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پژوهشکده‌ی آمار

Some Results Based on Entropy Properties of Progressive Type-II Censored Data

Jafar Ahmadi

Ferdowsi University of Mashhad

Abstract. In many life-testing and reliability studies, the experimenter might not always obtain complete information on failure times for all experimental units. One of the most common censoring schemes is progressive type-II censoring. The aim of this paper is characterizing the parent distributions based on Shannon entropy of progressive type-II censored order statistics. It is shown that the equality of the Shannon information in progressive type-II censored order statistics can determine the parent distribution uniquely. We establish some characterization through the difference of Shannon entropy of the parent distribution and respective progressive type-II censored order statistics. We also prove that the dispersive ordering of the parent distributions implies the entropy ordering of their respective progressive type-II censored order statistics.

Keywords. exponential distribution; Müntz-Szász theorem; order statistics; reliability properties; stochastic orders; Weibull distribution; Pareto distribution.

1 Introduction

Progressive censored samples arise in life-time and reliability studies when life items are removed at various stages from the experiment. Saving on costs and time may be the consequence of such a sampling scheme. For a detailed and comprehensive discussion of progressive censoring scheme we refer to Balakrishnan and Aggrawala (2000) and the literatures cited therein.

Consider a reliability experiment in which n identical units are placed on a life-time test. Let $X_1, X_2, \ldots, X_n$ denote the life-times of these experimental units, with life-time cumulative distribution function, c.d.f., $F(x)$ and probability density function, p.d.f., $f(x)$. Suppose that a censoring scheme $\widetilde{R} = (R_1, \ldots, R_m)$, $m \leqslant n$, is fixed, such that immediately after the first failure time of one of the units, R_1 units are randomly withdrawn from the $n - 1$ remaining surviving units; at the second failure time, R_2 surviving units are selected at random and taken out of the experiment, and so on; finally, at the time of the mth failure, the remaining $R_m = n - m - \sum_{i=1}^{m-1} R_i$ objects are removed. The withdrawal of units may be seen as a model describing drop-outs of units due to failures, which have causes other than the specific one under study. The ordered values of m observed failure times, which are denoted by $X_{1:m:n}^{\widetilde{R}}, \ldots, X_{m:m:n}^{\widetilde{R}}$, are referred to as *progressive type-II right censored order statistics* of size m from a sample of size n with progressive censoring scheme $\widetilde{R} = (R_1, \ldots, R_m)$. In this paper, we denote them by $X_{1:m:n}, \ldots, X_{m:m:n}$ for simplicity. For the censoring scheme $\widetilde{R} = (0, \ldots, 0, n - m)$, this scheme reduces to the conventional type-II right censoring scheme, in which case just the first m usual order statistics are observed. Also, for the censoring scheme $\widetilde{R} = (0, \ldots, 0)$ so that $m = n$, we obtain the ordinary order statistics of $X_1, \ldots, X_n$, where no withdrawals are made. Thus, usual order statistics form a special case of progressive type-II right censored order statistics. So any result established for progressive type-II right censored order statistics becomes a generalization of the corresponding result for the ordinary order statistics (see Balakrishnan and Aggrawala, 2000, and also Balakrishnan, 2007, for more details).

The concept of Shannon's information (Shannon, 1948) plays the central role in information theory, sometimes referred as measure of uncertainty. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. A little work has been done in the context of Shannon's information properties of ordinary order statistics (see Wong and Chen, 1990; Park, 1995; Ebrahimi et al., 2004; Baratpour et al., 2007). In this paper, we consider some characterizations based on Shannon information of progressive type-II right censored order statistics. Recently, some authors have used the Fisher information in the usual order statistics as well as the usual type-I and type-II censored data to characterize some families of distributions (see, for example, Gertsbakh and Kagan, 1999; Zheng, 2001; Hofmann et al., 2005). Now, we ask the following question: can the parent distribution be characterized by its entropy? This question was also addressed by Baratpour et al. (2007, 2008) in the sense of

Shannon's and Rényi's entropies of usual order statistics, and they obtained several characterization results. Here, we describe conditions under which the Shannon information of progressive type-II right censored order statistics can uniquely determine the parent distribution F.

The rest of the paper is organized as follows. Section 2 contains some preliminaries and auxiliary results. In section 3, we present several characterization results in terms of entropy of progressive type-II censored order statistics. It is shown that the equality of the Shannon information in progressive type-II censored order statistics can uniquely determine the parent distribution uniquely up to a location shift. We prove that the difference between the entropy of the parent distribution and the entropy of appropriately chosen subsequences of the progressive type-II censored order statistics characterizes the parent distribution, but for a change of location and scale. In section 4, we give some ordering results. It is proved that the dispersive ordering of the parent distributions implies the entropy ordering of their respective progressive type-II censored order statistics.

2 Auxiliary Results

If X is a random variable having an absolutely continuous c.d.f. F with p.d.f. f, then the basic uncertainty measure for distribution F is defined as

$$H(X) = -\int_{-\infty}^{\infty} f(x) \log f(x)\ dx. \tag{1}$$

$H(X)$ is commonly referred to as the *entropy* of X or *Shannon information* measure. The probability integral transformation provides the following useful representation of the entropy of X:

$$H(X) = -\int_{0}^{1} \log f\left(F^{-1}(x)\right)\ dx. \tag{2}$$

For a non-negative random variable X, we have

$$H(X) = 1 - \int_{0}^{\infty} f(x) \log h_X(x)\ dx, \tag{3}$$

where $h_X(t) = f(t)/\bar{F}(t)$ is the hazard rate function of X, and $\bar{F}(t) = 1 - F(t)$ is the survival function of X.

In this paper, we explore properties of the entropy for progressive type-II right censored order statistics. If the failure times $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) with common absolutely continuous c.d.f. $F(x)$ and p.d.f. $f(x)$, then the joint density of $X_{1:m:n}, \ldots, X_{m:m:n}$ is given by

$$f_{X_{1:m:n},\ldots,X_{m:m:n}}(x_1,\ldots,x_m) = c\prod_{i=1}^{m} f(x_i)[\bar{F}(x_i)]^{R_i}, \qquad x_1 < x_2 < \cdots < x_m \tag{4}$$

where $c = n(n-R_1-1)\cdots(n-R_1-\cdots-R_{m-1}-m+1)$. Kamps and Cramer (2001) derived a simpler expression for the marginal distribution by carrying out necessary integration. Their expression for the c.d.f. of $X_{r:m:n}$ is given by

$$F_{X_{r:m:n}}(x) = 1 - c_{r-1}\sum_{i=1}^{r}\frac{a_{i,r}}{\gamma_i}[\bar{F}(x)]^{\gamma_i}, \qquad r = 1,\ldots,m \tag{5}$$

where

$$\gamma_r = m - r + 1 + \sum_{i=r}^{m} R_i \quad \text{and} \quad c_{r-1} = \prod_{i=1}^{r}\gamma_i, \qquad r = 1,\ldots,m$$

and

$$a_{i,r} = \prod_{\substack{j=1 \\ j\neq i}}^{r}\frac{1}{\gamma_j - \gamma_i}, \qquad r = 1,\ldots,m$$

the empty product $\prod_{\phi}$ is defined to be 1.

Upon differentiating the expression in (5) with respect to x, we readily obtain the p.d.f. of $X_{r:m:n}$ as

$$f_{X_{r:m:n}}(x) = c_{r-1}\sum_{i=1}^{r} a_{i,r} f(x)[\bar{F}(x)]^{\gamma_i - 1}, \qquad r = 1,\ldots,m \tag{6}$$

Note that since $\gamma_1 = m + \sum_{i=1}^{m} R_i = n$, the p.d.f. of $X_{1:m:n}$ does not involve m and the distributional properties of $X_{1:m:n}$ is exactly the same as the distribution of the usual smallest order statistic $X_{1:n}$ (see Balakrishnan and Aggrawala, 2000, p. 7). Thus the results known in terms of $X_{1:n}$, will hold for $X_{1:m:n}$.

From (5) and (6) we get

$$h_{X_{r:m:n}}(t) = h_X(t)k_r(t), \qquad r = 1,\ldots,m \tag{7}$$

where

$$k_r(t) = \frac{\sum_{i=1}^{r} a_{i,r}[\bar{F}(t)]^{\gamma_i - 1}}{\sum_{i=1}^{r}\frac{a_{i,r}}{\gamma_i}[\bar{F}(t)]^{\gamma_i - 1}}, \qquad r = 1,\ldots,m$$

It can be shown that $k_r(t)$ is an increasing function in t, so by the aforementioned identity (7), we can obtain some reliability relations between the original variable X and the corresponding progressive type-II right censored order statistics (not presented here). Block et al. (1985) obtained a relation similar to equation (7) for two non-negative random variables and studied the conditions for transmitting some reliability relations between them (see also Kamps, 1995, chapter V).

For the standard uniform distribution, denoting the rth progressive type-II right censored order statistic by $U_{r:m:n}$, from (6) we have

$$f_{U_{r:m:n}}(u) = c_{r-1}\sum_{i=1}^{r} a_{i,r}(1-u)^{\gamma_i-1}, \qquad 0<u<1, \quad r=1,\dots,m \tag{8}$$

Using the expression in (1)

$$\begin{aligned} H(U_{r:m:n}) &= -\int_0^1 c_{r-1}\sum_{i=1}^{r} a_{i,r}u^{\gamma_i-1}\log\left(c_{r-1}\sum_{i=1}^{r} a_{i,r}u^{\gamma_i-1}\right)\,du \\ &= -\log c_{r-1} - c_{r-1}\sum_{i=1}^{r} a_{i,r}\int_0^1 u^{\gamma_i-1}\log\left(\sum_{i=1}^{r} a_{i,r}u^{\gamma_i-1}\right)\,du. \end{aligned} \tag{9}$$

For $r=1$, (9) is simplified as

$$H(U_{1:m:n}) = 1-\frac{1}{\gamma_1}-\log\gamma_1. \tag{10}$$

The transformation formula for the entropy applied to $X_{r:m:n}=F^{-1}(U_{r:m:n})$ gives the following representations of the entropy of progressive type-II right censored order statistics.

Lemma 1. *Let $X_1, X_2, \dots, X_n$ be i.i.d. continuous random variables from the common absolutely continuous c.d.f. $F(x)$, p.d.f. $f(x)$ and entropy $H(X)<\infty$. Then,*

$$H(X_{r:m:n}) = H(U_{r:m:n}) - E\left[\log f(F^{-1}(U_{r:m:n}))\right], \quad \text{for} \quad 1\leqslant r\leqslant m\leqslant n. \tag{11}$$

Proof. The proof is easy, so it is omitted.

Note that only the second term in (11) involves F. It is obvious that for $n=1$, the result in Lemma 1 reduces to the entropy of the parent distribution. For $\widetilde{R}=(0,\dots,0)$ where $m=n$, the result in Lemma 1 turns into the entropy of the ordinary order statistics of $X_1,\dots,X_n$.

3 Characterization Results

In this section, we assume that $\{n_j, j \geqslant 1\}$ is a subsequence of $\mathbb{N}$ with

$$0 < n_1 < n_2 < \cdots \quad \text{and} \quad \sum_{j=1}^{+\infty} {n_j}^{-1} = \infty. \tag{12}$$

We characterize the parent distribution on the basis of entropy properties of progressive type-II right censored order statistics.

Theorem 1. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with common strictly continuous c.d.f. $F(x)$, p.d.f. $f(x)$ and entropy $H(X) < \infty$, then for fixed r and m $(1 \leqslant r \leqslant m)$, the sequence $H(X_{r:m:n_j}) - H(X)$, $n_j \geqslant m$, such that the sequence $\{n_j, j \geqslant 1\}$ satisfies (12), characterizes the distribution function F up to location and scale parameters.*

Proof. Using Lemma 1, we find

$$H(X_{r:m:n}) = H(U_{r:m:n}) - c_{r-1} \sum_{i=1}^{r} a_{i,r} \int_0^1 u^{\gamma_i - 1} \log f(F^{-1}(1-u))\, du. \tag{13}$$

Suppose for two c.d.f.'s F and G with corresponding p.d.f.'s f and g, respectively,

$$H(X_{r:m:n}) - H(X) = H(Y_{r:m:n}) - H(Y),$$

where $X_{r:m:n}$ and $Y_{r:m:n}$ are the rth progressive type-II right censored order statistics of X and Y in a sample of size n, respectively. Then from (13) we have

$$c_{r-1} \int_0^1 \sum_{i=1}^{r} a_{i,r} u^{\gamma_i - 1} \log \left[\frac{g(G^{-1}(1-u))}{f(F^{-1}(1-u))} \right] du = \int_0^1 \log \left[\frac{g(G^{-1}(1-u))}{f(F^{-1}(1-u))} \right] du. \tag{14}$$

Let $c = \int_0^1 \left[\log g(G^{-1}(1-u)) - \log f(F^{-1}(1-u)) \right] du$. Noting that

$$c_{r-1} \int_0^1 \sum_{i=1}^{r} a_{i,r} u^{\gamma_i - 1}\, du = 1,$$

we can re-express (14) as follows.

$$0 = \int_0^1 \left\{ \left(c - \log \left[\frac{g(G^{-1}(1-u))}{f(F^{-1}(1-u))} \right] \right) \sum_{i=1}^{r} a_{i,r} u^{-\sum_{j=1}^{i-1} R_j - i} \right\} u^n\, du. \tag{15}$$

If (15) holds for $n = n_j$, $j \geqslant 1$, such that $n_1 < n_2 < \cdots$ and $\sum_{j=1}^{+\infty} n_j^{-1} = \infty$, then using the completeness property of the sequence of functions $\{u^{n_j}, 0 < u < 1, j \geqslant 1\}$, and the classical Müntz-Szász theorem (see, for example, Kamps, 1995, p. 103), we obtain

$$\log\left[\frac{g(G^{-1}(1-u))}{f(F^{-1}(1-u))}\right] = c, \quad \text{for all} \quad 0 < u < 1.$$

Then we can conclude that $f(F^{-1}(t)) = e^{-c}g(G^{-1}(t))$, for all $t \in (0,1)$. Since $\frac{d}{dt}F^{-1}(t) = 1/f(F^{-1}(t))$, it then follows that $F^{-1}(v) = e^{-c}G^{-1}(v) + d$, for all $v \in (0,1)$, where d is a constant. This means that F and G are identical but for a change of location and scale, thus the result follows.

Theorem 2. *Under the assumptions of Theorem 1, for fixed r and m, $(1 \leqslant r \leqslant m)$, the two following statements are equivalent:*

(*i*) *X is identical in distribution with Y, but for a location shift;*

(*ii*) *$H(X_{r:m:n_j}) = H(Y_{r:m:n_j})$, $n_j \geqslant m$, such that the sequence $\{n_j, j \geqslant 1\}$ satisfies* (12).

Proof. The first part, [(i) ⇒ (ii)], is obvious. We will prove that the second part, [(ii) ⇒ (i)], holds. By the assumptions and Lemma 1, for $n \geqslant m$ we have

$$\begin{aligned} H(U_{r:m:n}) - E\left[\log f(F^{-1}(U_{r:m:n}))\right] &= H(X_{r:m:n}) \\ &= H(Y_{r:m:n}) \\ &= H(U_{r:m:n}) - E\left[\log g(G^{-1}(U_{r:m:n}))\right]. \end{aligned}$$

Then

$$E\left[\log f(F^{-1}(U_{r:m:n}))\right] = E\left[\log g(G^{-1}(U_{r:m:n}))\right].$$

Thus from (8) we get

$$c_{r-1}\int_0^1 \sum_{i=1}^{r} a_{i,r}u^{\gamma_i - 1}\log\left[\frac{g(G^{-1}(1-u))}{f(F^{-1}(1-u))}\right]\, du = 0. \tag{16}$$

If (16) holds for $n = n_j$, $j \geqslant 1$, such that $n_1 < n_2 < \cdots$ and $\sum_{j=1}^{+\infty} n_j^{-1} = \infty$, then we conclude that

$$f(F^{-1}(1-u)) - g(G^{-1}(1-u)) = 0, \quad \text{a.e.} \quad u \in (0,1).$$

By proceeding as in the proof of Theorem 1, we deduce that $F^{-1}(t) = G^{-1} + c$; in other words, F and G are identical but for a location shift.

Corollary 1. *Suppose that the sequence* $\{n_j, j \geqslant 1\}$ *satisfies* (12) *and the assumptions of Theorem 2 hold. Then the following statements are equivalent:*

(i) X *is identical in distribution with* Y, *but for a location shift;*

(ii) $H(X_{1:m:n}) = H(Y_{1:m:n})$;

(iii) $H(X_{m:m:n}) = H(Y_{m:m:n})$;

(iv) $H(X_{m-r+1:m:n}) = H(Y_{m-r+1:m:n})$, *for fixed* r, $(1 \leqslant r \leqslant m)$.

For the censoring scheme $\widetilde{R} = (0, \ldots, 0)$, ordinary order statistics, Baratpour et al. (2008) obtained similar results of Theorems 1, 2 and Corollary 1 based on Rényi entropy of order statistics.

Using the classical Müntz-Szász Theorem, Theorem 2.1 of Baratpour et al. (2007) can be improved, which is stated in the following.

Theorem 3. *Let* $X_1, X_2, \ldots, X_n$ *be i.i.d. continuous non-negative random variables with common c.d.f.* $F(x)$, *p.d.f.* $f(x)$ *and entropy* $H(X) < \infty$, *then the two following statements are equivalent.*

(i) *The hazard rate satisfies* $h_X(t) = c$ *(positive constant),*

(ii) $H(X_{1:m:n_j}) - H(X) = -\log \gamma_1$, *for fixed* m *and* $n_j \geqslant m$, *such that* $n_1 < n_2 < \cdots$ *and* $\sum_{j=1}^{+\infty} n_j^{-1} = \infty$.

Noting that $h_X(t) = c$ (positive constant) if and only if X has exponential distribution with hazard rate c, then we have the following result.

Corollary 2. *Suppose the conditions of Theorem 3 hold. Then the following two statements are equivalent.*

(i) X *has exponential distribution;*

(ii) $H(X_{1:m:n_j}) - H(X) = -\log \gamma_1$, *for fixed* m *and* $n_j \geqslant m$, *such that* $n_1 < n_2 < \cdots$ *and* $\sum_{j=1}^{+\infty} n_j^{-1} = \infty$.

Suppose X has the Weibull distribution with scale and shape parameters α and β, respectively, i.e. $\bar{F}(x) = \exp\{-(x/\alpha)^{\beta}\}$, where $\alpha > 0$ and $\beta > 0$. Then it can be shown that $H(X_{1:m:n_j}) - H(X) = -\frac{1}{\beta} \log \gamma_1$ and hence by Theorem 1, it is the only distribution with this property. This covers Weibull distributions with location-shift as well. Thus we have the following result.

Corollary 3. *Suppose the conditions of Theorem 3 hold. Then the two following statements are equivalent.*

(i) X *has Weibull distribution with shape parameter* β;

(*ii*) $H(X_{1:m:n_j}) - H(X) = -\frac{1}{\beta}\log\gamma_1$, *for* $n_j \geqslant m$, *such that* $n_1 < n_2 < \cdots$ *and* $\sum_{j=1}^{+\infty} n_j^{-1}$ *is infinite.*

We have similar result for Pareto distribution. Let X be a random variable having the Pareto distribution with scale and shape parameters β and α respectively, i.e, $\bar{F}(x) = (\frac{\beta}{x})^{\alpha}$, where $\alpha > 0$ and $x > \beta > 0$. Then we have the following corollary.

Corollary 4. *Suppose the conditions of Theorem 3 hold. Then the two following statements are equivalent.*

(*i*) X *has Pareto distribution with shape parameter* α;

(*ii*) $H(X_{1:m:n_j}) - H(X) = -\log\gamma_1 + \frac{1-\gamma_1}{\alpha\gamma_1}$, *for* $n_j \geqslant m$, *such that* $n_1 < n_2 < \cdots$ *and* $\sum_{j=1}^{+\infty} n_j^{-1}$ *is infinite.*

By Corollaries 2, 3, and 4, we conclude that the uncertainty about the X is always more than the $X_{1:m:n}$ in cases of exponential, Weibull and Pareto models. We give general results in the next section.

4 Ordering Results

There are several notions of stochastic ordering among random variables with varying degree of strength. In the following, we briefly review some of these notions that will be used later on in this section (see Shaked and Shantikumar, 2007, for more details).

Definition 1. *Let* X *and* Y *be two random variables with distributions* F *and* G, *density functions* f *and* g, *and failure rate functions* h_X *and* h_Y, *respectively.*

(*a*) *A random variable* X *is said to have increasing [decreasing] failure rate (IFR)[DFR] if its failure rate function* $h_X(t) = f(t)/\bar{F}(t)$ *is increasing [decreasing] in* $t > 0$.

(*b*) *The random variable* X *is said to be stochastically less than or equal to* Y, *denoted by* $X \leqslant_{\text{st}} Y$, *if* $\bar{F}(t) \leqslant \bar{G}(t)$, *for all* t.

(*c*) *The random variable* X *is said to be smaller than* Y *in the hazard rate order, denoted by* $X \leqslant_{\text{hr}} Y$, *if* $h_X(t) \geqslant h_Y(t)$, *for* $t \geqslant 0$.

(*d*) *The random variable* X *is said to be smaller than* Y *in the dispersive order, denoted by* $X \leqslant_{\text{disp}} Y$, *if* $F^{-1}(\beta) - F^{-1}(\alpha) \leqslant G^{-1}(\beta) - G^{-1}(\alpha)$ *whenever* $0 < \alpha \leqslant \beta < 1$.

(*e*) *The random variable* X *is said to be smaller than* Y *in the star order, denoted by* $X \leqslant_* Y$, *if* $G^{-1}(F(x))/x$ *is increasing in* $x \geqslant 0$.

(*f*) *The random variable* X *is said to be smaller than* Y *in the super-additive order, denoted by* $X \leqslant_{\text{su}} Y$, *if* $G^{-1}(F(x+y)) \geqslant G^{-1}(F(x)) + G^{-1}(F(y))$, *for all* $x \geqslant 0$ *and* $y \geqslant 0$.

Note that in the cases (a), (c), (e), and (f) the random variables should be non-negative. For applications of stochastic orders in reliability and life-testing contexts, see Shaked and Shantikumar (1994, 2007).

Definition 2. *Let* X *and* Y *be two random variables with entropy functions* $H(X)$ *and* $H(Y)$, *respectively, such that* $H(X) \leqslant H(Y)$. *Then* X *is said to be smaller than* Y *in the entropy order, denoted by* $X \leqslant_{\text{en}} Y$.

The following theorem is an extension of Theorem 2.4 of Ebrahimi et al. (2004) for progressive type-II right censored order statistics.

Theorem 4. *Let* X *and* Y, *be two random variables with p.d.f.'s* $f(\cdot)$ *and* $g(\cdot)$, *and absolutely continuous c.d.f.'s* $F(\cdot)$ *and* $G(\cdot)$, *respectively. If* $X \leqslant_{\text{disp}} Y$, *then for fixed* r *and* m $(1 \leqslant r \leqslant m)$,

$$X_{r:m:n} \leqslant_{\text{en}} Y_{r:m:n}, \quad \textit{for all} \quad n \geqslant m.$$

Proof. When X and Y have densities f and g, respectively, $X \leqslant_{\text{disp}} Y$ if and only if

$$g(G^{-1}(y)) \leqslant f(F^{-1}(y)), \quad \text{for all} \quad 0 < y < 1. \tag{17}$$

From (17) we immediately conclude that

$$E\left[\log g(G^{-1}(U_{r:m:n}))\right] \leqslant E\left[\log f(F^{-1}(U_{r:m:n}))\right], \quad \text{for all} \quad n \geqslant m. \tag{18}$$

Then by Lemma 1 and (18) we get

$$H(X_{r:m:n}) \leqslant H(Y_{r:m:n}), \quad \text{for all} \quad n \geqslant m.$$

Thus the result follows by Definition 2.

Theorem 5. *Let* X *and* Y *be two non-negative random variables with p.d.f.'s* $f(\cdot)$ *and* $g(\cdot)$, *and absolutely continuous c.d.f.'s* $F(\cdot)$ *and* $G(\cdot)$, *respectively. If* $X \leqslant_{\text{st}} Y$ *and, for fixed* r *and* m $(1 \leqslant r \leqslant m)$, $Y_{r:m:n}$ *is DFR, then*

$$X_{r:m:n} \leqslant_{\text{en}} Y_{r:m:n}, \quad \textit{for all} \quad n \geqslant m.$$

Proof. Noting that the stochastic order is preserved by progressive type-II right censored order statistics, hence $X_{r:m:n} \leqslant_{\text{st}} Y_{r:m:n}$. Thus the result follows by Theorem 2.2 of Ebrahimi et al. (2004).

Corollary 5. *Let X and Y be two non-negative random variables and suppose one of the following conditions is satisfied:*

(a) $X \leqslant_{\rm hr} Y$, *and X or Y is DFR;*

(b) $e^X \leqslant_* e^Y$;

(c) $X \leqslant_{\rm st} Y$ *and* $X \leqslant_{\rm su} Y$;

(d) $\lim_{x\to 0}(G^{-1}F(x)/x) \geqslant 1$ *and* $X \leqslant_{\rm su} Y$.

Then, for fixed r and m, $(1 \leqslant r \leqslant m)$, the following statement holds:

$$X_{r:m:n} \leqslant_{\rm en} Y_{r:m:n}, \quad \text{for all} \quad n \geqslant m.$$

Proof. By Theorems 3.B.20, 4.B.1, 4.B.2, and 4.B.3 of Shaked and Shantikumar (2007), under the conditions (a), (b), (c), and (d), respectively, we get $X \leqslant_{\rm disp} Y$, thus the results follows by Theorem 4.

By (7) we conclude $X_{1:m:n} \leqslant_{\rm hr} X$, so for a DFR distribution, for example Weibull distribution with shape parameter less than one, Pareto distribution, and the mixtures of exponential distributions, we have $X_{1:m:n} \leqslant_{\rm en} X$.

Theorem 6. *Let $X_1, \ldots, X_n$ be non-negative i.i.d. DFR random variables, then for $n \geqslant m$,*

$$X_{1:m:n} \leqslant_{\rm en} X_{r:m:n}, \quad \text{for all} \quad 2 \leqslant r \leqslant m.$$

Proof. From equation (7) we have $h_{X_{1:m:n}}(t) = \gamma_1 h_X(t)$, so by the assumption, $X_{1:m:n}$ is DFR. Noting that $X_{1:m:n} \leqslant_{\rm hr} X_{r:m:n}$, $(r = 2, 3, \ldots, m)$, the condition (a) of Corollary 5 holds and the result immediately follows.

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Jafar Ahmadi
Department of Statistics,
Faculty Mathematical of Sciences,
Ferdowsi University of Mashhad,
P.O. Box 91775-1159,
Mashhad, Iran.
e-mail: *ahmadi-j@um.ac.ir*