مجله‌ی پژوهش‌های آماری ایران

J. Statist. Res. Iran 4 (2007): 149-159

دوره‌ی ۴، شماره‌ی ۲، پاییز و زمستان ۱۳۸۶، صص ۱۴۹-۱۵۹

پژوهشکده‌ی آمار

Ordering of Order Statistics Using Variance Majorization

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Abstract. In this paper, we study stochastic comparisons of order statistics of independent random variables with proportional hazard rates, using the notion of variance majorization.

Keywords. reverse hazard rate ordering; usual stochastic ordering; proportional hazard models; majorization.

1 Introduction

Let $X_1, \ldots, X_n$ be any set of random variables. We denote the corresponding ith order statistic by $X_{i:n}$, $i = 1, \ldots, n$. Order statistics have many applications in many fields of probability and statistics. For example, in nonparametric context, they specify the empirical distribution function. In actuary context, the k largest claim amount in a portfolio is the $(n-k+1)$th order statistic. In reliability, they are the lifetimes of k-out-of-n systems. A k-out-of-n system of n components functions if at least k components of the n components function. That is the lifetime of a k-out-of-n system of n components corresponds to the $(n-k+1)$th order statistic. In particular, a 1-out-of-n system corresponds to a parallel system, n-out-of-n system corresponds to a series system and $(n-1)$-out-of-n system corresponds to a fail-safe system. Thus the study of stochastic properties of order statistics is of great importance in many fields of probability and statistics. David and Nagaraja (2003) provided extensive and

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comprehensive details on the theory of order statistics and its applications in statistics.

Let X_i denotes the lifetime of the ith component of a reliability system with survival function $\overline{F}_i(t)$, $\quad i = 1, \ldots, n$. Then they have proportional hazard rate (PHR) if there exist constants $\lambda_1, \ldots, \lambda_n$ and a (cumulative hazard) function $R(t) \geqslant 0$ such that $\overline{F}_i(t) = e^{-\lambda_i R(t)}$ for $i = 1, \ldots, n$. There are many parametric models such as Weibull, Exponential, Reighly, Pareto, and Lomax which can be considered as a PHR model. These distributions are of great importance in reliability context. They are used to model and describe the lifetime of components of a reliability system.

Stochastic comparisons of order statistics when the original observations are independent with proportional hazard rates, have been studied by Pledger and Proschan (1971), Proschan and Sethuraman (1976), Boland, El-Neweihi, and Proschan (1994), Dykstra, Kochar, and Rojo (1997), and Khaledi and Kochar (2000), Linhong and Xinsheng (2005), and Khaledi and Kochar (2006), among others.

In this paper, using a new notion, called variance majorization (see Definition 1), we investigate and study the behavior of survival function and reverse hazard rate function of a parallel system and a fail-safe system, with respect to constants of proportionality λ_i's. That is, we are interested in visualizing that how these characteristics of the systems change if we replace the vector $(\lambda_1, \ldots, \lambda_n)$ with another set of parameters say $(\lambda_1^*, \ldots, \lambda_n^*)$, according to variance majorization.

First, we introduce notations and recall some definitions.

Majorization. The notion of majorization is used to obtain various inequalities in statistics and probability. Let $\{x_{(1)} \leqslant \cdots \leqslant x_{(n)}\}$ denote the increasing arrangement of the components of a vector $\mathbf{x} = (x_1, \ldots, x_n)$. A vector $\mathbf{x}$ is said to majorize another vector $\mathbf{y}$ (written $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. Functions that preserve the majorization ordering are called Schur-convex functions. For more details on majorization and its applications, the reader is refereed to Marshall and Olkin (1979).

Now, we provide the definition of the variance majorization.

Definition 1 *(Neubauer and Watkins, 2006) Let* $\Psi = \{\mathbf{x} \in \mathbb{R}^n | x_1 \leqslant \cdots \leqslant x_n\}$. *For any* $\mathbf{x} \in \Psi$, *define*

$$\overline{x[i]} = \frac{1}{i} \sum_{k=1}^{i} x_k \qquad \textit{and} \qquad \operatorname{var}(x[i]) = \frac{1}{i} \sum_{k=1}^{i} (x_k - \overline{x[i]})^2.$$

Now, let $\mathbf{x}, \mathbf{y} \in \Psi$, *such that* $\overline{x} = \overline{y}$ *and* $\operatorname{var}(\mathbf{x}) = \operatorname{var}(\mathbf{y})$. *Then* $\mathbf{x}$ *is said to be* *variance majorized by* $\mathbf{y}$ *(or* $\mathbf{y}$ *variance majorizes* $\mathbf{x}$ *) denoted by* $\mathbf{y} \overset{vm}{\succeq} \mathbf{x}$ *if*

$$\text{var}(x[i]) \leqslant \text{var}(y[i]) \qquad for \qquad i = 2, \ldots, n.$$

For $\overline{x} = m$ and $\text{var}(x) = v \geqslant 0$, variance majorization is a partial ordering on the set

$$S(m, v) = \{x \in \Psi | \, \overline{x} = m, \text{var}[x] = v\},$$

which is the intersection of Ψ with a sphere in $\mathbb{R}^n$ centered at $m(1, \ldots, 1)$ with radius $\sqrt{nv}$, and the hyperplane through $m(1, \ldots, 1)$ orthogonal to the vector $(1, \ldots, 1)$. Neubauer and Watkins (2006) pointed out that smallest and largest vectors with respect to variance majorization ordering, denoted by $\mathbf{x}_{\min}$ and $\mathbf{x}_{\max}$, are respectively

$$\mathbf{x}_{\min} = (\alpha_1, \ldots, \alpha_1, \beta_1) \qquad \text{and} \qquad \mathbf{x}_{\max} = (\alpha_2, \beta_2, \ldots, \beta_2),$$

where

$$\alpha_1 = m - \sqrt{\frac{v}{n-1}}, \qquad \beta_1 = m + \sqrt{(n-1)v}, \tag{1}$$

$$\alpha_2 = m - \sqrt{(n-1)v}, \qquad \text{and} \qquad \beta_2 = m + \sqrt{\frac{v}{n-1}}. \tag{2}$$

That is

$$\mathbf{x}_{\min} \overset{vm}{\preceq} \mathbf{x} \overset{vm}{\preceq} \mathbf{x}_{\max}. \tag{3}$$

Let I be a closed interval. A function $f : \Psi \bigcap I^n \to \mathbb{R}^n$ is called a variance monotone increasing (VMI) function if it preserves the variance majorization. That is,

$$\mathbf{x} \overset{vm}{\preceq} \mathbf{y} \quad \Rightarrow \quad f(\mathbf{x}) \leqslant f(\mathbf{y}).$$

The variance monotone decreasing function is defined similary and denoted by VMD.

The following theorem proved in Neubauer and Watkins (2006) will be used later in this paper.

Theorem 1 *Let I be a closed interval in $\mathbb{R}^n$. Let $f(z_1, \ldots, z_n)$ be a continuous real valued function on $\Psi \bigcap I^n$ which is differentiable on the interior points of $\Psi \bigcap I^n$ with gradient $\nabla f(\mathbf{z}) = (f_1(\mathbf{z}), \ldots, f_n(\mathbf{z}))$. If*

$$\frac{f_2(z) - f_1(z)}{z_2 - z_1} \geqslant \frac{f_3(z) - f_2(z)}{z_3 - z_2} \geqslant \cdots \geqslant \frac{f_n(z) - f_{n-1}(z)}{z_n - z_{n-1}},$$

then

$$\mathbf{x} \overset{vm}{\preceq} \mathbf{y} \quad \Rightarrow \quad f(\mathbf{x}) \leqslant f(\mathbf{y}).$$

Throughout this paper *increasing* means *nondecreasing*, and *decreasing* means *nonincreasing*; and we shall be assuming that all distributions under study are absolutely continuous.

Stochastic Orderings

Let X and Y be univariate random variables with distribution functions F and G, survival functions $\overline{F}$ and $\overline{G}$, and density functions f and g. Let $l_X\,(l_Y)$ and $u_X\,(u_Y)$ be the left and the right endpoints of the support of $X\,(Y)$. The random variable X is said to be *stochastically* smaller than Y (denoted by $X \leqslant_{st} Y$) if $\overline{F}(x) \leqslant \overline{G}(x)$ for all x. This is equivalent to saying that $E[g(X)] \leqslant E[g(Y)]$ for any increasing function g for which expectations exist.

The *reversed hazard rate* of a life distribution F is defined as $\tilde{r}_F(x) = f(x)/F(x)$. Let $\tilde{r}_G(x)$ denote the reversed hazard rate of G. Then X is said to be *smaller than* Y *in the reversed hazard rate order* (written as $X \leqslant_{rh} Y$) if $\tilde{r}_F(x) \leqslant \tilde{r}_G(x)$, for all x, or equivalently, if $F(x)/G(x)$ is decreasing in x. It is well-known that the reversed hazard rate ordering implies stochastic ordering. For many interesting applications of reverse hazard rate ordering see Nanda and Shaked (2001).

A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be smaller than another random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ according to multivariate stochastic ordering (denoted by $\mathbf{X} \overset{\text{st}}{\preceq} \mathbf{Y}$) if $h(\mathbf{X}) \leqslant_{\text{st}} h(\mathbf{Y})$ for all increasing functions h. It is easy to see that multivariate stochastic ordering implies component-wise stochastic ordering. For more details on univariate as well as multivariate stochastic orderings see Shaked and Shanthikumar (1994).

Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\overline{F}^{\lambda_i}(x), \quad i = 1, \ldots, n$, where $\overline{F}$ is a baseline survival function. In section 2, we prove that the reverse hazard rate of $X_{n:n}$, the largest observation, is a VMI function in $(\lambda_1, \ldots, \lambda_n)$ (Theorem 3). We also prove that for the case that $\overline{F}(x) = e^{-x}$, the survival function of $X_{2:n}$, the lifetime of a fail-safe system, is a decreasing function of $(\lambda_1, \ldots, \lambda_n)$ with respect to variance majorization (Theorem 5). Then we use these results to get the new lower and upper bounds on the survival functions of a parallel system and a fail-safe system, made of independent components with exponential distributions. We also compare the new bounds obtained in this paper with other bounds available in the literature.

2 Main Results

Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\overline{F}^{\lambda_i}(x)$, $i = 1, \ldots, n$. In this section, first, we study the stochastic properties of the largest order statistic associated with these random variables. It is of interest to investigate the effect on the survival function of the failure time of a system consisting of such components when we switch the vector $(\lambda_1, \ldots, \lambda_n)$ to another vector, say $(\lambda_1^*, \ldots, \lambda_n^*)$. This problem, for the first time has been considered by Pledger and Proschan (1971). They proved the following result for the PHR model which uses the exponential distribution as a special case.

Theorem 2 *Let* $(X_1, \ldots, X_n)$ *and* $(X_1^*, \ldots, X_n^*)$ *be two random vectors of independent lifetimes with proportional hazards and with* $(\lambda_1, \ldots, \lambda_n)$ *and* $(\lambda_1^*, \ldots \lambda_n^*)$ *as the constants of proportionality. Then*

$$\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow X_{i:n} \geqslant_{\rm st} X_{i:n}^*, \qquad i = 1, \ldots, n. \tag{4}$$

Proschan and Sethuraman (1976) extended this result from componentwise stochastic ordering to multivariate stochastic ordering. That is, under the assumptions of Theorem 2, they proved that

$$(X_{1:n}, \ldots, X_{n:n}) \overset{\rm st}{\succeq} (X_{1:n}^*, \ldots, X_{n:n}^*). \tag{5}$$

Khaledi and Kochar (2006) pointed out that in the case that Weibull distributions with a common shape parameter α and scale parameters $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda_1^*, \ldots \lambda_n^*)$ are used, (4) and (5) hold if $(\lambda_1^\alpha, \ldots, \lambda_n^\alpha) \overset{m}{\succeq} (\lambda_1^{*\alpha}, \ldots, \lambda_n^{*\alpha})$. Khaledi and Kochar (2006) also proved that a similar result holds in the Weibull case when the two original vectors of scale parameters majorize each other and the shape parameter $\alpha \in (0, 1)$.

Next, we compare two parallel systems with respect to variance majorization.

Theorem 3 *Let* $X_1, \ldots, X_n$ *be independent random variables with* X_i *having survival function* $\overline{F}^{\lambda_i}(x)$, $\quad i = 1, \ldots, n$. *Let* $Y_1, \ldots, Y_n$ *be another set of independent random variables with* Y_i *having survival function* $\overline{F}^{\lambda_i'}(x)$, $i = 1, \ldots, n$. *Then*

$$(\lambda_1, \ldots, \lambda_n) \overset{vm}{\succeq} (\lambda_1', \ldots, \lambda_n') \quad \Longrightarrow \quad X_{n:n} \geqslant_{rh} Y_{n:n}$$

Proof. First we prove the result for the case that $\overline{F}(x) = e^{-x}$. In this case the reveres hazard rates of $X_{n:n}$ and $Y_{n:n}$, respectively, can be written as

$$\widetilde{r}_{X_{n:n}}(x) = \sum_{i=1}^{n} \frac{\lambda_i e^{-\lambda_i x}}{1 - e^{-\lambda_i x}} \quad \text{and} \quad \widetilde{r}_{Y_{n:n}}(x) = \sum_{i=1}^{n} \frac{\lambda_i' e^{-\lambda_i' x}}{1 - e^{-\lambda_i' x}}. \tag{6}$$

Since for any $x > 0$

$$(\lambda_1, \ldots, \lambda_n) \overset{vm}{\succeq} (\lambda_1', \ldots, \lambda_n') \quad \Longleftrightarrow \quad (x\lambda_1, \ldots, x\lambda_n) \overset{vm}{\succeq} (x\lambda_1', \ldots, x\lambda_n'),$$

it is enough to show that the function $h(\mathbf{u}) = \sum_{i=1}^{n} \frac{u_i e^{-u_i}}{1-e^{-u_i}}$ is VMI in $(u_1, \ldots, u_n)$.

It follows from Theorem 1 that we have to show that for $u_{i-1} < u_i < u_{i+1}$, $i = 2, \ldots, n-1$, the inequality

$$\frac{h_i(\mathbf{u}) - h_{i-1}(\mathbf{u})}{(u_i - u_{i-1})} \geqslant \frac{h_{i+1}(\mathbf{u}) - h_i(\mathbf{u})}{(u_{i+1} - u_i)} \tag{7}$$

holds. Simplifying (7), it can be written as

$$\frac{\frac{e^{u_i}-1-u_i e^{u_i}}{(e^{u_i}-1)^2} - \frac{e^{u_{i-1}}-1-u_{i-1}e^{u_{i-1}}}{(e^{u_{i-1}}-1)^2}}{u_i - u_{i-1}} \geqslant \frac{\frac{e^{u_{i+1}}-1-u_{i+1}e^{u_{i+1}}}{(e^{u_{i+1}}-1)^2} - \frac{e^{u_i}-1-u_i e^{u_i}}{(e^{u_i}-1)^2}}{u_{i+1} - u_i} \tag{8}$$

Inequality (8) is equivalent to the fact that the function $g(v) = \frac{e^u-1-ue^u}{(e^u-1)^2}$ is a concave function. This is true, since

$$g''(v) =^{sgn} u_1(v) = -4ve^v - ve^{2v} + 3e^{2v} - 3 - v \quad \text{and}$$
$$u_1''(v) =^{sgn} u_2(v) = -8 - 4v + 8e^v - 4ve^v$$

then, $u_2''(v) < 0$ implies that $g''(v) \leqslant 0$, for $v > 0$. This proves the required result for the case that F is exponential distribution. Now, Let $H(x) = -\ln \overline{F}(x)$. Then, for $i = 1, \ldots, n$, $Z_i = H(X_i)$ and $W_i = H(Y_i)$ are exponential random variables with hazard rates λ_i and $\lambda_i^{'}$, respectively. Combining these observations, it follows that $Z_{n:n} \geqslant_{rh} W_{n:n}$. The right inverse function of H, denoted by H^{-1}, is an increasing function. It is known that reverse hazard rate ordering is preserved under increasing transformation. Using these facts, we obtain

$$\begin{aligned} X_{n:n} &=_{st} H^{-1}(Z_{n:n}) \\ &\geqslant_{rh} H^{-1}(W_{n:n}) \\ &=_{st} Y_{n:n}. \end{aligned}$$

This completes the proof of the required result.

Corollary 1 *Let $Z_1, \ldots, Z_n$ be independent exponential random variables with Z_i having hazard rate λ_i, $i = 1, \ldots, n$. Then*

(*a*) $(n-1)\frac{\alpha_1 e^{-\alpha_1 x}}{1-e^{-\alpha_1 x}} + \frac{\beta_1 e^{-\beta_1 x}}{1-e^{-\beta_1 x}} \leqslant \tilde{r}_{Z_{n:n}}(x) \leqslant (n-1)\frac{\beta_2 e^{-\beta_2 x}}{1-e^{-\beta_2 x}} + \frac{\alpha_2 e^{-\alpha_2 x}}{1-e^{-\alpha_2 x}}$ *and*

(*b*) $1 - (1-e^{\alpha_1 x})^{n-1}(1-e^{\beta_1 x}) \leqslant \overline{F}_{Z_{n:n}}(x) \leqslant 1 - (1-e^{\beta_2 x})^{n-1}(1-e^{\alpha_2 x})$,

where α_1, β_1, α_2 and β_2 are as those in (1) *and* (2).

Khaledi and Kochar (2006) proved that if $X_1, \ldots, X_n$ are independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$, then

$$\overline{F}_{Z_{n:n}}(x) \geqslant (1 - e^{-\tilde{\lambda} x})^n, \tag{9}$$

where $\widetilde{\lambda} = \sqrt[n]{\prod_{i=1}^{n} \lambda_i}$. Now, it is of interest to compare the lower bounds obtained in part (b) of Corollary 1 with that in (9).

Bon and Paltanea (2006) proved the following interesting characterization result.

Theorem 4 *Let $X_1, \ldots, X_n$ be a set of independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$; and let $Y_1, \ldots, Y_n$, be a random sample from exponential distribution with hazard rate μ. Then, for $k \in \{1, \ldots, n\}$,*

$$X_{k:n} \geqslant_{\text{st}} Y_{k:n} \quad \Longleftrightarrow \quad \mu \geqslant \left\{ \binom{n}{k}^{-1} \sum_{|J|=k} \prod_{i \in J} \lambda_i \right\}^{\frac{1}{k}},$$

where J is any subset of $\{1, \ldots, n\}$.

For $k = n$, it follows from this theorem that

$$1 - (1 - e^{\alpha_1 x})^{n-1}(1 - e^{\beta_1 x}) \geqslant (1 - e^{-\widetilde{\lambda} x})^n,$$

if and only if,

$$\prod_{i=1}^{n} \lambda_i \geqslant \alpha_1^{n-1} \beta_1.$$

But, it follows from Cohn's inequality (cf. Cohn, 1967) that

$$\prod_{i=1}^{n} \lambda_i \leqslant \alpha_1^{n-1} \beta_1,$$

where α_1 and β_1 are chosen so that the sequences $(\lambda_1, \ldots, \lambda_n)$ and $(\alpha_1, \ldots, \alpha_1, \beta_1)$ have the same means and variances. Combining these observations, we find out that for some sets of λ_i's the lower bounds obtained in part (b) of Corollary 1 may be sharper than that in (9).

To justify this observation let $\boldsymbol{\lambda} = (0.02, 0.1, 0.09)$, then $\alpha_1 = 0.044$, $\beta_1 = 0.12$, and $\mu = (\prod_{i=1}^{n} \lambda_i)^{1/n} = 0.056$. The survival functions of $X_{3:3}$ and the lower bounds mentioned above are graphed in Figure 1. In this graph we observe that the bound obtained in part (b) of Corollary 1 is not better than that in (9). On the other hand, let $\boldsymbol{\lambda} = (0.02, 0.1, 9)$, then $\alpha_1 = 0.059$, $\beta_1 = 9$, and $\mu = 0.026$. In this case we will see in Figure 2 that the bound obtained in part (b) of Corollary 1 is sharper than that in (9).

Remark. An interesting observation that one can figure out from Corollary 1 is that, the inequality on the right hand side will give us a computable convenient upper bound on survival function of the lifetime of a parallel system consisting of independent but non-identically distributed exponential components. This kind of bounds can not be obtained from Theorem 4 or from other related results in the literature.

In the next theorem we show that the result of Theorem 3 may not hold for other order statistics.

Theorem 5 *Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$ and let $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with Y_i having hazard rate $\lambda_i^{'}$, $i = 1, \ldots, n$. Then*

$$(\lambda_1, \ldots, \lambda_n) \overset{vm}{\succeq} (\lambda_1', \ldots, \lambda_n') \quad \Longrightarrow \quad X_{2:n} \leqslant_{\text{st}} Y_{2:n}$$

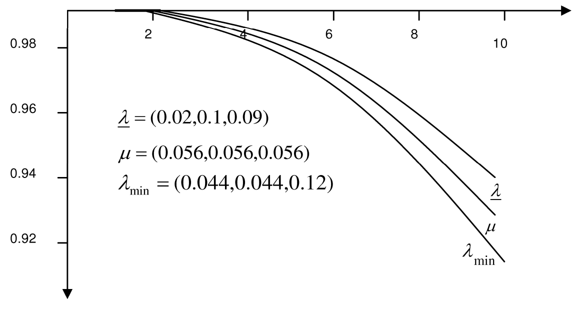

Figure 1. Graphs of survival functions of $X_{3:3}$.

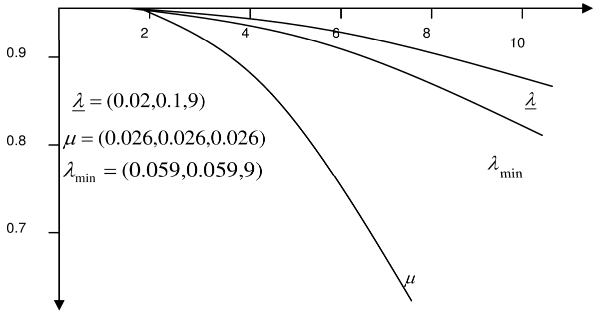

Figure 2. Graphs of survival functions of $X_{3:3}$.

Proof. The distribution function of $X_{2:n}$ is

$$F_{X_{2:n}}(t) = 1 - e^{-\sum_{i=1}^n \lambda_i t}\left[\sum_{j=1}^n e^{\lambda_j t} - (n-1)\right].$$

Variance majorization assumption between $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^{'}$ implies that $v = V(\boldsymbol{\lambda}) = V(\boldsymbol{\lambda}^{'})$ and $m = \overline{\lambda} = \overline{\lambda}^{'}$. Therefore, $X_{2:n} \leqslant_{st} Y_{2:n}$, if and only if, for $t > 0$, $\sum_{j=1}^n e^{\lambda_j t} \leqslant \sum_{j=1}^n e^{\lambda_j^{'} t}$. Now, since for $t > 0$

$$(\lambda_1, \ldots, \lambda_n) \overset{vm}{\succeq} (\lambda_1^{'}, \ldots, \lambda_n^{'}) \quad \Longleftrightarrow \quad (t\lambda_1, \ldots, t\lambda_n) \overset{vm}{\succeq} (t\lambda_1^{'}, \ldots, t\lambda_n^{'}),$$

it is enough to show that

$$S(\mathbf{u}) = \sum_{j=1}^n e^{u_j}$$

is a VMI function in $\mathbf{u}$. This follows from Theorem 1, since the second derivative of e^u is an increasing function.

Corollary 2 *Under the conditions of Theorem 5 we have*

$$\begin{aligned} &e^{-\{(n-1)\beta_2+\alpha_2\}t}\{(n-1)e^{\beta_2 t} + e^{\alpha_2 t} - (n-1)\} \leqslant \overline{F}_{X_{2:n}}(t) \\ &\leqslant e^{-((n-1)\alpha_1+\beta_1)t}\{(n-1)e^{\alpha_1 t} + e^{\beta_1 t} - (n-1)\}, \end{aligned} \tag{10}$$

where α_1, β_1, α_2 and β_2 are the same as those in (1) and (2).

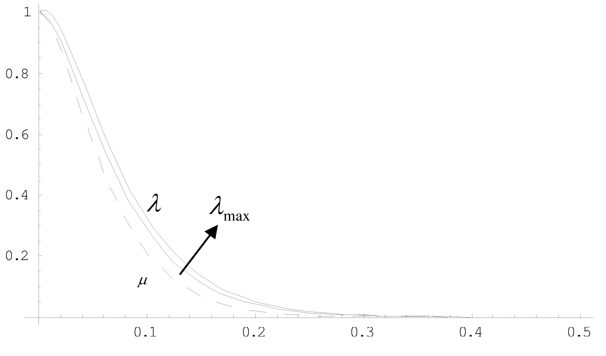

Figure 3. Graphs of survival functions of $X_{2:3}$.

Proof. Using (3), the required result follows from Theorem 5.

Now we show that the lower bound obtained in Corollary 2 is sharper than that obtained in Theorem 4. In Theorem 4, suppose $(\lambda_1, \ldots, \lambda_n) = (\alpha_2, \beta_2, \ldots, \beta_2)$ and $\mu = \{\binom{n}{r}^{-1} \sum_{|J|=k} \prod_{i\in J} \lambda_i\}^{\frac{1}{k}}$, then we have,

$$e^{-n\mu x}\{ne^{\mu x} - (n-1)\} \leqslant e^{-((n-1)\beta_2+\alpha_2)t}\{(n-1)e^{\beta_2 t} + e^{\alpha_2 t} - (n-1)\}.$$

On the other hand, it follows from Theorem 5 that

$$\overline{F}_{X_{2:n}}(t) \geqslant e^{-\{(n-1)\beta_2+\alpha_2\}t}\{(n-1)e^{\beta_2 t} + e^{\alpha_2 t} - (n-1)\}.$$

From this and the inequality on the right hand side of (10) it follows that the new lower bound obtained in (10) is sharper than the one obtained in Theorem 5. To justify this, let $\boldsymbol{\lambda} = (10, 8, 10.5, 2)$, then $\alpha_2 = 1.86$, $\beta_2 = 10.57$ and $\mu = (\prod_{i=1}^{n} \lambda_i)^{1/n} = 8.42$. In Figure 3, we observe that the new lower bound obtained in Corollary 2 is sharper than that in Theorem 4.

Acknowledgement

We thank the referees for comments on a previous draft of the paper. The comments led us to significantly improve the paper.

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