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پژوهشکده‌ی آمار

Comparison of Estimates Using Record Statistics From Lomax Model: Bayesian and Non Bayesian Approaches

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Abstract. This paper address the problem of Bayesian estimation of the parameters, reliability and hazard function in the context of record statistics values from the two-parameter Lomax distribution. The ML and the Bayes estimates based on records are derived for the two unknown parametesr and the survival time parameters, reliability and hazard functions. The Bayes estimates are obtained based on conjugate prior for the scale parameter and discrete prior for the shape parameter of this model. This is done with respect to both symmetric loss function (squared error loss), and asymmetric loss function (linear-exponential (LINEX)) loss function. The maximum likelihood and the different Bayes estimates are compared via Monte Carlo simulation study. A practical example consisting of real record values including in the data from an accelerated test on insulating fluid reported by Nelson was used for illustration and comparison. Finally, Bayesian predictive density function, which is necessary to obtain bounds for predictive interval of future record is derived and discussed using a numerical example.

Keywords. Lomax distribution; Bayes estimation; Record values; Symmetric and asymmetric loss functions; Bayes predeiction; Simulation.

1 Introduction

Record values and the associated statistics are of interest and importance in many areas of real life applications involving data relating to meteorology, sport, economics and lifetesting. Many authors have studied records and associated statistics. Among them are Resnick (1987), Nagaraja (1988), Ahsanullah (1993, 1995), Arnold et al. (1992, 1998), Raqab and Ahsanullah (2001), and Raqab (2002).

The Lomax distribution can be considered as a mixture of the exponential-gamma distribution. Lomax (1954) used this distribution in the analysis of business failure data. Balkema and De Haan (1974) showed that this distribution arises as a limit distribution of residual lifetime at great age. Lomax distribution includes increasing and decreasing hazard rates as well. Lomax distribution has been shown to be useful for modelling and analizing the life time data in medical and biological sciences, engineering, etc. So, it has been received the greatest attention from theoretical and applied statisticians primarily due to its use in reliability and lifetesting studies. Many statistical methods have been developed for this distribution, for a review of Lomax distribution see Habibullah and Ahsanullah (2000), Upadhyay and Peshwani (2003) and Abd Ellah (2003) and the references of them. A great deal of research has been done on estimating the parameters of a Lomax using both classical and Bayesian techniques, and a very good summary of this works can be found in Johnson et al. (1994).

Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables with *cdf* $F(x)$ and *pdf* $f(x)$. Set $Y_n = \max(X_1, X_2, X_3, \ldots X_n)$, $n \geqslant 1$, X_j is said to be an upper record and is denoted by $X_{U(j)}$ if $Y_j > Y_{j-1}$, $j > 1$.

Let $X_{U(1)}, X_{U(2)}, X_{U(3)}, \ldots X_{U(n)}$ be the first n upper record values arising from a sequence $\{X_i\}$ of i.i.d Lomax variables with pdf

$$f(x) = \alpha\beta^{\alpha}(x+\beta)^{-\alpha-1}, \qquad x \geqslant 0, \quad \alpha, \beta > 0, \tag{1}$$

and distribution function

$$F(x) = 1 - \beta^{\alpha}(x+\beta)^{-\alpha}, \qquad x \geqslant 0, \quad \alpha, \beta > 0. \tag{2}$$

Where β is the scale parameter and α is the shape parameter. This version of the Lomax distribution "separates" the two parameters and often simplifies the algebra in the subsequent Bayesian manipulations.

The reliability function $R(t)$, and the hazard (instantaneous failure rate) function $H(t)$ at mission time t for the Lomax distribution are respectively:

$$R(t) = \beta^{\alpha}(t+\beta)^{-\alpha}, \qquad t > 0, \tag{3}$$

and

$$H(t) = \alpha(t+\beta)^{-1} \qquad t > 0. \tag{4}$$

For most statisticians, mainly interested in controlling the amount of variability, it has become standard practice to use the squared error loss function ($s.e.l$) (symmetric). The symmetric nature of this function gives equal weight to overestimation and underestimation, while in the estimation of parameters of life time model, overestimation may be more serious than underestimation or vice-versa. For example, in the estimation of reliability and failure rate functions, an overestimate is usually much more serious than underestimate, in this case the use of symmetric loss function may be inappropriate as it has been recognized by Basu and Ebrahimi (1992). This leads to the idea that an asymmetrical loss function may be more appropriate.

A number of asymmetric loss functions are proposed. One of the most popular is linear-exponential loss function (LINEX) which was introduced by Varian (1975) and many other authors including Basu and Ebrahimi (1992), Pandey (1997), Soliman (2000, 2002) and Soliman et al. (2006) who have used this loss function in different estimation problems. This function rises approximately exponentially on one side of zero and approximately linearly on the other side. Under the assumption that the minimal loss occurs at $\overset{*}{\phi} = \phi$, the LINEX loss function for $\phi = \phi(\alpha, \beta)$ can be expressed as:

$$L(\Delta) \propto \exp\{c\Delta\} - c\Delta - 1; \qquad c \neq 0, \tag{5}$$

where $\Delta = (\overset{*}{\phi} - \phi)$ and $\overset{*}{\phi}$ is an estimate of ϕ. The sign and magnitude of the shape parameter c represents the direction and degree of symmetry respectively (If $c > 0$, the overestimation is more serious than underestimation, and vice-versa). For c closed to zero, the LINEX loss is approximately squared error loss and therefore it is almost symmetric.

The posterior expectation of the LINEX loss function (5) is

$$E_\phi[L(\overset{*}{\phi} - \phi)] \propto \exp\{c\overset{*}{\phi}\}E_\phi[\exp\{-c\phi\}] - c(\overset{*}{\phi} - E_\phi(\phi)) - 1 \tag{6}$$

where $E_\phi(\cdot)$ denotes the posterior expectation with respect to the posterior density of ϕ. The Bayes estimator of ϕ, denoted by $\overset{*}{\phi}_{BL}$ under the LINEX loss function is the value $\overset{*}{\phi}$ which minimizes (6), it is

$$\overset{*}{\phi}_{BL} = -\frac{1}{c}\ln[E_\phi\{\exp(-c\phi)\}] \tag{7}$$

provided that the expectation $E_\phi\{\exp(-c\phi)\}$ exists and is finite (Calabria and Pulcini, 1996).

The objective of this paper is to obtain and compare several types of estimation based on record statistics for the two unknown parameter of the Lomax distribution, and the survival time parameters, namely the hazard and Reliability functions. A discussion of the maximum likelihood estimators is also included in section (2). In section (3), the Bayes estimators of the parameters of the model as well as the reliability and hazard functions, are derived based on upper record values using the conjugate prior on the shape parameter and discretizing the scale parameter to a finite number of values. The estimates are obtained using both the symmetric loss function ($s.e.l.$) and the asymmetric loss function (varian's linear-exponential (LINEX)). The maximum likelihood and Bayes estimates are compared via Monte Carlo simulation study. The last section provides Bayes prediction for future record with applied example.

2 Maximum Likelihood Estimation

The joint density function of the first n upper record values $\mathbf{x} \equiv (x_{u(1)}, x_{u(2)}, \ldots, x_{u(n)})$ is given by

$$f_{1,2,\ldots,n}(x_{u(1)}, x_{u(2)}, x_{u(3)}, \ldots, x_{u(n)}) = f(x_{u(n)}) \prod_{i=1}^{n-1} \frac{f(x_{u(i)})}{1 - F(x_{u(i)})},$$

$$-\infty < x_{u(1)} < x_{u(2)} < \cdots < x_{u(n)} < \infty,$$

$$\begin{aligned} f &= \alpha\beta^{\alpha}(x_{u(n)} + \beta)^{-\alpha-1} \prod_{i=1}^{n-1} \frac{\alpha\beta^{\alpha}(x_{u(i)} + \beta)^{-\alpha-1}}{\beta^{\alpha}(x_{u(i)} + \beta)^{-\alpha}} \\ &= \alpha\beta^{\alpha}(x_{u(n)} + \beta)^{-\alpha-1} \times \alpha^{n-1} \prod_{i=1}^{n-1} (x_{u(i)} + \beta)^{-1} \\ &= \alpha^{n}\beta^{\alpha}(x_{u(n)} + \beta)^{-\alpha} u \qquad (8) \end{aligned}$$

where $f(\cdot)$, and $F(\cdot)$ are given, respectively, by (1) and (2) after replacing x by $x_{u(i)}$, and $u = \prod_{i=1}^{n}(x_{u(i)} + \beta)^{-1}$.

The Likelihood function (8) reduce to

$$\ell(\alpha, \beta \mid \mathbf{x}) = \alpha^{n} u (x_{u(n)} + \beta)^{-\alpha} \qquad (9)$$

where $u = \prod_{i=1}^{n} (x_{u(i)} + \beta)^{-1}$. The log-likelihood function is

$$L(\alpha, \beta \mid \mathbf{x}) \equiv \ln \ell = n \ln(\alpha) - \alpha \ln(x_{u(n)} + \beta) - \sum_{i=1}^{n} \ln(x_{u(i)} + \beta) \qquad (10)$$

Assuming that the scale parameter β is known, using equation (10) the maximum likelihood estimator (MLE), $\hat{\alpha}_{ML}$ of the shape parameter α can be shown to be

$$\hat{\alpha}_{ML} = \frac{n}{\ln(x_{u(n)} + \beta) - \ln \beta} \tag{11}$$

If both of the parameters α and β are unknown, their MLE's $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ can be obtained by solving the following likelihood equations:

$$\frac{\partial L}{\partial \alpha} = 0, \quad \frac{\partial L}{\partial \beta} = 0 \tag{12}$$

By eliminating β between the two equation of (12) we obtain

$$\frac{n}{\beta\{\ln(x_{u(n)} + \beta) - \ln \beta\}} - \frac{n}{(x_{u(n)} + \beta)\{\ln(x_{u(n)} + \beta) - \ln \beta\}} - \sum_{i=1}^{n} \frac{1}{x_{u(i)} + \beta} = 0, \tag{13}$$

which maybe solved using for example, Newton-Raphson iteration scheme. This yields the maximum likelihood estimate $\hat{\beta}_{ML}$ of β. With $\hat{\beta}_{ML}$ now known, α is estimated from the second equation as follows

$$\hat{\alpha}_{ML} = \frac{n}{\ln(x_{u(n)} + \hat{\beta}_{ML}) - \ln \hat{\beta}_{ML}} \tag{14}$$

The corresponding MLE's $\hat{R}_{ML}(t)$, and $\hat{H}_{ML}(t)$ of $R(t)$ and $H(t)$ are given respectively by replacing α and β by $\hat{\alpha}_{ML}$ and $\hat{\alpha}_{ML}$ in equations (3) and (4).

3 Bayes Estimation

3.1 Known Scale Parameter β

Under the assumption that the scale parameter β is known, we assume a gamma $\gamma(a, b)$ conjugate prior for α as:

$$\pi(\alpha) = \frac{b^a \alpha^{a-1} \exp\{-b\alpha\}}{\Gamma(a)} \qquad \alpha > 0, \ a, b > 0. \tag{15}$$

Combining the likelihood function (9) and the prior density (15), we obtain the posterior density of α in the form:

$$\pi^*(\alpha \mid \mathbf{x}) = \frac{v^{(n+a)} \alpha^{(n+a-1)} \exp\{-\alpha v\}}{\Gamma(n+a)} \tag{16}$$

where $v = b + \ln\{x_{u(n)} + \beta\}$

Bayes estimator based on squared error loss function
Under a squared error loss function, the Bayes estimator $\tilde{\phi}_{BS}$ of a function $\phi(\alpha)$ is the posterior mean.

The Bayes estimator $\tilde{\alpha}_{BS}$ of α is then given by

$$\tilde{\alpha}_{BS} = E(\alpha \mid \mathbf{x}) = \int_0^\infty \alpha \pi^*(\alpha \mid \mathbf{x})\, d\alpha = \frac{n+a}{v} \tag{17}$$

Similarly, the Bayes estimator for the reliability function $R(t)$ is

$$\tilde{R}_{BS}(t) = \left\{ 1 + \frac{\ln(1+\frac{t}{\beta})}{v} \right\}^{-(n+a)} \tag{18}$$

The Bayes estimator for the hazard function $H(t)$ follows as

$$\tilde{H}_{BS}(t) = \frac{n+a}{v(t+\beta)} \tag{19}$$

Bayes estimator based on LINEX loss function
Under the LINEX loss function the Bayes estimator $\tilde{\alpha}_{BL}$ for α, using (7) is

$$\tilde{\alpha}_{BL} = -\frac{1}{c} \ln \left[\int_0^\infty \{\exp(-c\alpha)\} \pi^*(\alpha \mid \mathbf{x})\, d\alpha \right] \tag{20}$$

Using (16), this simplifies to

$$\tilde{\alpha}_{BL} = \frac{n+a}{c} \ln\left(1 + \frac{c}{v}\right) \tag{21}$$

The Bayes estimators for $R(t)$ and $H(t)$ are given by

$$\tilde{R}_{BL}(t) = -\frac{1}{c} \ln \left[\sum_{i=0}^{\infty} \frac{(-c)^i}{i!} \left\{ 1 + \frac{i \ln(1+\frac{t}{\beta})}{v} \right\}^{-(n+a)} \right] \tag{22}$$

and

$$\tilde{H}_{BL}(t) = \frac{n+a}{c} \ln \left\{ 1 + \frac{c}{v(t+\beta)} \right\} \tag{23}$$

3.2 Unknown Scale and Shape Parameters β and α

It is well-known that, for Bayes estimators, the performance depends on the form of the assumed prior distribution and the loss function. Under the assumption that both the parameters α and β are unknown no analogous reduction via sufficiency is possible for the likelihood corresponding to a sample of record from the Lomax density (1). Also, specifying a general joint prior for α and β leads to computational complexities. In trying to solve this problem and simplifing the Bayesian analysis we use the Soland's method. Soland (1969) considered a family of joint prior distributions that places continuous distributions on the shape parameter and discrete distributions on the scale parameter.

We assume that the scale parameter β is restricted to a finite number of values $\beta_1, \beta_2, \ldots, \beta_k$ with respective prior probabilities $\eta_1, \eta_2, \ldots, \eta_k$ such that $0 \leqslant \eta_j \leqslant 1$, and $\sum_{j=1}^{k} \eta_j = 1$. [*i.e.* $\Pr(\beta = \beta_j) = \eta_j$]. Further, suppose that conditional upon $\beta = \beta_j, \alpha$ has natural conjugate prior with distribution gamma (a_j, b_j) with density

$$\pi(\alpha \mid \beta = \beta_j) = \frac{b_j^{a_j} \alpha^{a_j - 1}}{\Gamma(a_j)} \exp\{-b_j \alpha\}; \qquad a_j, b_j;\ \alpha > 0. \tag{24}$$

Where a_j and b_j are chosen so as to reflect prior beliefs on α given that $\beta = \beta_j$.

Then given the set of the first n upper record values $\mathbf{x}$, the conditional posterior pdf of α is

$$\pi^*(\alpha \mid \beta = \beta_j, \mathbf{x}) = \frac{\upsilon_j^{A_j} \alpha^{A_j - 1}}{\Gamma(A_j)} \exp\{-\upsilon_j \alpha\}; \qquad A_j, \upsilon_j; \alpha > 0, \tag{25}$$

which is a gamma (A_j, B_j), where

$$A_j = a_j + n, \qquad B_j = b_j + \ln(x_{u(n)} + \beta_j) - \ln \beta_j. \tag{26}$$

On applying the discrete version of Bayes' theorem the marginal posterior probability distribution of β_j is

$$\begin{aligned} P_j &= \Pr(\beta = \beta_j \mid \mathbf{x}) \\ &= A \int_0^\infty \frac{\eta_j b_j^{a_j} \alpha^{n + a_j - 1} u_j}{\Gamma(a_j)} \exp[-\alpha\{b_j + \ln(x_{u(n)} + \beta_j)\} - \ln \beta_j]\, d\alpha \\ &= A \frac{\eta_j b_j^{a_j} u_j \Gamma(A_j)}{\upsilon_j^{A_j} \Gamma(a_j)} \end{aligned} \tag{27}$$

where A is a normalized constant given by

$$A^{-1} = \sum_{j=1}^{k} \frac{\eta_j b_j^{a_j} u_j \Gamma(A_j)}{\upsilon_j^{A_j} \Gamma(a_j)}$$

and $u_j = \prod_{i=1}^{n} (x_{u(i)} + \beta)^{-1}$.

Estimators Based on Squared Error Loss Function

The Bayes estimators $\tilde{\alpha}_{BS}$, $\tilde{\beta}_{BS}$ of the parameters α and β, under the $(s.e.l.)$ function are obtained using the posterior pdf's (25) and (27). The Bayes estimator for the shape parameter α is

$$\tilde{\alpha}_{BS} = \int_0^\infty \sum_{j=1}^k P_j \alpha \pi^*(\alpha \mid \beta = \beta_j, \mathbf{x})\, d\alpha$$

Using (25) this simplifies to

$$\tilde{\alpha}_{BS} = \sum_{j=1}^k \frac{P_j A_j}{v_j} \tag{28}$$

The Bayes estimator of β is given by

$$\tilde{\beta}_{BS} = \sum_{j=1}^k P_j \beta_j \tag{29}$$

Similarly, the Bayes estimators $\tilde{R}(t)_{BS}$ and $\tilde{H}(t)_{BS}$of the reliability and hazard functions $R(t)$ and $H(t)$ are given respectively by

$$\begin{aligned} \tilde{R}(t)_{BS} &= \int_0^\infty \sum_{j=1}^k P_j \exp\left\{-\alpha \ln\left(1+\frac{t}{\beta_j}\right)\right\} \pi^*(\alpha \mid \beta = \beta_j, \mathbf{x})\, d\alpha \\ &= \sum_{j=1}^k P_j \left\{1 + \frac{\ln(1+\frac{t}{\beta_j})}{v_j}\right\}^{-A_j} \end{aligned} \tag{30}$$

and

$$\tilde{H}(t)_{BS} = \sum_{j=1}^k P_j \left\{\frac{A_j}{v_j(t+\beta_j)}\right\} \tag{31}$$

where A_j and B_j are as given in (26).

Estimators Based on LINEX Loss Function

Under the LINEX loss function (5), the Bayes estimator $\tilde{\phi}_{BL}$ of a function $\phi(\alpha, \beta)$ is given by equation (7), the Bayes estimator for the scale parameter α

is

$$\tilde{\alpha}_{BL} = -\frac{1}{c}\ln\left[\int_0^\infty \sum_{j=1}^{k} P_j \exp\{-\alpha c\}\pi^*(\alpha \mid \beta=\beta_j, \mathbf{x})\, d\alpha\right]$$
$$= -\frac{1}{c}\ln\left[\sum_{j=1}^{k} P_j \left(1+\frac{c}{\upsilon_j}\right)^{-A_j}\right] \tag{32}$$

The Bayes estimators for β is

$$\tilde{\beta}_{BL} = -\frac{1}{c}\ln\left[\sum_{j=1}^{k} P_j \exp\{-c\beta_j\}\right] \tag{33}$$

Similarly, the Bayes estimator for the reliability function $R(t)$ follows as

$$\tilde{R}(t)_{BL} = -\frac{1}{c}\ln\left[\int_0^\infty \sum_{j=1}^{k} P_j \exp\{-cR(t)\}\pi^*(\alpha \mid \beta=\beta_j, \mathbf{x})\, d\alpha\right]$$

where $R(t)$ as given in (3). By using the exponential series, after some simplification we obtain

$$\tilde{R}(t)_{BL} = -\frac{1}{c}\ln\left[\sum_{j=1}^{k}\sum_{i=0}^{\infty} P_j \frac{(-c)^i}{i!}\left\{1+\frac{i\ln(1+\frac{t}{\beta_j})}{\upsilon_j}\right\}^{-A_j}\right] \tag{34}$$

The Bayes estimator for the hazard function $H(t)$ is obtained as

$$\tilde{H}(t)_{BL} = -\frac{1}{c}\ln\left[\sum_{j=1}^{k} P_j \left\{1+\frac{c}{\upsilon_j(t+\beta_j)}\right\}^{-A_j}\right] \tag{35}$$

To implement the calculations in this section, it is necessary to elicit the values of (β_j, η_j) and the hyperparameters (a_j, b_j) in the conjugate prior (24), for $j = 1, 2, \ldots, k$. The former pairs of values are fairly straight forward to specify, but for (a_j, b_j) it is necessary to condition prior beliefs about α on each β_j in turn, and this can be difficult in practice. An alternative method for obtaining the values (a_j, b_j) can be based on the expected value of the reliability function $R(t)$ conditional on $\beta = \beta_j$, which is given using (24) by

$$E_{\alpha|\beta_j}[R(t)|\beta=\beta_j] = \int \exp\left\{-\alpha\ln\left(1+\frac{t}{\beta_j}\right)\right\}\frac{b_j^{a_j}\alpha^{a_j-1}}{\Gamma(a_j)}\exp\{-\alpha b_j\}\, d\alpha$$
$$= \left\{1+\frac{\upsilon_j}{b_j}\right\}^{-a_j} \tag{36}$$

Now, suppose that prior beliefs about the lifetime distribution enable one to specify two values $(R(t_1), t_1)$, $(R(t_2), t_2)$. Thus, for these two prior values $R(t = t_1)$ and $R(t = t_2)$, the values of a_j and b_j for each value β_j, can be obtained numerically from (36). If there is no prior beliefs, a nonparametric procedure can be used to estimate the corresponding two different values of $R(t)$, see Martz and Waller (1982-pp 105).

4 Simulation Study and Comparisons

The expressions of the various estimators show that an analytical comparison of these estimators is not possible. So, in order to assess the performances of the estimators, a Monte Carlo simulation study was used.

4.1 The Case of Known β

Random samples of upper record of different sizes are generated and the estimates obtained in the previous sections are computed and compared in the following steps:

1. For given values $(a = 2,\ b = 2)$ generate $\alpha = 3.004$ from the prior pdf (15).

2. Using the value $\alpha = 3.004$ from step 1., with $\beta = 1.007$ (Known), we generate $n = 3, 5, 7$. Upper record values from the Lomax $(\alpha = 3.004,\ \beta = 1.007)$ with pdf (1).

3. The ML estimate $\hat{\alpha}_{ML}$ of α is computed using equation (11), and the corresponding ML estimates $\hat{R}_{ML}$ and $\hat{H}_{ML}$ of the reliability and hazard functions at some t (chosen here as $t = 0.75$) are obtained as described in section (2).

4. The different Bayes estimates of α, $R(t)$, and $H(t)$ are computed using equations (17), (18), (19), (21), (22), and (23).

5. Steps 1-4 are repeated 10000 times, and the estimated variance (EV) is computed. The computation results are displayed in table (1).

Table 1. EV of the estimates of α, $R(t)$, and $H(t)$ with $(t = 0.75)$.

n	$EV(\hat{\alpha}_{ML})$	$EV(\tilde{\alpha}_{BS})$	$EV(\alpha_{BL})$		
			$c = 0.5$	$c = 1$	$c = 5$
3	0.2076	0.1764	0.1663	0.1582	0.1269
5	0.1603	0.1452	0.1413	0.1379	0.1219
7	0.1305	0.1286	0.1269	0.1253	0.1169
	$EV(\hat{R}_{ML})$	$EV(\tilde{R}_{BS})$	$EV(\tilde{R}_{BL})$		
3	0.1134	0.1076	0.1096	0.1091	0.1090
5	0.1058	0.1044	0.1045	0.1046	0.1053
7	0.1034	0.1031	0.1031	0.1032	0.1035
	$EV(\hat{H}_{ML})$	$EV(\tilde{H}_{BS})$	$EV(\tilde{H}_{BS})$		
3	0.6627	0.3177	0.3723	0.3409	0.30482
5	0.3717	0.3288	0.3109	0.2968	0.20435
7	0.2868	0.2814	0.2735	0.2669	0.20362

4.2 The Case of Unknown α and β

As indicated in section (3.2), specifying a general joint prior for α and β leads to computational complexities. Soland's methods provides a simpler alternative which can serve to give good approximations to the corresponding more general case of assuming continuous priors for α and for α given β. To illustrate the application of the results using this method we consider the following example:

Example 1. Let us consider the first seven upper record values simulated from a two-parameter Lomax distribution (1) with shape parameter $\alpha = 3$, and scale parameter $\beta = 3$, they are as follows:

$$0.418225, \quad 0.95362, \quad 1.57713, \quad 2.76785, \quad 3.12518, \quad 3.354, \quad 4.25777.$$

Using this record values the different estimates of α, β, $R(t)$, and $H(t)$ are computed according to the following steps:

1. We approximate the prior for α over the interval $(2.5, 3.4)$ by the discrete prior with β taking the ten values 2.5(0.1)3.4, each with probability 0.1. There is no further prior information about α. A nonparametric procedure can be use to estimate any two different values of the reliability function $R(t_1)$, and $R(t_2)$, see Martz and Waller (1982-pp 105).

2. Substituting the values of $(R(t_1), t_1), (R(t_2), t_2)$ obtained in step 1. into equation (36), where a_j and b_j are obtained numerically for each given β_j, $j = 1, 2, \ldots, 10$. using the Newton-Raphson method. The resulting values of the hyperparameters (a_j, b_j) in the gamma prior are given in Table 2 as well as the posterior probabilities for each β_j (see equation (27)).

3. The ML estimates $(\cdot)_{ML}$, and the Bayes estimates $((\cdot)_{BS}, (\cdot)_{BL})$ of α, β, $R(t)$, and $H(t)$, are computed using results in section (3.2). The results are presented in Table 3.

4. Based on the sample of upper record of size $n = 7$, for $(\alpha = 3, \beta = 1)$ the estimated reliability $\hat{R}_{ML}$, $\tilde{R}_{BS}$ and $\tilde{R}_{BL}$ are plotted against t with the true value (REX) of $R(t)$ in Figure 1. Also, for the same parameters , in Figure 2. the estimated failure rate $\hat{H}_{ML}, \tilde{H}_{BS}$, and $\tilde{H}_{BL}$ with the true value (HEX) of $H(t)$ are plotted.

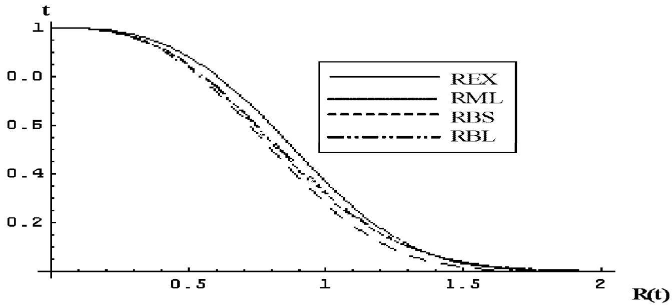

Figure 1. Estimated reliabilities as compared with actual $R(t)$.

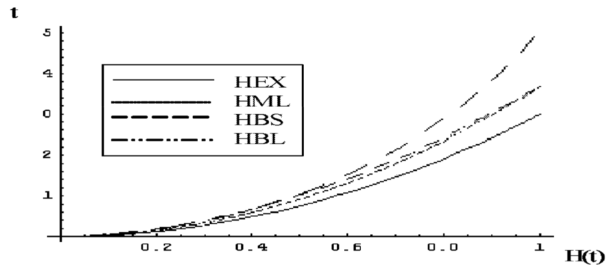

Figure 2. estimated failure function as compared with actual $H(t)$.

Table 2. Prior information, Hyper parameter values of the gamma and the posterior probabilities

j	**1**	**2**	**3**	**4**	**5**
β_j	2.5	2.6	2.7	2.8	2.9
η_j	0.1	0.1	0.1	0.1	0.1
a_j	1.823	0.9855	0.8735	0.7646	0.6577
b_j	1.873	0.8671	0.7532	0.6432	0.5357
u_j	45.396	65.4596	77.2167	87.4141	95.5195
P_j	0.14256	0.13345	0.12341	0.11318	0.10318
j	**6**	**7**	**8**	**9**	**10**
β_j	3.0	3.1	3.2	3.3	3.4
η_j	0.1	0.1	0.1	0.1	0.1
a_j	0.523	0.479	0.442	0.411	0.385
b_j	0.299	0.253	0.216	0.186	0.161
u_j	104.506	142.427	154.662	174.239	189.453
P_j	0.09362	0.08462	0.07623	0.06845	0.06130

Table 3. Estimates and the true value of α, β, $R(t)$ and $H(t)$ with ($t = 0.75$)

	True value	$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$		
				$c = 0.7$	$c = 1$	$c = 3$
β	2.0	2.118	2.0671	2.083	1.9786	1.967
				$c = -3$	$c = -1$	$c = 1$
α	4.0	3.7928	3.8741	3.9931	3.9140	3.8367
				$c = -2$	$c = -1$	$c = 1$
$R(t)$	0.16558	0.1619	0.1635	0.1649	0.1642	0.1626
				$c = 0.3$	$c = 0.4$	$c = 1$
$H(t)$	1.758	1.8171	1.761	1.7079	1.6857	1.573

Example 2. As another example we choose the real data set which was also used in Lawless (1982-pp 185).

These data are from Nelson (1982) concerning the data on time to breakdown of an insulating fluid between electrodes at a voltage of 34 k.V. (minutes). The 19 times to breakdown are:

0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91,
32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71, 72.89.

table 4. Prior information, Hyper parameter values of the gamma and the posterior probabilities

j	**1**	**2**	**3**	**4**	**5**
β_j	1.5	1.6	1.7	1.8	1.9
η_j	0.1	0.1	0.1	0.1	0.1
a_j	2.826	2.558	2.428	2.350	2.298
b_j	4.943	3.184	2.95	2.822	2.485
u_j	4	9.257	37.147	244.930	1532.46
P_j	0.566	0.247	0.109	0.048	0.021
j	**6**	**7**	**8**	**9**	**10**
β_j	2	2.1	2.7	3.3	3.9
η_j	0.1	0.1	0.1	0.1	0.1
fa_j	2.260	2.165	2.124	2.099	2.084
b_j	2.245	2.045	2.0397	2.0265	2.0184
u_j	9588.22	9.2×10^9	8.8×10^{11}	5.5×10^{14}	4.1×10^{21}
P_j	0.0089	0.0001	0	0	0

Therefore, we observe the upper record values from the observed data as follows:

$$0.96,\ 4.15,\ 8.01,\ 31.75,\ 33.91,\ 36.71,\ 72.89.$$

A model suggested by engineering considerations is that, for a fixed voltage level, time to breakdown has a Lomax distribution.

Based on these seven upper record values, the illustration of the results in this section is done for the conjugate prior on the scale parameter. The hyperparameters of the gamma prior (24) are derived, using (36). From Table 1 in Zimmer et al. (1998), we assume that the reliability for times $t_{1=}0.96$, and $t_2 = 6.50$ are respectively $R(t_1) = 0.85$, and $R(t_2) = 0.50$. As mention in example 1, these two prior values are substituted into (36), where a_j and b_j are solved numerically for each given β_j, $j = 1, 2, \ldots, 10$, using the Newton-Raphson method. Table 4 gives the values of the hyperparameters and the posterior probabilities derived for each β_j. The ML estimates $(\cdot)_{ML}$, and the Bayes estimates $((\cdot)_{BS}, (\cdot)_{BL})$ of α, β, $R(t)$, and $H(t)$, are computed and the results are displayed in Table 5.

Table 5. Estimates of α, β, $R(t)$ and $H(t)$ with $(t = 5)$

	$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$		
			$c = -1$	$c = 3$	$c = 5$
$\boldsymbol{\alpha}$	3.465	3.381	3.361	3.278	3.179
$\boldsymbol{\beta}$	1.599	1.748	1.809	1.680	1.525
$\boldsymbol{R(t)}$	2.451	2.6313	2.6571	2.428	2.325
$\boldsymbol{H(t)}$	1.281	1.32897	1.3942	1.098	1.089

5 Prediction of the Future Record

In the context of prediction of the future record observations, the prediction intervals provide bounds to contain the results of a future record, based upon the results of the previous records observed from the same sample. This section is devoted to drive the Bayes predictive density function, which is necessary to obtain bounds for predictive interval of future record. Suppose that we observe only the first n upper record observations $\mathbf{x} \equiv (x_{u(1)}, x_{u(2)}, \ldots, x_{u(n)})$, and the goal is to obtain the Bayes predictive interval for the s^{th} future upper record, where $1 \leqslant n < s$. Let $Y \equiv X_{U(s)}$ be the s^{th} upper record value, the conditional density function of Y for given $x_n = X_{U(n)}$ is given, by Ahsanullah (1995)

$$f(y|x_n;\theta) = \frac{\{w(y) - w(x_n)\}^{s-n-1}}{\Gamma(s-n)} \cdot \frac{f(y)}{1 - F(x_n)}, \tag{37}$$

where $w(\cdot) = -\ln[1 - F(\cdot)]$.

Using the Lomax distribution, with pdf given by (1), the conditional density function (37) is

$$f(y|x_n;\alpha,\beta) = \frac{\alpha^{s-n}\left\{\ln\left(\frac{y+\beta}{x+\beta}\right)\right\}^{s-n-1}}{\Gamma(s-n)(y+\beta)} \cdot \left(\frac{y+\beta}{x+\beta}\right)^{-\alpha}. \tag{38}$$

The Bayes predictive density function of y given the observed record $\mathbf{x}$ is given by

$$f(y|\mathbf{x}) = \int_\theta f(y|x_n;\alpha,\beta) \sum_{j=1}^{k} P_j \pi^*(\alpha|\beta = \beta_j, \mathbf{x})\, d\alpha. \tag{39}$$

Substituting from (38) and (25) into (39) we get

$$f(y|\mathbf{x}) = \sum_{j=1}^{k} \frac{P_j v_j^{A_j} u \left\{\ln\left(\frac{y+\beta_j}{x+\beta_j}\right)\right\}^{s-n-1}}{Bet(n+a_j, s-n)\left\{v_j + \ln\left(\frac{y+\beta_j}{x+\beta_j}\right)\right\}^{s+a_j}(y+\beta_j)}, \tag{40}$$

where A_j and B_j are as given by (26), $Bet(.,.)$ is the beta function of the second kind

$$Bet(z_1, z_2) = \int_0^\infty \frac{t^{z_1-1}}{(1+t)^{z_1+z_2}}\ dt. \tag{41}$$

It follows that the lower and upper $100\tau\%$ prediction bounds for $Y = X_{U(s)}$, given the past record values $\mathbf{x}$, can be derived using the predictive survival function defined by

$$\begin{aligned} f(Y \geqslant \lambda|\mathbf{x}) &= \int_\lambda^\infty f(y|\mathbf{x})\ dy \\ &= \sum_{j=1}^{k} \frac{P_j}{Bet(n+a_j, s-n)}\ InBet(n+a_j, s-n; \delta_j) \end{aligned} \tag{42}$$

where $\delta_j = \ln\left(\frac{\lambda+\beta_j}{x+\beta_j}\right)/v_j$, and $InBet(z_1, z_2, \xi)$ is the incomplete beta function defined by $InBet(z_1, z_2, \xi) = \int_\xi^\infty t^{z_1-1}/(1+t)^{z_1+z_2}\ dt$.

Iterative numerical methods are required to obtain the lower and upper $100\tau\%$ prediction bounds for $Y = X_{U(s)}$ by finding λ from equation (42), using

$$\Pr[LL(\mathbf{x}) < Y < UL(\mathbf{x})] = \tau$$

where $LL(\mathbf{x})$ and $UL(\mathbf{x})$ are the lower and upper limits respectively, satisfying

$$\Pr[Y > LL(\mathbf{x})|\mathbf{x}] = \frac{(1+\tau)}{2} \qquad \Pr[Y > UL(\mathbf{x})|\mathbf{x}] = \frac{(1-\tau)}{2} \tag{43}$$

It is often important to predict the first unobserved record value $X_{U(n+1)}$; the predictive survival function for $Y_{n+1} = X_{U(n+1)}$ is given from (42) by setting $s = n+1$ as

$$f(Y_{n+1} \geqslant \lambda|\mathbf{x}) = \sum_{j=1}^{k} P_j(1+\delta_j)^{-(n+a_j)}. \tag{44}$$

Iterative numerical methods are also required to obtain prediction bounds for Y_{n+1}.

5.1 An Illustration Examples

Example 3. The above prediction procedure is demonstrated by using a simulated sets of record from the Lomax model (1). A samples of upper record values of size $n = 3, 5, 7$ are simulated from the Lomax distribution with $(\alpha, \beta) = (2, 2)$, $(3, 4)$, $(4, 4)$, which include the exponential, Rayeligh, and Weibull models, respectively. Using our results in equation (44), the lower and

Table 6. The lower LL, the upper (UL) and the width of the 95% prediction intervals for the future upper record $X_{U(n+1)}$, ($n = 3, 5, 7$).

n	(α, β)	previous record values	LL	UL	Width
3	$(2, 2)$	0.1964, 2.2065, 6.2152	6.3529	673.39	667.04
5		3.40728, 3.83727, 5.62398, 7.2552, 7.46057	7.5766	121.70	114.12
7		.0489, 2.0905, 3.7928, 4.0789, 6.1966, 6.5367, 9.0686	9.1768	69.024	59.847
3	$(3, 4)$	.3323, 2.2286, 4.90786	5.0195	418.42	413.40
5		1.17076, 2.6028, 4.59304, 7.16314, 8.6684	8.806	155.24	146.43
7		.1542, 3.2405, 5.9598, 7.6089, 10.8167, 10.9072, 12.579	12.74	110.25	97.51
3	$(4, 4)$	.21733, 4.57068, 4.74837	4.8562	392.04	387.18
5		.9887, 3.01356, 7.90616, 8.9098, 9.72644	9.8835	187.13	177.25
7		.20008, 3.5655, 3.960, 5.3299, 8.8052, 9.292, 14.078	14.257	130.13	115.87

upper 95% prediction bounds for the next record values $X_{U(n+1)}$, for the three cases ($n = 3, 5, 7$) are obtained and displayed in Table 6.

Example 4. Based on the seven record values from Example 2, with the corresponding hyperparameter values obtained in the same example Table 4, and using the results in (43) and (44), the lower and upper 95% prediction bounds for the next record values $X_{U(8)}$ are respectively 73.125 and 116.725.

6 Conclusion

Based on the set of the upper record values the present paper proposed classical and Bayesian approaches to estimate the two unknown parameters as well as the reliability and hazard functions for Lomax model. We also consider the problem of predicting future record in a Bayesian setting. Bayes estimators are obtained using both symmetric and asymmetric loss functions. It appears to be clear from this study that the Bayes method of estimation based on record statistics is superior to the ML method. Comparisons are made between different estimators based on simulation study and practical example using a set of real record values. The effect of symmetric and asymmetric loss functions was examined and the following were observed:

1. Table 1 (the case of known β) shows that the Bayes estimates related to LINEX loss function have the smallest (EV) as compared with quadratic Bayes estimates or the MLE's. This is valid for all number of record values n, and the estimated variances decrease as n increasing.

2. For the case of unknown shape and scale parameters, the use of a discrete distribution for the shape parameter resulted in closed form expression for the posterior pdf. The equal probabilities chosen in the discrete distributions caused an element of uncertainly, which can be desirable in some cases. Table 3 shows that the Bayes estimates based on symmetric and asymmetric loss functions are perform better than the MLE's, and the asymmetric Bayes estimates are sensitive to the values of the shape parameter c of the LINEX loss function. The problem of choosing the value of the parameter c is discussed in Calabria and Pulcini (1996).

3. The analytical ease with which results can be obtained using asymmetric loss functions makes them attractive to use in applied problems and in assessing the effects of departures from assumed symmetric loss functions.

4. It is clear that the variance of the Lomax (α, β) distribution tends to be zero as β tends to infinity. This implies that as β gets larger, the observations concentrate on a shorter domain. It then follows that the width of the predictive interval decrease as β increase, see Table 6.

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