha)}}{(m+1)}+n-s\Big\}\Gamma(s-r)}{\Gamma\Big\{\frac{\frac{k-\alpha p}{(1+\alpha)}}{(m+1)}+n-r\Big\}}$$

$$= \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\Big(-\frac{\alpha}{\beta}\Big)^j \sum_{p=0}^{\infty}\frac{(j)_p}{p!}\Big(\frac{\alpha x+\beta}{\beta}\Big)^p$$

$$\times \frac{(m+1)^{s-r}\Gamma(s-r)}{\prod_{w=1}^{s-r}\Big\{\Big(k-\frac{\alpha p}{1+\alpha}\Big)+(n-r-w)(m+1)\Big\}}$$

$$= \frac{C_{s-1}}{C_{r-1}}\Big(-\frac{\alpha}{\beta}\Big)^j \sum_{p=0}^{\infty}\frac{(j)_p}{p!}\Big(\frac{\alpha x+\beta}{\beta}\Big)^p \frac{1}{\prod_{w=1}^{s-r}\Big(\frac{\gamma_{r+w}-\alpha p}{(1+\alpha)}\Big)},$$

and hence the result given in (20).
When $m=-1$, we have that

$$E\Big\{\frac{1}{X^j(s,n,-1,k)}|X(r,n,-1,k)=x\Big\} = \frac{k^{s-r}}{(s-r-1)!\{\bar{F}(x)\}^k}$$
$$\times \int_x^{\infty} y^{-j}[\ln\{\bar{F}(x)\}-\ln\{\bar{F}(y)\}]^{s-r-1}\{\bar{F}(y)\}^{k-1}f(y)dy. \tag{24}$$

Setting $w=\ln\{\bar{F}(x)\}-\ln\{\bar{F}(y)\}$ in (24) and integrating the resulting expression we get the result given in (21). □

Special cases

i) Putting $m=0$, $k=1$ in (20), the exact expression for the conditional quotient moments of order statistics of the generalized Pareto distribution is obtained as

$$E\Big(\frac{1}{X^j_{s:n}}|X_{r:n}=x\Big)=\Big(-\frac{\alpha}{\beta}\Big)^j\sum_{p=0}^{\infty}\frac{(j)_p}{p!}\Big(\frac{\alpha x+\beta}{\beta}\Big)^p$$
$$\times\prod_{w=1}^{s-r}\left(\frac{n-r-w+1}{n-r-w+1-\frac{\alpha p}{(1+\alpha)}}\right).$$

ii) Putting $k=1$ in (21), we deduce the explicit expression for the conditional quotient moments of upper record values for generalized Pareto distribution in the form

$$E\Big(\frac{1}{X^j_{U(s)}}|X_{U(r)}=x\Big)=\Big(\frac{\alpha}{\beta}\Big)^j\sum_{p=0}^{\infty}\frac{(j)_p}{p!}\Big\{\frac{1+\alpha}{\alpha(p+j)}\Big\}^{s-r}\Big(\frac{\beta}{\alpha x+\beta}\Big)^{p+j}.$$

Making use of (7), we can derive recurrence relations for the conditional quotient moments of *gos*.

Theorem 4. *For $1\leqslant r\leqslant s-2$, $k\geqslant 1$, $i=0,1,2,\dots$ and $j=1,2,\dots$*

$$\Big\{1+\frac{\alpha(j+1)}{(1+\alpha)\gamma_s}\Big\}E\Big\{\frac{X^i(r,n,m,k)}{X^{j+1}(s,n,m,k)}|X(r,n,m,k)=x\Big\}$$
$$=E\Big\{\frac{X^i(r,n,m,k)}{X^{j+1}(s-1,n,m,k)}|X(r,n,m,k)=x\Big\}$$
$$-\frac{\beta(j+1)}{(1+\alpha)\gamma_s}E\Big\{\frac{X^i(r,n,m,k)}{X^{j+2}(s,n,m,k)}|X(r,n,m,k)=x\Big\}. \tag{25}$$

Proof. From (4), we have

$$E\Big\{\frac{X^i(r,n,m,k)}{X^{j+1}(s,n,m,k)}|X(r,n,m,k)=x\Big\}=\frac{x^iC_{s-1}I(x)}{(s-r-1)!C_{r-1}\{\bar{F}(x)\}^{\gamma_{r+1}}} \tag{26}$$

where $I(x)$ is defined in (18). Substituting the value of $I(x)$ from (19) in (26) and simplifying the resulting expression we get the result given in (25). $\square$

Remark 3. Putting $m=0,\ k=1$ in (25), we obtain a recurrence relation

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for conditional quotient moment of order statistics as

$$\Big\{1+\frac{\alpha(j+1)}{(1+\alpha)(n-s+1)}\Big\}E\Big(\frac{X_{r:n}^{i}}{X_{s:n}^{j+1}}|X_{r:n}=x\Big)=E\Big(\frac{X_{r:n}^{i}}{X_{s-1:n}^{j+1}}|X_{r:n}=x\Big)$$
$$-\frac{\beta(j+1)}{(1+\alpha)(n-s+1)}$$
$$\times E\Big(\frac{X_{r:n}^{i}}{X_{s:n}^{j+2}}|X_{r:n}=x\Big).$$

Remark 4. Setting $m=-1$ and $k\geqslant 1$ in Theorem 4, we get a recurrence relation for quotient moment of upper k records as

$$\Big\{1+\frac{\alpha(j+1)}{(1+\alpha)k}\Big\}E\Big(\frac{X_{U(r):k}^{i}}{X_{U(s):k}^{j+1}}|X_{U(r)}=x\Big)=E\Big(\frac{X_{U(r):k}^{i}}{X_{U(s-1):k}^{j+1}}|X_{U(r)}=x\Big)$$
$$-\frac{\beta(j+1)}{(1+\alpha)k}E\Big(\frac{X_{U(r):k}^{i}}{X_{U(s):k}^{j+2}}|X_{U(r)}=x\Big).$$

4 Characterization

Theorem 5. *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, then. If $m\neq -1$*

$$E\Big\{\frac{1}{X^{j}(s,n,m,k)}|X(r,n,m,k)=x\Big\}=\Big(-\frac{\alpha}{\beta}\Big)^{j}\sum_{p=0}^{\infty}\frac{(j)_p}{p!}\Big(\frac{\alpha x+\beta}{\beta}\Big)^{p}$$
$$\times\prod_{w=1}^{s-r}\Big(\frac{\gamma_{r+w}}{\gamma_{r+w}-\frac{\alpha p}{(1+\alpha)}}\Big),\qquad (27)$$

and if $m=-1$

$$E\Big\{\frac{1}{X^{j}(s,n,-1,k)}|X(r,n,-1,k)=x\Big\}=\Big(\frac{\alpha}{\beta}\Big)^{j}\sum_{p=0}^{\infty}\frac{(j)_p}{p!}\Big\{\frac{(1+\alpha)k}{\alpha(p+j)}\Big\}^{s-r}$$
$$\times\Big(\frac{\beta}{\alpha x+\beta}\Big)^{p+j},\qquad (28)$$

if and only if

$$\bar{F}(x) = \Big(\frac{\beta}{\alpha x + \beta}\Big)^{(\frac{1}{\alpha})+1}, \qquad x > 0,\ \alpha, \beta > 0.$$

Proof. The necessary part follows immediately from (20). On the other hand if (27) is satisfied, then on using equation (4), we have

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_x^\infty y^{-j}[\{\bar{F}(x)\}^{m+1} - \{\bar{F}(y)\}^{m+1}]^{s-r-1}$$

$$\times \{\bar{F}(y)\}^{\gamma_s - 1} f(y)dy = \{\bar{F}(x)\}^{\gamma_{r+1}} H_r(x), \tag{29}$$

where

$$H_r(x) = \Big(-\frac{\alpha}{\beta}\Big)^j \sum_{p=0}^{\infty} \frac{(j)_p}{p!} \Big(\frac{\alpha x + \beta}{\beta}\Big)^p \prod_{w=1}^{s-r} \left(\frac{\gamma_{r+w}}{\gamma_{r+w} - \frac{\alpha p}{(1+\alpha)}}\right).$$

Differentiating (29) both the sides with respect to x , and rearranging the terms, we get

$$-\frac{C_{s-1}\{\bar{F}(x)\}^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}} \int_x^\infty y^{-j}[\{\bar{F}(x)\}^{m+1} - \{\bar{F}(y)\}^{m+1}]^{s-r-2}$$

$$\times \{\bar{F}(y)\}^{\gamma_s - 1} f(y)dy = H_r'(x)\{\bar{F}(x)\}^{\gamma_{r+1}} - \gamma_{r+1} H_r(x)\{\bar{F}(x)\}^{\gamma_{r+1}-1} f(x)$$

or

$$\gamma_{r+1} H_{r+1}(x)\{\bar{F}(x)\}^{\gamma_{r+2}+m} f(x) = H_r'(x)\{\bar{F}(x)\}^{\gamma_{r+1}}$$

$$- \gamma_{r+1} H_r(x)\{\bar{F}(x)\}^{\gamma_{r+1}-1} f(x).$$

Therefore

$$\frac{f(x)}{\bar{F}(x)} = -\frac{H_r'(x)}{\gamma_{r+1}\{H_{r+1}(x) - H_r(x)\}} = \frac{(1+\alpha)}{\alpha x + \beta},$$

where

$$H_r'(x) = \Big(-\frac{\alpha}{\beta}\Big)^j \sum_{p=0}^{\infty} \frac{\alpha p \ \ (j)_p}{p! \ \beta} \Big(\frac{\alpha x + \beta}{\beta}\Big)^{p-1} \prod_{w=1}^{s-r} \left(\frac{\gamma_{r+w}}{\gamma_{r+w} - \frac{\alpha p}{(1+\alpha)}}\right)$$

and

$$H_{r+1}(x) = \Big(-\frac{\alpha}{\beta}\Big)^j \sum_{p=0}^{\infty} \frac{(j)_p}{p!} \Big(\frac{\alpha x + \beta}{\beta}\Big)^p \prod_{w=1}^{s-(r+1)} \left(\frac{\gamma_{r+1+w}}{\gamma_{r+1+w} - \frac{\alpha p}{(1+\alpha)}}\right),$$

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which proves that

$$\bar{F}(x) = \Big(\frac{\beta}{\alpha x + \beta}\Big)^{(\frac{1}{\alpha})+1}, \qquad x > 0,\ \alpha, \beta > 0.$$

When $m = -1$ the necessary part follows immediately from (21). On the other hand if (28) is satisfied, then on using equation (24), we have

$$\frac{k^{s-r}}{(s-r-1)!}\int_x^\infty y^{-j}[\ln\{\bar{F}(x)\} - \ln\{\bar{F}(y)\}]^{s-r-1}\{\bar{F}(y)\}^{k-1} f(y)dy$$
$$= \{\bar{F}(y)\}^k G_r(x) \qquad (30)$$

where

$$G_r(x) = \Big(\frac{\alpha}{\beta}\Big)^j \sum_{p=0}^{\infty} \frac{(j)_p}{p!}\Big\{\frac{(1+\alpha)k}{\alpha(p+j)}\Big\}^{s-r}\Big(\frac{\beta}{\alpha x + \beta}\Big)^{p+j}.$$

Differentiating (30) both sides with respect to x and rearranging the terms, we get

$$\frac{k^{s-r} f(x)}{(s-r-2)!\bar{F}(x)}\int_x^\infty y^{-j}[\ln\{\bar{F}(x)\} - \ln\{\bar{F}(y)\}]^{s-r-2}\{\bar{F}(y)\}^{k-1} f(y)dy$$
$$= -k\{\bar{F}(y)\}^{k-1} f(x) G_r(x) + \{\bar{F}(y)\}^k G_r'(x)$$

or

$$-kG_{r+1}(x)\{\bar{F}(y)\}^{k-1} f(x) = \{\bar{F}(y)\}^k G_r'(x) - k\{\bar{F}(y)\}^{k-1} f(x) G_r(x).$$

Therefore

$$\frac{f(x)}{\bar{F}(x)} = -\frac{G_r'(x)}{\gamma_{r+1}\{G_{r+1}(x) - G_r(x)\}} = \frac{(1+\alpha)}{\alpha x + \beta}$$

which proves that

$$\bar{F}(x) = \Big(\frac{\beta}{\alpha x + \beta}\Big)^{(\frac{1}{\alpha})+1}, \qquad x > 0,\ \alpha, \beta > 0.$$

□

5 Conclusion

This paper deals with the *gos* from the generalized Pareto distribution. Exact expressions and recurrence relations satisfied by the quotient moments of *gos* for a generalized Pareto distribution are derived. Characterization of this distribution by considering the conditional expectation for generalized Pareto distribution satisfied by the quotient moment of the *gos*. Special cases are also deduced.

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