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Testing Exponentiality Based on Renyi Entropy of Transformed Data

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Abstract. In this paper, we introduce new tests for exponentiality based on estimators of Renyi entropy of a continuous random variable. We first consider two transformations of the observations which turn the test of exponentiality into one of uniformity and use a corresponding test based on Renyi entropy. Critical values of the test statistics are computed by Monte Carlo simulations. Then, we compare powers of the tests for various alternatives and sample sizes with exponentiality tests based on Kullback-Leibler information proposed by Ebrahimi et al. (1992) and Choi et al. (2004). Our simulation results show that the proposed tests have higher powers than the competitor tests.

Keywords. Entropy; Renyi distance; goodness of fit test; uniformity; spacing.

MSC 2010: 62G10, 94A17.

1 Introduction

Many current results in reliability studies and engineering are based on the assumption that the life of a product has an exponential distribution. If this assumption is wrong, then the results obtained may be invalid. One way to deal with this problem is to check the distribution assumption carefully. Therefore different goodness of fit tests of exponentiality have been developed.

Standard procedures for testing exponentiality are the tests of Kolmogorov-Smirnov and Cramer-Von Mises, which utilize the empirical distribution function. Lilliefors (1969) suggested a modified Kolmogorov-Smirnov test. Van-Soest (1969) studied a modified goodness of fit test based on Cramer-Von Mises statistic. Finkelstein and Schafer (1971) improved the previous Kolmogorov-Smirnov statistic. Harris (1976) compared Finkelstein and Schafer's tests with those of Frecho and Ringer (1972). Also, many methods were reviewed by Ascher (1990). Another test statistic of exponentiality was prepared by Jammalamadaka and Taufer (2003) based on comparing the empirical distribution function of the normalized spacings with that of the original data.

Recent years have witnessed an increasing interest in using alternative methods, in constructing goodness of fit tests for exponentiality, such as methods based on entropy. Ebrahimi et al. (1992) introduced a test procedure which exploit the Kullback-Leibler information and estimated the test statistic by entropy estimator of Vasicek (1976). Following their method, more test statistics were proposed by Grzegorzewski and Wieczorkowski (1999), Park and Park (2003), Choi et al. (2004), Yousefzadeh and Arghami (2008), Gurevich and Davidson (2008), Alizadeh Noughabi and Arghami (2011b) and Zamanzade and Arghami (2011) based on different entropy estimators including estimators of Vasicek (1976), Van-Es (1992), Ebrahimi et al. (1994), Correa (1995) and Alizadeh Noughabi (2010).

Vexler and Gurevich (2010) and Gurevich and Vexler (2011) developed empirical likelihood ratio tests for goodness of fit and demonstrated that the well-known goodness of fit tests based on sample entropy and Kullback-Leibler information are a product of the proposed empirical likelihood methodology.

Furthermore, some researchers have shown interest in developing goodness of fit tests based on transformed data including Taufer (2002) and Alizadeh Noughabi and Arghami (2011a, 2011c). A wide selection of classical and recent tests for exponentiality including some mentioned tests have been discussed and compared by Henze and Meintanis (2005).

Recently, interest has also increased in the applications of other information measures in goodness of fit tests. For example, test statistics based on Renyi distance and Lin-Wong information have been studied by Abbasnejad (2011) and Abbasnejad et al. (2013).

The rest of the paper is organized as follows. In Section 2, we use two characterizations of the exponential distribution to prepare test statistics based on Renyi information. Two transformations of the sample data are considered and the test of exponentiality turns to uniformity test. For each transformation, we prepare four test statistics based on different estimators of Renyi entropy introduced by Wachowiak et al. (2005) and Abbasnejad (2012). In Section 3, we obtain the powers of the proposed tests by simulation on a wide variety of alternatives and for sample sizes 10 and 20 and compare them with other tests which are based on Kullback-Leibler information.

2 Test Statistics

Let $X_1, \ldots, X_n$ be a random sample from a continuous non-negative probability distribution function F with a density function f and suppose it is of interest to verify

$$H_0: \; f(x) = f_0(x;\theta),$$

against

$$H_1: \; f(x) \neq f_0(x;\theta),$$

where $f_0(x;\theta) = \theta e^{-\theta x}, \; x > 0$ and $\theta > 0$ is unknown.

In order to obtain a test statistic, we use the following theorem.

***Theorem* 1.** *Let X_1 and X_2 be two independent observations from a continuous distribution function F. Then*

(i) $W = \frac{X_1}{X_1+X_2}$ *is distributed as $U(0,1)$ if and only if F is exponential.*

(ii) $Z = \frac{X_1-X_2}{X_1+X_2}$ *is distributed as $U(-1,1)$ if and only if F is exponential.*

Proof. The first and second parts were proved by Kotz and Stentel (1988) and Alizadeh Noughabi and Arghami (2011c), respectively. □

Let $X_{1:n}, \ldots, X_{n:n}$ be an ordered random sample of size n. Consider the following transformations of the sample data as

$$W_{ij} = \frac{X_{i:n}}{X_{i:n} + X_{j:n}}, \qquad i \neq j, \; j = 1, \ldots, n,$$

$$Z_{ij} = \frac{X_{i:n} - X_{j:n}}{X_{i:n} + X_{j:n}}, \qquad i > j, \; j = 1, \ldots, n.$$

Under the null hypothesis, each W_{ij} and Z_{ij} have uniform distribution on $(0, 1)$. It seems to be appropriate to use a test of uniformity based on transformed data W_{ij} or Z_{ij} instead of testing exponentiality based on the original data. Hence, to test H_0, we transform X_i to W_{ij} or Z_{ij} and test $H_0' : g(z) = g_0(z)$, where $g_0(z) = 1,\ 0 < z < 1$.

The asymmetric Renyi distance of g from g_0 is:

$$\begin{aligned} D_r(g, g_0) &= \frac{1}{r-1} \ln \int_0^1 \left[\frac{g(z)}{g_0(z)}\right]^{r-1} g(z)dz \\ &= \frac{1}{r-1} \ln \int_0^1 g^r(z)dz \\ &= -H_r(g), \end{aligned} \tag{1}$$

where the $H_r(g)$ is the Renyi entropy of order r of distribution with density g. $D_r(g, g_0) \geqslant 0$ and the equality holds if and only if H_0' holds.

Large values of $D_r(g, g_0)$ lead us to reject the null hypothesis H_0' in favor of the alternative hypothesis H_1'.

To derive a test statistic by evaluating the information function (1), a density g must be completely specified, which is not operational. Thus, it is necessary to estimate the information function (1) from a sample. Toward this end, we use four estimators of Renyi entropy proposed by Wachowiak et al. (2005) and Abbasnejad (2012).

Let $X_{1:n} \leqslant, \ldots, \leqslant X_{n:n}$, be order statistics of a random sample $X_1, \ldots, X_n$ from a distribution function F with density function f. The estimators of Wachowiak et al. (2005) are:

- The Vasicek-type estimator

  $$HV_{r,m,n} = -\frac{1}{r-1} \ln \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\frac{n}{2m}(X_{i+m:n} - X_{i-m:n})\right]^{1-r} \right\},$$

  where m is a positive integer, $m \leqslant \frac{n}{2}$, and $X_{i:n} = X_{1:n}$ if $i < 1$, $X_{i:n} = X_{n:n}$ if $i > n$.
  This estimate was based on the fact that using $p = F(x)$, $H_r(f)$ can be expressed as

  $$H_r(f) = -\frac{1}{r-1} \ln \int_0^1 \left[\frac{dF^{-1}(p)}{dp}\right]^{1-r} dp.$$

  Similar to Vasicek estimator, their estimator was constructed by replacing the distribution function F by the empirical distribution function F_n, and using a difference operator instead of the differential operator.

- The Correa-type estimator

$$HC_{r,m,n} = -\frac{1}{r-1}\ln\left\{\frac{1}{n}\sum_{i=1}^{n} b_i^{r-1}\right\},$$

where $b_i = \frac{\sum_{j=i-m}^{i+m}(X_{j:n}-\bar{X}_i)(j-i)}{n\sum_{j=i-m}^{i+m}(X_{j:n}-\bar{X}_i)^2}$ and $\bar{X}_i = \frac{1}{2m+1}\sum_{j=i-m}^{i+m} X_{j:n}$.
This estimator was obtained using a local linear regression based on the set $\{X_{i-m:n}, \ldots, X_{i+m:n}\}$.

Abbasnejad (2012) introduced two modifications of Correa-type estimator of Renyi entropy, in order to account the fact that the differences are truncated around the smallest and the largest data points (i, e. $X_{i-m:n}$ is replaced by $X_{1:n}$ for $i \leqslant m$ and $X_{i+m:n}$ is replaced by $X_{n:n}$ for $i \geqslant n-m+1$). The modified estimators are:

- The first modified Correa-type estimator

$$HC1_{r,m,n} = -\frac{1}{r-1}\ln\left\{\frac{1}{n}\sum_{i=1}^{n} c_i^{r-1}\right\},$$

where

$$c_i = \frac{\sum_{j=k_1(i)}^{k_2(i)}(X_{j:n}-\tilde{X}_i)\{F_n(j)-\tilde{F}_n(i)\}}{\sum_{j=k_1(i)}^{k_2(i)}(X_{j:n}-\tilde{X}_i)^2}, \quad i=1,\ldots,n,$$

$$k_1(i) = \begin{cases} 1 & 1 \leqslant i \leqslant m \\ i-m & i \geqslant m+1 \end{cases}, \qquad k_2(i) = \begin{cases} i+m & 1 \leqslant i \leqslant n-m \\ n & i \geqslant n-m+1 \end{cases}$$

$$\tilde{X}_i = \sum_{j=k_1(i)}^{k_2(i)} \frac{X_{j:n}}{k_2(i)-k_1(i)+1}, \qquad \tilde{F}_n(i) = \sum_{j=k_1(i)}^{k_2(i)} \frac{F_n(j)}{k_2(i)-k_1(i)+1}.$$

It can be easily shown that

$$c_i = \begin{cases} \frac{\sum_{j=1}^{i+m}(X_{j:n}-\tilde{X}_i)(\frac{j}{n}-\frac{m+i+1}{2n})}{\sum_{j=1}^{i+m}(X_{j:n}-\tilde{X}_i)^2} & 1 \leqslant i \leqslant m \\ b_i & m+1 \leqslant i \leqslant n-m \\ \frac{\sum_{j=i-m}^{n}(X_{j:n}-\tilde{X}_i)(\frac{j}{n}-\frac{n+i-m}{2n})}{\sum_{j=i-m}^{n}(X_{j:n}-\tilde{X}_i)^2} & n-m+1 \leqslant i \leqslant n. \end{cases}$$

- The second modified Correa-type estimator

$$HC2_{r,m,n} = -\frac{1}{r-1}\ln\left\{\sum_{i=1}^{n} w_i c_i^{r-1}\right\},$$

where

$$w_i = \frac{F_n(X_{i+m:n}) - F_n(X_{i-m:n})}{\sum_{i=1}^{n}[F_n(X_{i+m:n}) - F_n(X_{i-m:n})]}, \quad i = 1, \ldots, n.$$

It can be easily shown that

$$w_i = \begin{cases} \frac{i+m-1}{m(2n-m-1)} & 1 \leqslant i \leqslant m \\ \frac{2}{2n-m-1} & m+1 \leqslant i \leqslant n-m \\ \frac{n-i+m}{m(2n-m-1)} & n-m+1 \leqslant i \leqslant n. \end{cases}$$

Abbasnejad (2012) compared her estimators with Correa-type estimator and by simulation, showed that her estimators have smaller bias and mean squared error (MSE).

In practice, to test H_0', by computing $HV_{r,m,n}$, $HC_{r,m,n}$, $HC1_{r,m,n}$ and $HC2_{r,m,n}$ on $W_{i:n'}$, $n' = n(n-1)$ and $Z_{i:n''}$, $n'' = n(n-1)/2$ and estimating (1), we obtain the following test statistics:

$$\begin{aligned} TV^W &= -HV_{r,m,n'} \quad \text{and} \quad & TV^Z &= -HV_{r,m,n''}, \\ TC^W &= -HC_{r,m,n'} \quad \text{and} \quad & TC^Z &= -HC_{r,m,n''}, \\ TC1^W &= -HC1_{r,m,n'} \quad \text{and} \quad & TC1^Z &= -HC1_{r,m,n''}, \\ TC2^W &= -HC2_{r,m,n'} \quad \text{and} \quad & TC2^Z &= -HC2_{r,m,n''}. \end{aligned}$$

We reject the null hypothesis for large values of the above test statistics. It is obvious that the test statistics are invariant with respect to transformations of scale.

The next theorem states that the tests based on TV^W and TV^Z are consistent and it can easily be proved by following the line of argument in Alizadeh Noughabi and Arghami (2011c).

Theorem 2. *Let G be an unknown continuous distribution with the support $(0,1)$ and G_0 be the uniform distribution. Then under H_1', the tests TV^W and TV^Z are consistent.*

Proof. As $n, m \to \infty$ and $m/n \to 0$, we have

$$\begin{aligned}
\frac{2m}{n'} &= G_{n'}(W_{i+m:n'}) - G_{n'}(W_{i-m:n'}) \\
&\simeq G(W_{i+m:n'}) - G(W_{i-m:n'}) \\
&\simeq \frac{g(W_{i+m:n'}) + g(W_{i-m:n'})}{2}(W_{i+m:n'} - W_{i-m:n'}),
\end{aligned}$$

where $G_{n'}$ is the empirical distribution function. The second equality is obtained by the consistency of the empirical distribution function and the third equality follows trapezoidal rule in approximating a definite integral. Therefore

$$\begin{aligned}
TV^W &= \frac{1}{r-1}\ln\left\{\frac{1}{n'}\sum_{i=1}^{n'}\left[\frac{n'}{2m}(W_{i+m:n'} - W_{i-m:n'})\right]^{1-r}\right\} \\
&\simeq \frac{1}{r-1}\ln\left\{\frac{1}{n'}\sum_{i=1}^{n'}\left[\frac{g(W_{i+m:n'}) + g(W_{i-m:n'})}{2}\right]^{r-1}\right\} \\
&= \frac{1}{r-1}\ln\left\{\frac{1}{n'}\sum_{i=1}^{n'}[g(W_{i:n'})]^{r-1}\right\} \\
&= \frac{1}{r-1}\ln\left\{\frac{1}{n'}\sum_{i=1}^{n'}[g(W_i)]^{r-1}\right\} \\
&\overset{n'\to\infty}{\to} \frac{1}{r-1}\ln E[(g(W_1))^{r-1}] \\
&= \frac{1}{r-1}\ln\int_0^1 g^r(w)dw \\
&= -H_r(g),
\end{aligned}$$

where the limit holds by the law of large numbers. So under H_0', $TV^W \overset{Pr.}{\to} -H_r(g_0) = 0$. Also note that the uniform distribution maximizes Renyi entropy, that is under H_1'

$$TV^W \overset{Pr.}{\to} -H_r(g) > -H_r(g_0) = 0.$$

Similar result holds for TV^Z. □

Critical points are determined by the quantiles of the distribution of the test statistics under hypothesis H_0. Since the sampling distribution of the

Table 1. Critical values of TV^W, TC^W, $TC1^W$ and $TC2^W$ for $r = 1.2$ and $\alpha = 0.01$ (0.05).

n	n'	m	TV^W	TC^W	$TC1^W$	$TC2^W$
5	20	4	1.2387 (0.8222)	1.0930 (0.6750)	0.9917 (0.5734)	1.0041 (0.5744)
10	90	9	0.5748 (0.4004)	0.4940 (0.3273)	0.4432 (0.2847)	0.4462 (0.2803)
15	210	14	0.3815 (0.2770)	0.3303 (0.2287)	0.3058 (0.2019)	0.3028 (0.1930)
20	380	19	0.3051 (0.2154)	0.2676 (0.1804)	0.2462 (0.1619)	0.2376 (0.1552)
30	870	29	0.2101 (0.1437)	0.1890 (0.1208)	0.1763 (0.1099)	0.1714 (0.1082)
40	1560	39	0.1756 (0.1121)	0.1547 (0.0956)	0.1443 (0.0881)	0.1356 (0.0853)
50	2450	49	0.1378 (0.0894)	0.1185 (0.0601)	0.1054 (0.0561)	0.1994 (0.0305)

Table 2. Critical values of TV^Z, TC^Z, $TC1^Z$ and $TC2^Z$ for $r = 1.2$ and $\alpha = 0.01$ (0.05).

n	n''	m	TV^Z	TC^Z	$TC1^Z$	$TC2^Z$
5	10	3	1.4876 (1.0335)	1.2870 (0.8337)	1.0995 (0.6504)	1.0464 (0.6265)
10	45	7	0.6490 (0.4848)	0.5523 (0.3828)	0.4722 (0.3101)	0.4692 (0.2987)
15	105	10	0.4802 (0.3296)	0.4047 (0.2622)	0.3638 (0.2187)	0.3338 (0.2088)
20	190	14	0.3605 (0.2494)	0.3083 (0.2025)	0.2754 (0.1720)	0.2608 (0.1625)
30	435	21	0.2576 (0.1729)	0.2211 (0.1412)	0.1985 (0.1224)	0.1808 (0.1169)
40	780	28	0.1955 (0.1323)	0.1710 (0.1087)	0.1532 (0.0952)	0.1401 (0.0884)
50	1225	35	0.1588 (0.1087)	0.1389 (0.0893)	0.1281 (0.0785)	0.1180 (0.0737)

test statistics are intractable, we determine the critical points by means of Monte Carlo simulations for α equal to 0.01 and 0.05. We generated 10000 samples of size n from uniform distribution and computed the test statistics. The upper α quantile of the empirical distribution of the test statistics gives the critical values. The results are given in Tables 1 and 2.

Unfortunately, there is no choice criterion of r^* and m^* (the optimum values of r and m). In general they depends on the type of the alternatives. Totally, we suggest to choose $s^* = 1.2$ and value of $m^* = [\sqrt{n} + 0.5]$ which produced the maximum power for the most of alternative distributions based on simulation results.

3 Numerical Results

To study the behavior of the proposed tests, a simulation study was performed. We consider two competitor tests based on Kullback-Leibler information.

The asymmetric Kullback-Leibler distance of f from f_0 is:

$$D(f, f_0) = \int_0^{+\infty} f(x) \ln \left[\frac{f(x)}{\theta e^{-\theta x}} \right] dx$$
$$= -H(f) - \ln \theta + 1,$$

where $H(f) = -\int_0^{+\infty} f(x) \ln f(x) dx$ is the Shannon entropy of f. $D(f, f_0) \geqslant 0$ and the equality holds if and only if H_0 holds. By estimating θ by its maximum likelihood estimator $1/\bar{X}$ and estimating $H(f)$, Ebrahimi et al. (1992) and Choi et al. (2004) proposed following test statistics:

- Ebrahimi et al. (1992) test

$$KLV = \frac{\exp(HV_{mn})}{\exp(\ln \bar{X} + 1)},$$

where $HV_{mn} = \frac{1}{n} \sum_{i=1}^{n} \ln \left[\frac{n}{2m} (X_{i+m:n} - X_{i-m:n}) \right]$ is Vasicek estimator.

- Choi et al. (2004) test

$$KLC = \frac{\exp(HC_{mn})}{\exp(\ln \bar{X} + 1)},$$

where $HC_{mn} = -\frac{1}{n} \sum_{i=1}^{n} \ln b_i$ is Correa estimator.

For power comparisons, we consider the following alternatives:

- Monotone decreasing hazard: Gamma with shape parameter 0.5, 0.8, Weibull with shape parameter 0.5, 0.8, Generalized exponential (GE) with shape parameter 0.5, 0.8.

- Monotone increasing hazard: Uniform, Weibull with shape parameter 2, Gamma with shape parameter 1.5, 2, 3, Beta with shape parameters 1 and 2, 2 and 1.

- Non-monotone hazard: Log normal with shape parameter 0.5, 1, 1.2 and Beta with shape parameters 0.5 and 1.

Under each alternative we generated 10000 samples of size 10 and 20 and then we obtained transformed samples. We evaluated for each transformed sample the test statistics and power of each test was estimated by the frequency of the event “the test statistic is larger than the critical value”.

Table 3. Power comparisons of various tests for $n = 10$ and $\alpha = 0.05$.

Alternatives	KLV	KLC	TV^W	TC^W	$TC1^W$	$TC2^W$	TV^Z	TC^Z	$TC1^Z$	$TC2^Z$
Gamma(0.5)	.0204	.0108	.3515	.3363	.3054	.2129	**.4138**	.4036	.3816	.3460
Gamma(0.8)	.0270	.0251	.00789	.0731	.0637	.0379	**.0918**	.0894	.0820	.0676
Weibull(0.5)	.1113	.0634	.6182	.6036	.5682	.4623	**.6252**	.6178	.6027	.5763
Weibull(0.8)	.0152	.0127	.0913	.0846	.0720	.0421	**.1097**	.1058	.0994	.0822
GE(0.2)	.4427	.3405	.9521	.9479	.9383	.9098	**.9630**	.9617	.9594	.9524
GE(0.8)	.0264	.0240	.0755	.0699	.0623	.0428	**.0906**	.0849	.0785	.0631
Uniform	.4877	**.4953**	.2952	.2864	.2832	3065	.3029	.2964	.2913	.2957
Weibull(2)	**.6897**	.6846	.6088	.6240	.6410	.6822	.5568	.5821	.5953	.6415
Gamma(1.5)	**.1647**	.1624	.1305	.1385	.1510	.1758	.1072	.1177	.1243	.1429
Gamma(2)	.3219	3280	.2728	.2890	.3154	**.3586**	.2446	.2556	.2708	.3047
Gamma(3)	.6108	.6204	.6030	.6196	.6453	**.6929**	.5535	.5734	.5879	.6419
Beta(1,2)	.1922	**.2031**	1378	.1380	.1472	.1669	.1311	.1308	.1362	.1416
Beta(2,1)	**.9824**	.9841	.9068	.8981	.8972	.9205	.8962	.8893	.8830	.8841
Lognormal(0.5)	.8535	.8420	.8811	.8954	.9096	**.9311**	.8455	.8567	.8717	.9014
Lognormal(1)	.0973	.0899	.0684	.0745	.0828	**.1004**	.0587	.0622	.0669	.0808
Lognormal(1.2)	**.0449**	.0367	.0289	.0297	.0308	.0341	.0231	.0249	.0266	.0313
Beta(0.5,1)	.0555	.0503	.2559	.2274	.1998	.1341	**.2875**	.2686	.2507	.2022

Table 4. Power comparisons of various tests for $n = 20$ and $\alpha = 0.05$.

Alternatives	KLV	KLC	TV^W	TC^W	$TC1^W$	$TC2^W$	TV^Z	TC^Z	$TC1^Z$	$TC2^Z$
Gamma(0.5)	.1264	.0869	**.6789**	.6666	.6403	.5571	.6725	.6724	.6626	.6287
Gamma(0.8)	.0251	.0231	.1307	.1252	.1142	.0797	.1369	**.1373**	.1290	.1091
Weibull(0.5)	.5499	.4595	**.9093**	.9057	.8974	.8601	.9055	.9063	.9000	.8887
Weibull(0.8)	.0306	.0236	.1941	.1903	.1727	.1200	**.1969**	.1948	.1847	.1620
GE(0.2)	.9319	.8944	**.9993**	.9992	.9990	.9980	.9991	.9990	.9989	.9988
GE(0.8)	.0310	.0272	.1247	.1191	.1080	.0722	.1371	**.1380**	.1310	.1138
Uniform	.8676	**.8759**	.4881	.4742	.4647	.4984	..4750	.4616	.4614	.4671
Weibull(2)	.9165	.9162	.8959	.9018	.9078	**.9331**	.8782	.8859	.8916	.9130
Gamma(1.5)	.2232	.2190	.1923	.2065	.2257	**.2696**	.1663	.1728	.1829	.2232
Gamma(2)	.5018	.5020	.4957	.5108	.5358	**.6047**	.4260	.4390	.4626	.5354
Gamma(3)	.8905	.8787	.9104	.9184	.9279	**.9496**	.8888	.8951	.9041	.9355
Beta(1,2)	.3391	**.3587**	.1744	.1790	.1916	.2094	.1760	.1796	.1807	.1978
Beta(2,1)	1	1	.9979	.9970	.9966	.9981	.9975	.9968	.9959	.9967
Lognormal(0.5)	.9921	.9872	.9986	.9989	.9992	**.9998**	.9980	.9982	.9990	.9994
Lognormal(1)	**.1546**	.1334	.0848	.0876	.1050	.1407	.0734	.0767	.0864	.1133
Lognormal(1.2)	**.1136**	.0825	.0151	.0158	.0191	.0233	.0140	.0144	.0165	.0213
Beta(0.5,1)	.1897	.1642	.4830	.4571	.4279	.3514	.5142	**.5477**	.4765	.4217

Tables 3 and 4 show the estimated powers of the proposed tests and those of the competing tests, at the significance level $\alpha = 0.05$, based on the results of 10000 simulations (of sample sizes 10, 20).

It is evident from Tables 3 and 4 that the tests based on transformed data behaves better than the tests based on the original data. Also it can be seen that for $n = 10$, the test statistic TV^Z has maximum powers for the alternatives with decreasing hazard rates. However, for $n = 20$, TV^W behaves better than other test statistics for the alternatives with decreasing hazard rates and $TC2^W$ has maximum powers for the alternatives with increasing hazard rates.

Example 1. As an example of application we use the data set given by Proschan (1963), which has also been used by Choi et al. (2004). The data are recording times to failure of air conditioning equipment for an aircraft. The failure times are

12 21 26 27 29 29 48 57 59 70 74 153 326 386 502.

By transforming data set of size $n = 15$ to two data sets of sizes $n' = 240$ and $n'' = 120$ and using corresponding $m = 15$ and $m = 11$, respectively, the test statistics are computed as follows:

$$\begin{aligned} TV^W &= 0.1164\ (0.2770), & TC^W &= 0.0649\ (0.2287),\\ TC1^W &= 0.0452\ (0.2019), & TC2^W &= 0.0333\ (0.1930),\\ TV^Z &= 0.1527\ (0.3296), & TC^Z &= 0.0885\ (0.2622),\\ TC1^W &= 0.0562\ (0.2187), & TC2^W &= 0.0491\ (0.2088), \end{aligned}$$

where the values in parentheses are the critical values of the test statistics at $\alpha = 0.05$ found from Table 1 and Table 2. None of the test statistics are greater than the corresponding critical values and so, we can not reject the hypothesis that the failure time is exponentiality distributed. The same result was obtained by Choi et al. (2004).

4 Conclusion

In this paper, to construct a goodness of fit test for exponentiality, we consider two transformations of the observations which turn the test of exponentiality into one of uniformity and use a corresponding test based on Renyi entropy. Four different estimators of Renyi entropy including estimators of Wachowiak et al. (2005) and Abbasnejad (2012), were used to estimate the test statistics. A Monte Carlo simulation was carried out for power comparisons under several alternatives. Based on the simulation results, we observed that the proposed tests have better powers than those of competing tests based on Kullback-Leibler information of the original data especially for the alternatives with monotone hazard rates.

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