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On Performance of Reconstructed Middle Order Statistics in Exponential Distribution

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Abstract. In a number of life-testing experiments, there exist situations where the monitoring breaks down for a temporary period of time. In such cases, some parts of the ordered observations, for example the middle ones, are censored and the only outcomes available for analysis consist of the lower and upper order statistics. Therefore, the experimenter may not gain the complete information on failure times for all experimental units. So, the accuracy of some statistical inferences may be decreases. In this paper, the effect of reconstructing missing order statistics on the performance of the maximum likelihood estimator (MLE) of the mean of the exponential distribution is investigated. To illustrate the proposed procedure in the paper, a real example is presented and using a simulation study, it will be shown that the reconstructing missing order statistics improves the estimation of the parameter of interest.

Keywords. Fisher information (FI); conditional median (mean) reconstructor; convex combination reconstructor.

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1 Introduction

There are some situations in life-testing and reliability experiments that the failure times are not monitoring continuously. For example, consider the

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situations that the data points are not observable when they fall inside an interval. Suppose n units are placed on test simultaneously; the first few failure times may be recorded at the beginning of the experiment, then the monitoring equipment breaks down for a temporary period of time due to negligence or problems, while the last few observations are observed. In such situations, the experimenter may not acquire the complete information on failure times for all experimental units. Therefore, the performance of some statistical inferences may be significantly decreased. In this case, the main question arises is that "How can one recover the lost information?". This model introduced by Razmkhah et al. (2010) in which they obtained some point and interval reconstructors for the missing order statistics. Recently, the reconstruction of missing ordered data have been studied by some authors. Klimczak and Rychlik (2005) provided the optimal upper bounds for the increments of order and record statistics under condition that the values of future order statistics and records are known. Balakrishnan et al. (2009) determined various point and interval reconstructors of previous record values in two-parameter exponential and Pareto models. Asgharzadeh et al. (2012) studied reconstruction of the past failure times in the left-censored set-up. In this paper, we investigate the effect of reconstructing missing data on the precision of the statistical inferences, when the underlying distribution is one-parameter exponential.

To determine the amount of lost information due to censored sample, we consider the FI criterion. Under certain regularity conditions, the FI about the real parameter θ contained in a random variable X with probability density function (pdf) $f_\theta(x)$ is defined by $I_X(\theta) = -\mathrm{E}\big(\frac{\partial^2}{\partial\theta^2}\log f_\theta(X)\big)$ (see, for example Lehmann and Casella, 1998, p. 115 or Rao, 1973, p. 329). It is well-known that the FI plays an important role in statistical inference through the information (Cramer-Rao) inequality and its association with the asymptotic properties, specially the asymptotic variance of the MLE.

The rest of this paper is as follows: In Section 2, some preliminaries and auxiliary results are presented. The amount of FI contained in order statistics about the mean of the exponential distribution has been given in Section 3. Section 4 focuses on the estimation for the mean of exponential distribution. In Section 5, a real example is presented and the performance of the MLE of θ is investigated in view of the mean squared error (MSE) criterion via a simulation study. In Section 6, some concluding remarks are presented.

2 Preliminaries and Auxiliary Results

Suppose n independent and identical units are placed on a life test with corresponding lifetimes $X_1, \ldots, X_n$ with pdf f and cumulative distribution function (cdf) F. Denote the ith order statistic of the sample $X_1, \ldots, X_n$ by $X_{i:n}$ and assume that some of the middle order statistics are lost, i.e., we only have observed the data set

$$\mathbf{X}_{r,s,n} = \{X_{1:n}, \ldots, X_{r:n}, X_{s:n}, \ldots, X_{n:n}\},$$

where $0 \leqslant r < s \leqslant n+1$. For convenience of notation, we let $X_{0:n} = 0$ and $X_{n+1:n} = \infty$. If $r = 0$, we indeed observe $\{X_{s:n}, \ldots, X_{n:n}\}$ which coincides with the left censoring model and in the case of $s = n+1$, the data set $\{X_{1:n}, \ldots, X_{r:n}\}$ is observed where is the same as Type-II censored data. The main goal of this paper is to use the reconstructed missing data in order to improve the performance of the MLE of the parameter of interest. Toward this end, we use the results in Razmkhah et al. (2010) to recover the lost information in the sample. They introduced three point reconstructors for the missing order statistics from a middle part of the random sample based on the data set $\mathbf{X}_{r,s,n}$. Here, we briefly present their results and focus on the case of exponential distribution which will be used in what follows.

It is well-known that a random variable X is said to have an exponential distribution with mean θ, denoted by $\exp(\theta)$, if its cdf is

$$F(x) = 1 - e^{-\frac{x}{\theta}}, \qquad x > 0, \quad \theta > 0. \tag{1}$$

From (1), we immediately find that $F^{-1}(x) = -\theta \log(1-x),\ 0 < x < 1$. The exponential distribution is the simplest and most important distribution in reliability studies, and is applied in a wide variety of statistical procedures, especially in life testing problems. See, Balakrishnan and Basu (1995) for some researches based on this distribution.

2.1 Conditional Median (CM) Reconstructor

It is reasonable to consider the median of the conditional density of $X_{l:n}$ given $\mathbf{X}_{r,s,n}$ as a reconstructor of $X_{l:n}$. Hence, we say that $\hat{X}_{l:n}^{CM}$ is the CM reconstructor of $X_{l:n}$, if $P\{X_{l:n} \leqslant \hat{X}_{l:n}^{CM} | \mathbf{X}_{r,s,n}\} = P\{X_{l:n} \geqslant \hat{X}_{l:n}^{CM} | \mathbf{X}_{r,s,n}\}$. Razmkhah *et al.* (2010) showed that

$$\hat{X}_{l:n}^{CM} = F^{-1}\Big[F(X_{r:n}) + \widetilde{m}(r,l,s)\big\{F(X_{s:n}) - F(X_{r:n})\big\}\Big], \tag{2}$$

where $\widetilde{m}(r,l,s)$ is the median of beta distribution with parameters $l-r$ and $s-r$, denoted by $Beta(l-r,s-r)$.
For the $\exp(\theta)$ distribution, using (1) and (2), we find

$$\hat{X}_{l:n}^{CM} = X_{r:n} - \theta \log\left\{1 - \widetilde{m}(r,l,s)\big(1 - e^{-W_{r,s}/\theta}\big)\right\}, \tag{3}$$

where $W_{r,s} = X_{s:n} - X_{r:n}$ is a quasirange of the sample.

Using (2), the CM reconstructor for other parametric distributions can be easily found. Some examples are presented here:

- The CM reconstructor of the lth order statistic for the case of Pareto distribution with cdf $F(x) = 1 - (\frac{\beta}{x})^{\alpha},\ x \geqslant \beta$, is given by

$$\hat{X}_{l:n}^{CM} = X_{r:n}\left[1 - \widetilde{m}(r,l,s)\left\{1 - \left(\frac{X_{r:n}}{X_{s:n}}\right)^{\alpha}\right\}\right]^{\frac{1}{\alpha}}. \tag{4}$$

- For the Weibull distribution with cdf $F(x) = 1 - e^{-\lambda x^{\alpha}},\ x > 0$, we have

$$\hat{X}_{l:n}^{CM} = \left[X_{r:n}^{\alpha} - \frac{1}{\lambda}\log\left\{1 - \widetilde{m}(r,l,s)\left(1 - e^{-\lambda(X_{s:n}^{\alpha} - X_{r:n}^{\alpha})}\right)\right\}\right]^{\frac{1}{\alpha}}. \tag{5}$$

- Consider the Fréchet distribution with cdf $F(x) = e^{-\frac{\theta}{x}}$, $x > 0$, then,

$$\hat{X}_{l:n}^{CM} = \left[\frac{1}{X_{r:n}} - \frac{1}{\theta}\log\left\{1 - \widetilde{m}(r,l,s)\left(1 - e^{-\theta(\frac{1}{X_{s:n}} - \frac{1}{X_{r:n}})}\right)\right\}\right]^{-1}. \tag{6}$$

2.2 Unbiased Conditional (UC) Reconstructor

The mean of the conditional density of $X_{l:n}$ given $\mathbf{X}_{r,s,n}$ is called the UC reconstructor of $X_{l:n}$, denoted by $\hat{X}_{l:n}^{UC}$. Razmkhah et al. (2010) considered the UC reconstructor of $X_{l:n}$ as follows

$$\hat{X}_{l:n}^{UC} = F^{-1}\left[F(X_{r:n}) + \frac{l-r}{s-r}\big\{F(X_{s:n}) - F(X_{r:n})\big\}\right]. \tag{7}$$

Using (7), for the case of $\exp(\theta)$ distribution, we have

$$\hat{X}_{l:n}^{UC} = X_{r:n} - \theta\log\left\{1 - \frac{l-r}{s-r}\big(1 - e^{-W_{r,s}/\theta}\big)\right\}. \tag{8}$$

Corollary 1. Using (7), the UC reconstructor of the lth order statistic in a sample of size n from Pareto, Weibull and Fréchet distributions can be obtained by replacing $\frac{l-r}{s-r}$ instead of $\widetilde{m}(r,l,s)$ in (4), (5) and (6), respectively.

2.3 Convex Combination (CC) Reconstructor

A convex combination of $X_{r:n}$ and $X_{s:n}$ has been considered as a distribution-free reconstructor for $X_{l:n}$, denoted by $\hat{X}_{l:n}^{CC}$, i.e.,

$$\hat{X}_{l:n}^{CC} = w(r,l,s,n)X_{r:n} + \big\{1 - w(r,l,s,n)\big\}X_{s:n}, \qquad r < l < s, \quad 0 < w(r,l,s,n) < 1. \tag{9}$$

Notice that the optimal value of $w(r,l,s,n)$ in (9) can be determined by minimizing the MSE of $\hat{X}_{l:n}^{CC}$; Therefore, it depends on the distribution of the underlying population. For the case of $\exp(\theta)$ distribution, the optimal value of $w(r,l,s,n)$ can be determined as follows (see, Razmkhah et al., 2010)

$$w_{opt}(r,l,s,n) = \frac{\varphi_2(l,s,n) + \varphi_1(r,l,n)\varphi_1(l,s,n)}{\varphi_2(r,l,n) + \varphi_2(l,s,n) + 2\varphi_1(r,l,n)\varphi_1(l,s,n)}, \tag{10}$$

where

$$\varphi_1(r,l,n) = \sum_{i=r+1}^{l} \frac{1}{n-i+1}$$

and

$$\varphi_2(r,l,n) = \sum_{i=r+1}^{l} \frac{1}{(n-i+1)^2} + \varphi_1^2(r,l).$$

3 Fisher Information

The FI contained in ordered data has been studied by several authors. For example, Park et al. (2008) studied the decomposition of FI in hybrid censored data and showed that the additivity rule of FI is satisfied in the case of hybrid censored data. Hofmann and Nagaraja (2003) investigated FI in record data. Balakrishnan et al. (2008) studied the optimal plans in progressively Type II censored samples based on FI. We refer the readers for more details regarding the FI in ordered data to the recent survey paper by Zheng et al. (2009) and the references cited therein.

Suppose that $\mathbf{X} = (X_1, \ldots, X_n)$ is a random sample from exponential distribution with cdf (1) and assume we only observe the data set $\mathbf{X}_{r,s,n}$. On the other words, suppose that a middle block of order statistics, $X_{r+1:n}, \ldots, X_{s-1:n}$, is lost. In order to compute the FI contained in the data set $\{X_{r+1:n}, \ldots, X_{s-1:n}\}$, we use the results obtained by Park and Kim (2006). They showed that

$$I_{X_{r+1:n},\ldots,X_{s-1:n}}(\theta) = \frac{s-r-2}{\theta^2} + I_{X_{r:n}}(\theta). \tag{11}$$

By considering the exact expression for $I_{X_{r:n}}(\theta)$ in Arnold et al. (1992, p. 166), we have

$$\theta^2 I_{X_{r:n}}(\theta) = \begin{cases} 1, & r=1 \\ 1+2n(n-1)\sum_{j=1}^{\infty}\frac{1}{(n+j)^3}, & r=2 \\ 1+\frac{n(n-r+1)}{r-2}\left\{\left(\sum_{i=1}^{r-2}\frac{1}{n-i}\right)^2+\sum_{i=1}^{r-2}\frac{1}{(n-i)^2}\right\}, & r\geqslant 3. \end{cases} \tag{12}$$

To investigate the amount of lost FI about θ, we take the quantity

$$\tau(n,r,s) = \frac{I_{X_{r+1:n},\ldots,X_{s-1:n}}(\theta)}{I_{\mathbf{X}}(\theta)}.$$

Tables 1 and 2 contain the numerical values of $\tau(n,r,s)$ for $n = 10$ and $n = 20$, respectively, and some selected values of r and s.

Table 1. Values of $\tau(10, r, s)$ for some selected r and s.

	r				
s	1	2	3	4	5
3	0.1				
4	0.2	0.1994			
5	0.3	0.2994	0.2975		
6	0.4	0.3994	0.3975	0.393	
7	0.5	0.4994	0.4975	0.493	0.484
8	0.6	0.5994	0.5975	0.593	0.584
9	0.7	0.6994	0.6975	0.693	0.684
10	0.8	0.7994	0.7975	0.793	0.784

Table 2. Values of $\tau(20, r, s)$ for some selected r and s.

	r									
s	1	2	3	4	5	6	7	8	9	10
3	0.05									
4	0.10	0.0999								
5	0.15	0.1499	0.1497							
6	0.20	0.1999	0.1997	0.1993						
7	0.25	0.2499	0.2497	0.2493	0.2484					
8	0.30	0.2999	0.2997	0.2993	0.2984	0.2971				
9	0.35	0.3499	0.3497	0.3493	0.3484	0.3471	0.3451			
10	0.40	0.3999	0.3997	0.3993	0.3984	0.3971	0.3951	0.3921		
11	0.45	0.4499	0.4497	0.4493	0.4484	0.4471	0.4451	0.4421	0.4379	
12	0.50	0.4999	0.4997	0.4993	0.4984	0.4971	0.4951	0.4921	0.4879	0.4821
13	0.55	0.5499	0.5497	0.5493	0.5484	0.5471	0.5451	0.5421	0.5379	0.5321
14	0.60	0.5999	0.5997	0.5993	0.5984	0.5971	0.5951	0.5921	0.5879	0.5821
15	0.65	0.6499	0.6497	0.6493	0.6484	0.6471	0.6451	0.6421	0.6379	0.6321
16	0.70	0.6999	0.6997	0.6993	0.6984	0.6971	0.6951	0.6921	0.6879	0.6821
17	0.75	0.7499	0.7497	0.7493	0.7484	0.7471	0.7451	0.7421	0.7379	0.7321
18	0.80	0.7999	0.7997	0.7993	0.7984	0.7971	0.7951	0.7921	0.7879	0.7821
19	0.85	0.8499	0.8497	0.8493	0.8484	0.8471	0.8451	0.8421	0.8379	0.8321
20	0.90	0.8999	0.8997	0.8993	0.8984	0.8971	0.8951	0.8921	0.8879	0.8821

From Tables 1 and 2, it is observed that for fixed n, $\tau(n, r, s)$ is decreasing in r and increasing in s as we expect. It is also deduced that often (when s is close to n) the missing order statistics contain plausible FI that motivate us to reconstruct them. It seems reasonable that the performance of our statistical inference increases by reconstructing missing order statistics. In the next section, we study the effect of reconstructing missing order statistics on the performance of the MLE of θ, which is related to the FI about θ via Cramer-Rao inequality.

4 Estimation for Mean of the Exponential Distribution

In this section, we assume that $X_1, \ldots, X_n$ is a random sample of exponential distribution with cdf (1). The MLE of θ is derived in different cases and their performances are compared in view of MSE criterion. Toward this end, we consider three different data sets namely: complete sample, data set $\mathbf{X}_{r,s,n} = \{X_{1:n}, \ldots, X_{r:n}, X_{s:n}, \ldots, X_{n:n}\}$ and modified sample which is built by adding the reconstructed missing order statistics to $\mathbf{X}_{r,s,n}$.

4.1 Complete Sample

If the complete sample is available, then it is obvious that the MLE of θ is the sample mean, i.e.,

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i, \tag{13}$$

and also $\mathrm{MSE}(\hat{\theta}) = \frac{\theta^2}{n}$.

4.2 Data Set $\mathbf{X}_{r,s,n}$ (Censored Sample)

Suppose that the available data set is $\mathbf{X}_{r,s,n}$ and we are interested to estimate the mean of the exponential distribution. It can be shown that in this case the MLE of θ, denoted by $\hat{\theta}_m = \varphi(\mathbf{X}_{r,s,n})$, is the solution of the following non-linear equation in terms of θ

$$\theta(n+r-s+1)\left(1-e^{-W_{r,s}/\theta}\right) + e^{-W_{r,s}/\theta}\left\{\sum_{i\in\Delta_{r,s}} X_{i:n} + (s-r-1)X_{s:n}\right\} = \sum_{i\in\Delta_{r,s}} X_{i:n} + (s-r-1)X_{r:n}, \tag{14}$$

where $\Delta_{r,s} = \{1,\ldots,r,s,\ldots,n\}$.

4.3 Modified Sample

If we reconstruct the lost order statistics, the modified MLE of θ can be obtained by substituting the reconstructed ones in (13). Suppose that the data set

$$\{X_{1:n},\ldots,X_{r:n},\hat{X}_{r+1:n},\ldots,\hat{X}_{s-1:n},X_{s:n},\ldots,X_{n:n}\}$$

is available, then we can use the modified form of (13) as follows

$$\hat{\hat{\theta}} = \frac{1}{n}\left\{\sum_{i\in\Delta_{r,s}} X_{i:n} + \sum_{i=r+1}^{s-1}\hat{X}_{i:n}\right\}, \tag{15}$$

where $\hat{X}_{i:n}$ is the reconstructor of $X_{i:n}$. To obtain the modified MLE's of θ, we use three reconstructors presented in Section 2.
Substituting the CM reconstructor (3) in (15), we have

$$\hat{\theta}_{CM} = \frac{1}{n}\Bigg[\sum_{i\in\Delta_{r,s}} X_{i:n} + (s-r-1)X_{r:n} - \varphi(\mathbf{X}_{r,s,n})\sum_{l=r+1}^{s-1}\log\left\{1-\widetilde{m}(r,l,s)\left(1-e^{-\frac{W_{r,s}}{\varphi(\mathbf{X}_{r,s,n})}}\right)\right\}\Bigg], \quad (16)$$

where $\varphi(\mathbf{X}_{r,s,n})$ is the solution of the Eq. (14) in terms of θ.
By considering the UC reconstructor (8), we get

$$\hat{\theta}_{UC} = \frac{1}{n}\Bigg[\sum_{i\in\Delta_{r,s}} X_{i:n} + (s-r-1)X_{r:n} - \varphi(\mathbf{X}_{r,s,n})\sum_{l=r+1}^{s-1}\log\left\{1-\frac{l-r}{s-r}\left(1-e^{-\frac{W_{r,s}}{\varphi(\mathbf{X}_{r,s,n})}}\right)\right\}\Bigg]. \quad (17)$$

Using the CC reconstructor (9), we find

$$\hat{\theta}_{CC} = \frac{1}{n}\Bigg\{\sum_{i\in\Delta_{r-1,s+1}} X_{i:n} + a(r,s,n)X_{r:n} + b(r,s,n)X_{s:n}\Bigg\}, \quad (18)$$

where

$$b(r,s,n) = s-r-a(r,s,n) \quad (19)$$

and

$$a(r,s,n) = 1 + \sum_{l=r+1}^{s-1} w_{opt}(r,l,s,n), \quad (20)$$

where $w_{opt}(r,l,s,n)$ is as defined in (10).

To compare the performance of the mentioned estimators based on different reconstructors, we use the MSE criterion. The MSE of $\hat{\theta}_{CC}$ can be derived as follows

$$\theta^{-2}\mathrm{MSE}(\hat{\theta}_{CC}) = \xi_1(n,r,s) + \big\{\xi_2(n,r,s)-1\big\}^2, \quad (21)$$

where

$$\xi_1(n,r,s) = \frac{\theta^2}{n^2}\Bigg[\sum_{i=1}^{r-1}(2n-2i-1)\psi_1(n,i) + \sum_{i=s+1}^{n}(2n-2i+1)\psi_1(n,i) + \Big\{a^2(r,s,n) + 2(n-s)a(r,s,n) + 2a(r,s,n)b(r,s,n)\Big\}\psi_1(n,r) + \Big\{b^2(r,s,n) + 2(n-s)b(r,s,n)\Big\}\psi_1(n,s)\Bigg],$$

such that $b(r,s,n)$ and $a(r,s,n)$ are as defined in (19) and (20), respectively, and $\psi_1(n,i) = \sum_{j=1}^{i}\frac{1}{(n-j+1)^2}$. Also,

$$\xi_2(n,r,s) = \frac{\theta}{n}\Bigg\{\sum_{i\in\Delta_{r-1,s+1}}\psi_2(n,i) + a(r,s,n)\psi_2(n,r) + b(r,s,n)\psi_2(n,s)\Bigg\},$$

where $\psi_2(n,i) = \sum_{j=1}^{i}\frac{1}{n-j+1}$.

The MSE of $\hat{\theta}_{CM}$ and $\hat{\theta}_{UC}$ cannot be derived theoretically. Therefore, we use a simulation study in the next section.

5 Computational Results

To illustrate the performance of the proposed procedure in this paper, we first present a real example and then carry out a simulation study.

5.1 Real Example

Consider a data set concerning the rock crushing machine which was reported by Dunsmore (1983) as a real data set where comes from the exponential distribution. A rock crushing machine has to be reset if at any operation the size of the rock being crushed is larger than the rock that has been crushed before. The following data are the size with up to the third time that the machine has been reset:

9.3 0.6 24.4 18.1 6.6 9.0 14.3 6.6 13.0 2.4 5.6 33.8

Using the above data set, the sample mean or equivalently the estimated value for θ based on the complete sample is $\hat{\theta} = 11.975$. Now, we first assume that $X_{5:12}$ and $X_{6:12}$ are lost, that is the data set $\mathbf{X}_{4,7,12} =$

$\{X_{1:12}, \ldots, X_{4:12}, X_{7:12}, \ldots, X_{12:12}\}$ is just observed and then use the modified estimators presented in Section 4 to estimate θ. Using (14), we get $\hat{\theta}_m = 11.9916$. Therefore, the CM, UC and CC reconstructors of $X_{5:12}$ and $X_{6:12}$ can be obtained by using (3), (8) and (9), respectively. The results are tabulated in Table 3.

Table 3. Values of various reconstructors of $X_{5:12}$ and $X_{6:12}$.

l	$\hat{X}^{CM}_{l:12}$	$\hat{X}^{UC}_{l:12}$	$\hat{X}^{CC}_{l:12}$
5	7.4037	7.5191	7.4306
6	8.6410	8.5144	8.4101

Using the results of Table 3, the modified estimators of θ can be derived by using (16)–(18) to be $\hat{\theta}_{CM} = 11.9895$, $\hat{\theta}_{UC} = 11.9888$ and $\hat{\theta}_{CC} = 12.5231$, respectively.

5.2 Simulation Study

Here, a simulation study is carried out in order to assess the performances of the various mentioned estimators of θ. With this in mind, the following algorithm has been applied:

(i) For given n, a random sample is generated from the standard exponential distribution.

(ii) The values of $\hat{\theta}_m$ and $\hat{\theta}_{CC}$ are calculated from Eqs. (14) and (18), respectively.

(iii) Using the value of $\hat{\theta}_m$ in part (ii), the values of $\hat{\theta}_{CM}$ and $\hat{\theta}_{UC}$ are obtained from Eqs. (16) and (17), respectively.

(iv) The quantity $e_i^2 = (\hat{\theta}_i - 1)^2$ is computed.

(v) The steps (i)–(iv) are repeated 10^5 times and the MSE of $\hat{\theta}_i$ is calculated by averaging e_i^2.

The simulated results are presented in Tables 4 and 5 for $n = 10$ and $n = 20$, respectively, and some selected values of r and s. In these tables, the corresponding columns of $\hat{\theta}^*_{CC}$ represent the numerical values of MSE of $\hat{\theta}_{CC}$, which is calculated using Eq. (21), directly.

Table 4. alues of MSE of $\hat{\theta}_i$ for $n = 10$ and some selected r and s.

	$s = 9$					$s = 10$				
r	$\hat{\theta}_m$	$\hat{\theta}_{CM}$	$\hat{\theta}_{UC}$	$\hat{\theta}_{CC}$	$\hat{\theta}^*_{CC}$	$\hat{\theta}_m$	$\hat{\theta}_{CM}$	$\hat{\theta}_{UC}$	$\hat{\theta}_{CC}$	$\hat{\theta}^*_{CC}$
1	0.1272	0.1240	0.1213	0.1176	0.1175	0.1820	0.1716	0.1636	0.1480	0.1491
2	0.1196	0.1168	0.1147	0.1125	0.1128	0.1588	0.1502	0.1443	0.1368	0.1380
3	0.1140	0.1117	0.1101	0.1088	0.1088	0.1421	0.1351	0.1307	0.1273	0.1281
4	0.1093	0.1074	0.1063	0.1055	0.1055	0.1290	0.1234	0.1203	0.1189	0.1197
5	0.1054	0.1040	0.1033	0.1029	0.1030	0.1191	0.1147	0.1126	0.1120	0.1126
6	0.1029	0.1018	0.1015	0.1014	0.1012	0.1119	0.1085	0.1072	0.1068	0.1071
7	0.1010	0.1004	0.1004	0.1004	0.1003	0.1063	0.1037	0.1032	0.1029	0.1031

Table 5. Values of MSE of $\hat{\theta}_i$ for $n = 20$ and some selected r and s.

	$s = 19$					$s = 20$				
r	$\hat{\theta}_m$	$\hat{\theta}_{CM}$	$\hat{\theta}_{UC}$	$\hat{\theta}_{CC}$	$\hat{\theta}^*_{CC}$	$\hat{\theta}_m$	$\hat{\theta}_{CM}$	$\hat{\theta}_{UC}$	$\hat{\theta}_{CC}$	$\hat{\theta}^*_{CC}$
1	0.0826	0.0806	0.0784	0.0754	0.0752	0.1317	0.1257	0.1196	0.1058	0.1062
2	0.0783	0.0765	0.0745	0.0727	0.0724	0.1179	0.1126	0.1075	0.1004	0.1006
3	0.0743	0.0726	0.0709	0.0700	0.0698	0.1064	0.1018	0.0976	0.0951	0.0952
4	0.0708	0.0693	0.0678	0.0675	0.0673	0.0970	0.0930	0.0894	0.0900	0.0901
5	0.0677	0.0664	0.0650	0.0652	0.0650	0.0893	0.0858	0.0828	0.0852	0.0853
6	0.0650	0.0638	0.0626	0.0630	0.0629	0.0829	0.0798	0.0773	0.0806	0.0807
7	0.0628	0.0617	0.0607	0.0611	0.0608	0.0778	0.0751	0.0730	0.0765	0.0765
8	0.0606	0.0596	0.0587	0.0592	0.0590	0.0731	0.0707	0.0689	0.0725	0.0725
9	0.0587	0.0579	0.0571	0.0575	0.0573	0.0692	0.0671	0.0656	0.0689	0.0688
10	0.0570	0.0562	0.0556	0.0560	0.0558	0.0657	0.0639	0.0626	0.0655	0.0654
11	0.0555	0.0548	0.0543	0.0546	0.0544	0.0627	0.0611	0.0601	0.0624	0.0623
12	0.0542	0.0536	0.0532	0.0533	0.0532	0.0601	0.0587	0.0578	0.0596	0.0595
13	0.0531	0.0526	0.0522	0.0523	0.0522	0.0579	0.0566	0.0559	0.0572	0.0570
14	0.0521	0.0517	0.0514	0.0515	0.0514	0.0559	0.0548	0.0542	0.0551	0.0549
15	0.0513	0.0510	0.0508	0.0509	0.0507	0.0542	0.0532	0.0528	0.0533	0.0532
16	0.0508	0.0505	0.0504	0.0504	0.0503	0.0527	0.0520	0.0517	0.0519	0.0518
17	0.0503	0.0502	0.0502	0.0502	0.0501	0.0516	0.0510	0.0508	0.0509	0.0508

From Tables 4 and 5, it is observed that:

(i) Comparing the simulated values of MSE of $\hat{\theta}_{CC}$ and the corresponding exact values confirms the performance of the simulation results.

(ii) Reconstructing the missing order statistics improves the performance of the estimators of θ. On the other words, the MSE of $\hat{\theta}_{CM}$, $\hat{\theta}_{UC}$ and $\hat{\theta}_{CC}$ is less than that of $\hat{\theta}_m$.

6 Concluding Remarks

In this paper, we considered the situations in life-testing and reliability experiments in which a middle part of the sample is lost or not saved during the experiment. We studied the effect of reconstructing the missing data points on estimating the mean of the exponential distribution. We considered three various data sets and showed that reconstructing the missing order statistics improves the performance of the estimators of the parameter of interest.

The results of subsections 2.1 and 2.2 can be developed for other life distributions if their cdf have an explicit form. We have presented the CM and UC reconstructors for the missing order statistics of Pareto, Weibull and Fréchet distributions. In general, let us show the CM reconstructor in (3) by $\Phi(X_{r:n}, X_{s:n})$, moreover, suppose that X has $\exp(\theta)$ distribution and $Y = h(X)$, where $h(\cdot)$ is a monotone function. Assuming the data set $\{Y_{1:n}, \ldots, Y_{r:n}, Y_{s:n}, \ldots, Y_{n:n}\}$ is available, it can be shown that the CM reconstructor of $Y_{l:n}$ is as follows

$$\hat{Y}_{l:n}^{CM} = h\Big[\Phi\big\{h^{-1}(Y_{r:n}), h^{-1}(Y_{s:n})\big\}\Big], \qquad r < l < s,$$

where $h^{-1}(\cdot)$ is the inverse function of $h(\cdot)$.

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