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پژوهشکدهی آمار

Number of Minimal Path Sets in a Consecutive-k-out-of-n:F System

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Abstract. In this paper the combinatorial problem of determining the number of minimal path sets of a consecutive-k-out-of-n:F system is considered. For the cases where $k = 2, 3$ the explicit formulae are given and for $k \geqslant 4$ a recursive relation is obtained. Direct computation for determining the number of minimal path sets of a consecutive-k-out-of-n:F system for $k \geqslant 4$ remains a difficult task.

Keywords. minimal path set; consecutive-k-out-of-n:F system; generating function.

1 Introduction and Notations

Minimal path sets and minimal cut sets of a system play a crucial role in the reliability analysis of the system. A consecutive-k-out-of-n:F (con$|k|n$:F) system consists of n linearly ordered and interconnected components. The system fails if and only if there are at least k consecutive failed components. If components are arranged on a circle we have a circular con$|k|n$:F system. This system has been studied by various authors. Kontoleon (1980) presented applications in telecommunication and pipeline networks, vacuum systems in accelerators, computer networks, design of integrated circuits etc. In a survey article by Chao et al. (1995) on the reliability aspect of this system alone, more than hundred papers have been cited.

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A minimal set of components whose functioning (failure) causes the functioning (failure) of the system is called a minimal path(cut) set of the system. Hence, a minimal cut set of a con$|k|n$:F system is of the form $\{i, i+1, \ldots, i+k-1\}$, $i = 1, 2, \ldots, n-k+1$. However, there is no such simple representation for a minimal path set of the system. For example all minimal path sets of a con$|2|5$:F system are $\{1, 3, 5\}$, $\{1, 3, 4\}$, $\{2, 4\}$ and $\{2, 3, 5\}$.

A con$|k|n$:F system is always more reliable than a conventional k-out-of-n:F system in which the system fails if and only if at least k components fail, since the family of minimal cut sets of the former is a subset of the family of minimal cut sets of the latter. However for $k = 1$ and $k = n$ both systems reduce to the usual series and parallel systems.

Using the number of path sets of a coherent system, Ramamurthy et al., (2003) developed a combinatorial approach to determine the structural matrix of the system and then obtained a number of structural measures of component importance. Seth and Sadegh (1999) obtained an expression for the number of path sets with known size in a con$|k|n$:F system, which is used to determine system reliability and the Barlow-Proschan measure of component importance. Seth and Sadegh (2001) have given some expressions to determine the number of minimal path sets with known size in a con$|2|n$:F system.

In this note the combinatorial problem of determining the number of minimal path sets of a con$|k|n$:F system is considered. In sections 2 and 3 explicit formulae for determining these number in con$|2|n$:F and con$|3|n$:F systems are given, respectively. For a con$|k|n$:F system ($k \geqslant 4$) a recursive relation is obtained in section 4. It may add that direct computation to determine the number of minimal path sets of a con$|k|n$:F system when $k \geqslant 4$ is a difficult task.

We shall use the following notations.

$\alpha_k(n)$: collection of all minimal path sets of a con$|k|n$:F system.

$p_n^L(k)(p_n^C(k))$: number of minimal path sets of a linear (circular) con$|k|n$:F system.

$p_n^{r,L}(2)$ $(p_n^{r,C}(2))$: number of minimal path sets of size r in a linear (circular) con$|2|n$:F system.

$\bar{r}_k(n)$ $(\bar{\bar{r}}_k(n))$: minimum (maximum) size of a minimal path set in a con$|k|n$:F system.

$p_n^i(k)$: number of minimal path sets of a con$|k|n$:F system that include component i.

$|A|$: cardinality of set A.

2 Consecutive-2-out-of-n:F System

For the sake of completeness, in this section we give the results of Seth and Sadegh (2001).

Suppose $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. We assume that $\binom{m}{r} = 0$ for $m < r$ or $r < 0$.

Theorem 1. *For $n \geqslant 0$ we have*

$$p_n^L(2) = \lfloor \rho^n (1+\rho)^2/(2\rho+3) + 0.5 \rfloor$$

where ρ is the unique real root of the cubic equation $x^3 - x - 1 = 0$.

Theorem 2. *For $n \geqslant 0$ we have*

$$p_n^L(2) = \sum_{i=\lfloor n/3 \rfloor}^{\lfloor n/2 \rfloor} \sum_{j=n-2i-2}^{n-2i} \binom{i}{j}.$$

Lemma 1. *Suppose $\bar{\bar{r}}_k(n)$ and $\bar{r}_k(n)$ denote the maximum and minimum size of a minimal path set in a linear con$|k|n$:F system respectively. We then have*

$$\bar{\bar{r}}_k(n) = \begin{cases} 2\left\lfloor \frac{n}{k+1} \right\rfloor, & \text{if } \frac{n+1}{k+1} > \left\lfloor \frac{n+1}{k+1} \right\rfloor \\ 2\left\lfloor \frac{n}{k+1} \right\rfloor + 1, & \text{if } \frac{n+1}{k+1} = \left\lfloor \frac{n+1}{k+1} \right\rfloor \end{cases} \quad \text{and} \quad \bar{r}_k(n) = \lfloor n/k \rfloor.$$

Theorem 3. *For $\bar{r}_2(n) \leqslant r \leqslant \bar{\bar{r}}_2(n)$ and $n \geqslant 2$ we have*

$$p_n^{r,L}(2) = \binom{n-r+1}{2n-3r}.$$

Remark 1. Using Theorem 3 we have

$$p_n^L(2) = \sum_{r=\bar{r}_2(n)}^{\bar{\bar{r}}_2(n)} p_n^{r,L}(2) = \sum_{r=\bar{r}_2(n)}^{\bar{\bar{r}}_2(n)} \binom{n-r+1}{2n-3r}.$$

Theorem 4. *For $n \geqslant 10$ we have*

$$p_n^C(2) = \lfloor \rho^n + 0.5 \rfloor$$

where ρ is defined in Theorem 1.

Theorem 5. *For $n \geqslant 3$ we have*

$$p_n^{r,C}(2) = \frac{n}{2r-n} \binom{n-r-1}{2n-3r}.$$

3 Consecutive-3-out-of-n:F System

In this section, we confine our attention to the minimal path sets of a con|3|n:F system. We show that $p_n^L(3)$, the number of minimal path sets of a linear con|3|n:F system satisfies the recursive relation $p_n^L(3) = p_{n-2}^L(3) + p_{n-3}^L(3) + p_{n-4}^L(3) - p_{n-6}^L(3)$, $n \geqslant 6$. We also show that the same relation holds for $p_n^C(3)$, the number of minimal path sets of a circular con|3|n:F system, but with different initial values. We give a direct formula for determination of $p_n^L(3)$.

Lemma 2. *Let $\alpha_3^L(n)$ denote the collection of all minimal path sets of a linear con|3|n:F system. We then have*

(*i*) $\{S : S \in \alpha_3^L(n)$ *and* $n \in S\} = \{S : S = T \cup \{n-3, n\}$ *and* $T \in \alpha_3^L(n-4)\}$, $n \geqslant 4$;

(*ii*) $\{S : S \in \alpha_3^L(n)$ *and* $n-2 \in S\} = \{S : S = T \cup \{n-2\}$ *and* $T \in \alpha_3^L(n-3)\}$, $n \geqslant 3$;

(*iii*) $\{S : S \in \alpha_3^L(n)$ *and* $n-1 \in S\} = \{S : S = T \cup \{n-1\}$ *where* $T \in \alpha_3^L(n-2)$ *and* $n-2 \notin T\}$, $n \geqslant 3$.

Proof. In view of the system structure defined in section 1, the proof of the lemma is easy and omitted.

Lemma 3. *In a linear con|3|n:F system we have*

(*i*) $|\{S : S \in \alpha_3^L(n), 1 \in S\}| = |\{S : S \in \alpha_3^L(n), n \in S\}| = p_{n-4}^L(3)$ *for* $n \geqslant 4$;

(*ii*) $|\{S : S \in \alpha_3^L(n)$ *and* $1 \in S, n \in S\}| = p_{n-8}^L(3)$ *for* $n \geqslant 8$;

(*iii*) $|\{S : S \in \alpha_3^L(n)$ *and* $1 \notin S, n \notin S\}| = p_n^L(3) - 2p_{n-4}^L(3) + p_{n-8}^L(3)$ *for* $n \geqslant 8$.

Proof. Using part (i) of Lemma 2, the proof of this lemma follows.

Theorem 6. *For $n \geqslant 6$ we have*

$$p_n^L(3) = p_{n-2}^L(3) + p_{n-3}^L(3) + p_{n-4}^L(3) - p_{n-6}^L(3)$$

where

$$p_0^L(3) = p_1^L(3) = p_2^L(3) = 1, \quad p_3^L(3) = p_4^L(3) = 3 \quad \text{and} \quad p_5^L(3) = 4.$$

Proof. We know that

$$\begin{aligned}\alpha_3^L(n) = \{S : S \in \alpha_3^L(n) \text{ and } n \in S\} \\ \cup \{S : S \in \alpha_3^L(n) \text{ and } n-1 \in S\} \\ \cup \{S : S \in \alpha_3^L(n) \text{ and } n-2 \in S\}\end{aligned}$$

and the collections on the right hand side are disjoint. By part (i) of Lemma 2, we have $| \{S : S \in \alpha_3^L(n) \text{ and } n \in S\} | = | \alpha_3^L(n-4) |= p_{n-4}^L(3)$. Using part (ii) of Lemma 2, we have $| \{S : S \in \alpha_3^L(n) \text{ and } n-2 \in S\} | = | \alpha_3^L(n-3) |= p_{n-3}^L(3)$. And in view of part (iii) and part (i) of Lemma 2, we have

$$| \{S : S \in \alpha_3^L(n) \text{ and } n-1 \in S\} |=| \alpha_3^L(n-2) | - | \alpha_3^L(n-6) |= p_{n-2}^L(3)-p_{n-6}^L(3).$$

Therefore we get the result $p_n^L(3) = |\alpha_3^L(n)| = p_{n-4}^L(3) + p_{n-3}^L(3) + p_{n-2}^L(3) - p_{n-6}^L(3)$. This completes the proof of the theorem.

Remark 2. By the similar argument given in Lemma 2 and Lemma 3 and in view of the structure of a circular con|3|n:F system it can be shown that for $n \geqslant 5$

$$p_n^C(3) = p_{n-2}^C(3) + p_{n-3}^C(3) + p_{n-4}^C(3) - p_{n-6}^C(3)$$

where $p_{-1}^C(3) = 0$, $p_0^C(3) = 6$, $p_1^C(3) = 0$, $p_2^C(3) = 2$, $p_3^C(3) = 3$ and $p_4^C(3) = 6$.

We now consider direct computation of $p_n^L(3)$, the number of minimal path sets of a linear con|3|n:F system. In Theorem 6, we showed that for $n \geqslant 6$

$$p_n^L(3) = p_{n-2}^L(3) + p_{n-3}^L(3) + p_{n-4}^L(3) - p_{n-6}^L(3)$$

where $p_0^L(3) = p_1^L(3) = p_2^L(3) = 1, p_3^L(3) = p_4^L(3) = 3$ and $p_5^L(3) = 4$.

Let $g_L(x)$ denote the generation function of $p_n^L(3)$, that is, $g_L(x) = \sum_{n=0}^{\infty} p_n^L(3)x^n$. Then in view of Theorem 6, we have

$$g_L(x)(1 - x^2 - x^3 - x^4 + x^6) = 1 + x + x^3 - x^5.$$

Hence, we get the result

$$g_L(x) = \frac{1 + x + x^3 - x^5}{1 - x^2 - x^3 - x^4 + x^6}.$$

To obtain partial fraction expansion of $g_L(x)$, we need to find the roots of the equation $1-x^2-x^3-x^4+x^6 = 0$. We know that this equation has two positive real roots and four complex roots. Also note that if x is a root of this equation then $1/x$ is also another root. Hence, we denote x_1 and $1/x_1$ as the real roots and $x_2, 1/x_2, x_3$ and $1/x_3$ as the complex roots.

We have $1-x^2-x^3-x^4+x^6 = (x-x_1)(x-\frac{1}{x_1})(x-x_2)(x-\frac{1}{x_2})(x-x_3)(x-\frac{1}{x_3})$. It implies that

$$\begin{cases} \left(x_3+\frac{1}{x_3}\right)+\left(x_2+\frac{1}{x_2}+x_1+\frac{1}{x_1}\right)=0, \\ 3+\left(x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2}\right)\left(x_3+\frac{1}{x_3}\right)+\left(x_1+\frac{1}{x_1}\right)\left(x_2+\frac{1}{x_2}\right)=-1, \\ -\left(x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2}\right)-\left[2+\left(x_1+\frac{1}{x_1}\right)\left(x_2+\frac{1}{x_2}\right)\right] \\ \times\left(x_3+\frac{1}{x_3}\right)-\left(x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2}\right)=-1. \end{cases}$$

From the first equation, we get $x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2}=-(x_3+\frac{1}{x_3})$. Using this and the second and third equations, we get

$$\begin{cases} 3+\left(x_2+\frac{1}{x_2}\right)\left(x_3+\frac{1}{x_3}\right)-\left(x_1+\frac{1}{x_1}\right)^2=-1, \\ -\left(x_1+\frac{1}{x_1}\right)\left(x_2+\frac{1}{x_2}\right)\left(x_3+\frac{1}{x_3}\right)=-1. \end{cases}$$

$$\Longrightarrow \begin{cases} 3+\left(x_2+\frac{1}{x_2}\right)\left(x_3+\frac{1}{x_3}\right)-\left(x_1+\frac{1}{x_1}\right)^2=-1, \\ \left(x_2+\frac{1}{x_2}\right)\left(x_3+\frac{1}{x_3}\right)=\frac{1}{x_1+\frac{1}{x_1}}. \end{cases}$$

It implies that $3+\frac{1}{x_1+\frac{1}{x_1}}-(x_1+\frac{1}{x_1})^2=-1$. Therefore

$$\left(x_1+\frac{1}{x_1}\right)^3-4\left(x_1+\frac{1}{x_1}\right)-1=0.$$

Similarly we can obtain

$$\left(x_2+\frac{1}{x_2}\right)^3-4\left(x_2+\frac{1}{x_2}\right)-1=0,$$
$$\left(x_3+\frac{1}{x_3}\right)^3-4\left(x_3+\frac{1}{x_3}\right)-1=0.$$

Let $s=x+\frac{1}{x}$ where x is a root of equation $1-x^2-x^3-x^4+x^6=0$. From the last three equations we have $s^3-4s-1=0$. Note that the cubic equation $s^3-4s-1=0$ has three real roots.

Using Cardan formula we have $s_1 \simeq 2.114907542$, $s_2 \simeq -1.86080585$, and $s_3 \simeq -0.25410168$. We note that $s_1 = x_1 + \frac{1}{x_1}$, $s_2 = x_2 + \frac{1}{x_2}$, and $s_3 = x_3 + \frac{1}{x_3}$. For $s = s_1$, x_1, and $\frac{1}{x_1}$ can then be obtained from the quadratic equation $x^2 - s_1 x + 1 = 0$, as follows:

$$x_1 = \frac{s_1 + \sqrt{s_1^2 - 4}}{2} \simeq 0.713639174$$

and

$$x_2 = \frac{1}{x_1} = \frac{s_1 + \sqrt{s_1^2 - 4}}{2} \simeq 1.401268368.$$

For $s = s_2$ we have $x^2 - s_2 x + 1 = 0$. This equation has two complex roots, x_3 and $x_4 = \frac{1}{x_3}$, on the unit circle as follows:

$$x_3 = \frac{s_2 + \sqrt{s_2^2 - 4}}{2} = \frac{s_2 + i\sqrt{4 - s_2^2}}{2} \simeq 0.93040292 + i\ 0.36653897,$$

where $i^2 = -1$ and $x_4 = \frac{1}{x_3} = \bar{x}_3$ (conjugate of x_3).

Suppose $x_3 = \text{cis}(\theta_3) = \cos\theta_3 + i\ \sin\theta_3$, where $\theta_3 = \text{Arctg}\left(\frac{I(x_3)}{R(x_3)}\right) + \pi \simeq 158°, 49'$.

It implies that $x_4 = \bar{x}_3 = \text{cis}(\theta_4)$, where

$$\theta_4 = \text{Arctg}\left(\frac{I(x_4)}{R(x_4)}\right) + \pi = -\text{Arctg}\left(\frac{I(x_3)}{R(x_3)}\right) + \pi = 2\pi - \theta_3 \simeq 201°, 51'.$$

Similarly for $s = s_3$, we have $x^2 - s_3 x + 1 = 0$ which implies that $x_5 \simeq -0.127050844 + i\ 20.991896206$, and $x_6 = \frac{1}{x_5} = \bar{x}_5$. Suppose $x_5 = \text{cis}(\theta_5)$, where $\theta_5 = \text{Arctg}\left(\frac{I(x_5)}{R(x_5)}\right) + \pi \simeq 97°, 29'$. And $x_6 = \text{cis}(\theta_6)$ with $\theta_6 = 2\pi - \theta_5 \simeq 262°, 71'$.

Recall that, $g_L(x) = \sum_{n=0}^{\infty} p_n^L(3) x^n = \frac{1+x+x^3-x^5}{1-x^2-x^3-x^4+x^6} = \frac{U(x)}{V(x)}$. It is known that

$$p_n^L(3) = \frac{\rho_1}{{x_1}^{n+1}} + \frac{\rho_2}{{x_2}^{n+1}} + \cdots + \frac{\rho_6}{{x_6}^{n+1}},$$

where $\rho_i = \frac{-U(x_i)}{V'(x_i)}$, $i = 1, 2, \ldots, 6$. We also note that $\rho_4 = \bar{\rho}_3$ and $\rho_6 = \bar{\rho}_5$. Now suppose $\rho_3 = a_3 + ib_3$, $\rho_5 = a_5 + ib_5$, $e_{n,3} = \frac{\rho_3}{x_3^{n+1}} + \frac{\rho_4}{x_4^{n+1}}$ and $e_{n,5} = \frac{\rho_5}{x_5^{n+1}} + \frac{\rho_6}{x_6^{n+1}}$.

We have $e_{n,3} = \frac{\rho_3}{{x_3}^{n+1}} + \frac{\bar{\rho}_3}{\bar{x}_3^{n+1}}$ and $e_{n,5} = \frac{\rho_5}{{x_5}^{n+1}} + \frac{\bar{\rho}_5}{\bar{x}_5^{n+1}}$. Therefore

$$\begin{aligned} e_{n,3} &= (a_3 + ib_3)[\text{cis}(-n-1)\theta_3] + (a_3 - ib_3)[\text{cis}(n+1)\theta_3] \\ &= 2[a_3 \cos(n+1)\theta_3 + b_3 \sin(n+1)\theta_3]. \end{aligned}$$

Similarly $e_{n,5} = 2[a_5 \cos(n+1)\theta_5 + b_5 \sin(n+1)\theta_5]$.

Theorem 7. *For $n \geqslant 0$ we have*

$$|e_{n,3} + \frac{\rho_2}{{x_2}^{n+1}}| < 1 \quad \text{and} \quad |e_{n,5} + \frac{\rho_2}{{x_2}^{n+1}}| < 1.$$

Hence

$$p_n^L(3) = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + e_{n,3} + 0.5 \right\rfloor = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + e_{n,5} + 0.5 \right\rfloor,$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x, and $|x|$ is the absolute value of x.

Proof. Using values x_3 and x_5 we can compute $\rho_3 = \frac{-U(x_3)}{V'(x_3)}$ and $\rho_5 = \frac{-U(x_5)}{V'(x_5)}$ and then show that

$$\left| e_{n,3} + \frac{\rho_2}{{x_2}^{n+1}} \right| = \left| 2[a_3 \cos(n+1)\theta_3 + b_3 \sin(n+1)\theta_3] + \frac{\rho_2}{{x_2}^{n+1}} \right|$$

is less than or equal to $2(|a_3| + |b_3|) + \frac{\rho_2}{x_2} \leqslant 0.39 + 0.014 < 1$,
where $\rho_3 = a_3 + i\, b_3$ and $\rho_2 = \frac{-U(x_2)}{V'(x_2)}$. Similarly we have

$$\left| e_{n,5} + \frac{\rho_2}{{x_2}^{n+1}} \right| = \left| 2[a_5 \cos(n+1)\theta_5 + b_5 \sin(n+1)\theta_5] + \frac{\rho_2}{{x_2}^{n+1}} \right|$$
$$\leqslant 2(\mid a_5 \mid + \mid b_5 \mid) + \frac{\rho_2}{x_2} \leqslant 0.73 + 0.014 < 1.$$

Therefore, we get the result

$$p_n^L(3) = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + \frac{\rho_3}{{x_3}^{n+1}} + \frac{\rho_4}{{x_4}^{n+1}} + 0.5 \right\rfloor = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + \frac{\rho_5}{{x_5}^{n+1}} + \frac{\rho_6}{{x_6}^{n+1}} + 0.5 \right\rfloor.$$

That is

$$p_n^L(3) = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + e_{n,3} + 0.5 \right\rfloor = \left\lfloor \frac{\rho_1}{{x_1}^{n+1}} + e_{n,5} + 0.5 \right\rfloor.$$

This completes the proof of the theorem.

Remark 3. Using the approach described and in view of Remark 2, we can obtain an expression for $p_n^C(3)$, the number of minimal path sets of a circular con$|3|n$:F system. This approach becomes cumbersome for $k \geqslant 4$ and leads to introduction of a general recursive relation for determination of the number of minimal path sets of a con$|k|n$:F system, which is considered in the next section.

4 Consecutive-k-out-of-n:F System

In this section we provide a recursive relation to determine the number of minimal path sets of a linear con$|k|n$:F system. For some special cases we give closed form formulae. We may add that direct computation of the number of minimal path sets of a con$|k|n$:F system is still a difficult task.

Let $\alpha_k(n)$ denote the collection of all minimal path sets of a con$|k|n$:F system and suppose $P \in \alpha_k(n)$ is a minimal path set of the system. We note that $|P \cap \{1, 2, \ldots, k\}| = 1$. Therefore we can partition $\alpha_k(n)$ into k disjoint subcollections, that is $\alpha_k(n) = \bigcup_{i=1}^{k} \alpha_k^i(n)$, where $\alpha_k^i(n)$ is the collection of all minimal path sets of a con$|k|n$:F system that contain component i, $i = 1, 2, \ldots, k$.

Suppose $p_n(k) = |\alpha_k(n)|$ and $p_n^i(k) = |\alpha_k^i(n)|$. We then have $p_n(k) = \sum_{i=1}^{k} p_n^i(k)$.

The following lemmas are required in the sequel.

Lemma 4. $p_n^i(k) = p_n^{n-i+1}(k)$ *for all* $i = 1, 2, \ldots, \lfloor \frac{n+1}{2} \rfloor$.

Proof. Consider another con$|k|n$:F system in which the components are numbered in the reverse order. Hence the proof follows.

Component $n - i + 1$ is called the mirror image of component i.

Remark 4. We note that for $1 \leqslant n < k$, a con$|k|n$:F system is always working, irrespective of the states of the components. In this case $\alpha_k(n) = \{\emptyset\}$. Hence we assume that $p_n(k) = 1$ for $n = 1, 2, \ldots, k - 1$. We also assume that $p_0(k) = 1$.

Lemma 5. *For* $1 \leqslant i \leqslant k$*, we have*

$$\{S : \ S \in \alpha_k^i(n)\} = \{S : \ S = \{i\} \cup T\}$$

where T *is a minimal path set of a con*$|k|n-i$*:F subsystem that consists of last* $n-i$ *components of the original system, such that,* $T \cap \{i+1, i+2, \ldots, k\} = \emptyset$. *We assume that* $\{i+1, i+2, \ldots, k\} = \emptyset$ *if* $i = k$.

Proof. The proof follows from the definition of a minimal path set of a con$|k|n$:F system.

Theorem 8. *For* $1 \leqslant i \leqslant k$*, we have*

$$p_n^i(k) = p_{n-i}(k) - \sum_{x=1}^{k-i} p_{n-i}^x(k) = \sum_{x=k-i+1}^{k} p_{n-i}^x(k).$$

We assume that $\sum_{x=1}^{k-i} p_{n-i}^x(k) = 0$ *if* $i = k$.

Proof. Note that $p_{n-i}(k) = |\alpha_k(n-i)| = \left|\bigcup_{x=1}^{k} \alpha_k^x(n-i)\right| = \sum_{x=1}^{k} |\alpha_k^x(n-i)|$.
Hence, we have

$$|\{T : T \in \alpha_k(n-i),\ T \cap \{1,2,\ldots,k-i\} = \emptyset\}| = \sum_{x=k-i+1}^{k} |\alpha_k^x(n-i)|$$
$$= p_{n-i}(k) - \sum_{x=1}^{k-i} |\alpha_k^x(n-i)|.$$

On the other hand in view of Lemma 5, we have

$$|\{T : T \in \alpha_k(n-i),\ T \cap \{1,2,\ldots,k-i\} = \emptyset\}| = \sum_{x=k-i+1}^{k} p_{n-i}^x(k).$$

Therefore we get the result

$$p_n^i(k) = p_{n-i}(k) - \sum_{x=1}^{k-i} |\alpha_k^x(n-i)| = p_{n-i}(k) - \sum_{x=1}^{k-i} p_{n-i}^x(k) = \sum_{x=k-i+1}^{k} p_{n-i}^x(k).$$

This completes the proof of the theorem.

Remark 5. We note that to get $p_n^i(k)$ for $i = 1,2,\ldots,k$, in view of Theorem 8 we first compute $p_n^k(k) = p_{n-k}(k)$. Using this, compute $p_n^1(k) = p_{n-1}^k(k) = p_{n-k-1}(k)$ and then $p_n^{k-1}(k) = p_{n-k+1}(k) - p_{n-k+1}^1(k) = p_{n-k+1}(k) - p_{n-2k}(k)$ and $p_n^2(k)$ and so on. We note that the last term is $p_n^{\bar{r}}(k)$ where $\bar{r} = \lfloor (k+1)/2 \rfloor$.

Example.

(a) If $k = 2$, we have $p_n^2(2) = p_{n-2}(2)$ and $p_n^1(2) = p_{n-1}^2(2) = p_{(n-1)-2}(2) = p_{n-3}(2)$. We get $p_n(2) = p_n^1(2) + p_n^2(2) = p_{n-2}(2) + p_{n-3}(2)$, as given in section 2.

(b) If $k = 3$, we have $p_n^3(3) = p_{n-3}(3)$, $p_n^1(3) = p_{n-1}^3(3) = p_{n-4}(3)$, and $p_n^2(3) = p_{n-2}(3) - p_{n-2}^1(3) = p_{n-2}(3) - p_{n-6}(3)$. Hence, we get $p_n(3) = p_n^1(3) + p_n^2(3) + p_n^3(3) = p_{n-2}(3) + p_{n-3}(3) + p_{n-4}(3) - p_{n-6}(3)$, as given in section 3.

(c) If $k = 4$, we have $p_n^4(4) = p_{n-4}(4)$, $p_n^1(4) = p_{n-5}(4)$, $p_n^3(4) = p_{n-3}(4) - p_{n-8}(4)$, and $p_n^2(4) = p_{n-5}(4) - p_{n-10}(4) + p_{n-6}(4)$. Using these, we have $p_n(4) = \sum_{x=1}^{4} p_n^x(4) = p_{n-3}(4) + p_{n-4}(4) + 2p_{n-5}(4) + p_{n-6}(4) - p_{n-8}(4) - p_{n-10}(4)$.

(d) For $k = 5, 6$ and 7 we have the following recursive relations.

$$\begin{aligned} p_n(5) &= p_{n-3}(5) + p_{n-4}(5) + p_{n-5}(5) + 2p_{n-6}(5) + p_{n-7}(5) - 2p_{n-9}(5) \\ &\quad - 2p_{n-10}(5) - p_{n-12}(5) + p_{n-15}(5). \\ p_n(6) &= p_{n-4}(6) + p_{n-5}(6) + p_{n-6}(6) + 3p_{n-7}(6) + 2p_{n-8}(6) + p_{n-9}(6) \\ &\quad - 2p_{n-11}(6) - 2p_{n-12}(6) - 3p_{n-14}(6) - 2p_{n-15}(6) + p_{n-18}(6) \\ &\quad + p_{n-21}(6). \\ p_n(7) &= p_{n-4}(7) + p_{n-5}(7) + p_{n-6}(7) + p_{n-7}(7) + 3p_{n-8}(7) + 2p_{n-9}(7) \\ &\quad + p_{n-10}(7) - 3p_{n-12}(7) - 4p_{n-13}(7) - 3p_{n-14}(7) - 3p_{n-16}(7) \\ &\quad - 2p_{n-17}(7) + 3p_{n-20}(7) + 3p_{n-21}(7) + p_{n-24}(7) - p_{n-28}(7). \end{aligned}$$

Using Theorem 8, similar expressions can be for $p_n(k)$ when $k \geqslant 8$.

Remark 6. Regarding Theorem 8 and Remark 5, $p_n^i(k)$ is of the form $\sum_{x=1}^{d_i} c_i(x) p_{n-n_i(x)}(k)$ where d_i, $c_i(x)$'s, and $n_i(x)$'s are integers such that $0 < n_i(1) < n_i(2) < \cdots < n_i(d_i) = \bar{n}_i$ and $d_i > 0$. We note that this expression holds if $n \geqslant \bar{n}_i$. For example we have $p_n^k(k) = p_{n-k}(k)$ if $n \geqslant k$, $p_n^1(k) = p_{n-k-1}(k)$ if $n \geqslant k+1$, $p_n^{k-1}(k) = p_{n-k+1}(k) - p_{n-2k}(k)$ if $n \geqslant 2k$ and $p_n^2(k) = p_{n-k-1}(k) + p_{n-k-2}(k) - p_{n-2k-2}(k)$ if $n \geqslant 2k+2$. We get $\bar{n}_k = k$, $\bar{n}_1 = k+1$, $\bar{n}_{k-1} = 2k$, and $\bar{n}_2 = 2k+2$. Using Remark 5, it is easy to verify that $\bar{n}_k < \bar{n}_1 < \bar{n}_{k-1} < \bar{n}_2 < \cdots < \bar{n}_{\bar{r}}$ where $\bar{r} = \lfloor (k+1)/2 \rfloor$. Therefore the recursive relation $p_n(k) = \sum_{i=1}^{k} p_n^i(k)$ holds if $n \geqslant \bar{n}_{\bar{r}}$.

Lemma 6. $\bar{n}_{\bar{r}} = \binom{k+1}{2}$.

Proof. If $k = 2s+1$, for some integer s, $0 \leqslant s \leqslant \lfloor k/2 \rfloor$, we then have $\bar{r} = \lfloor (k+1)/2 \rfloor = s+1$. On the other hand, we have $\bar{n}_k = k$, $\bar{n}_1 = k+1$, $\bar{n}_{k-1} = 2k$, $\bar{n}_2 = 2k+2$, $\bar{n}_{k-2} = 3k$, $\bar{n}_3 = 3k+3$, $\ldots, \bar{n}_{k-(s-1)} = sk$, $\bar{n}_s = sk+s$, and $\bar{n}_{k-s} = (s+1)k$. Note that $\bar{n}_{k-s} = \bar{n}_{s+1} = \bar{n}_{\bar{r}} = (s+1)k = (k+1)k/2 = \binom{k+1}{2}$. If $k = 2s$, we have $\bar{n}_{k-(s-1)} = sk$, $\bar{n}_s = sk+s$. In this case $\bar{r} = \lfloor (k+1)/2 \rfloor = s$. Hence, $\bar{n}_{\bar{r}} = \bar{n}_s = sk+s = s(k+1) = k(k+1)/2 = \binom{k+1}{2}$. This completes the proof of the lemma.

Remark 7. We have $p_n(k) = 1$ for $0 \leqslant n \leqslant k-1$ and $p_k(k) = k$. We note that we first need to find $p_n(k)$ for $k < n < \binom{k+1}{2}$. We then can use the recursive equation $p_n(k) = \sum_{i=1}^{k} p_n^i(k)$ to find $p_n(k)$ for $n \geqslant \binom{k+1}{2}$, where $p_n^i(k)$ can be obtained using Theorem 8.

In the remainder of this section we consider some special cases.

Lemma 7. *If* $k \leqslant n \leqslant 2k$ *then* $p_n(k) = 2k - n + \frac{(n-k)(n-k+1)}{2}$.

Proof. Suppose $n = k + t$ for some integer t, $0 \leqslant t \leqslant k$. In view of Lemma 1, we have $\bar{r}_k(n) \geqslant 1$ and $\bar{\bar{r}}_k(n) \leqslant 2$. If $t < k$ then minimal path sets of size 1 are given by $\{k\}$, $\{k-1\}$, $\ldots$, $\{t+1\}$. If $t > 0$, then minimal path sets of size 2 are given by

$$\begin{aligned}
&\{1, k+1\}\\
&\{2, k+2\},\ \{2, k+1\}\\
&\vdots\\
&\{t, k+t\},\quad \{t, k+t-1\}, \ldots, \{t, k+1\}.
\end{aligned}$$

Therefore the number of minimal path sets of size 1 equals to $k - t$ and the number of minimal path sets of size 2 is $1 + 2 + \cdots + t = t(t+1)/2$. Hence, $p_n(k) = k - t + \frac{t(t+1)}{2}$, where $t = n - k$. This completes the proof of the lemma.

Theorem 9. *If $2k \leqslant n \leqslant 3k+1$ then $p_n(k)$ is given by*

$$\frac{(3k-n)(3k-n+1)}{2} + \frac{(n-2k)\left[3k(k+1) - 2\left((n-2k)^2 - 1\right)\right]}{6} + \frac{(n-2k)^2\left[(n-2k)^2 - 1\right]}{12}.$$

Proof. If $2k + 1 \leqslant n \leqslant 3k + 1$, then regarding Lemma 1, we have $\bar{r}_k(n) \geqslant 2$ and $\bar{\bar{r}}_k(n) \leqslant 4$.

Using a simple enumeration process, it can be shown that the number of minimal path sets of size 2 is

$$1 + 2 + 3 + \cdots + 3k - n = \frac{(3k-n)(3k-n+1)}{2}.$$

The number of minimal path sets of size 3 is given by

$$\sum_{t=1}^{s} \left[t(k-t+1) + (k-t) + (k-t-1) + \cdots + (n-2k-t+1)\right] = \frac{s\left[3k(k+1) - 2(s^2-1)\right]}{6},$$

where $s = n - 2k$.

And finally the number of minimal path sets of size 4 is given by

$$\sum_{j=2}^{s} \left[1 + 2^2 + 3^2 + \cdots + (j-1)^2\right] = \sum_{j=2}^{s}\sum_{i=1}^{j-1} i^2 = \frac{s^2(s^2-1)}{12}.$$

Now using the last three equations the proof follows.

Remark 8. If $3k \leqslant n \leqslant 4k+1$, we note that $\bar{r}_k(n) \geqslant 3$ and $\bar{\bar{r}}_k(n) \leqslant 6$. It can be verified that in this case, the number of minimal path sets of size 3, is given by

$$\sum_{j=1}^{k-s}\sum_{i=1}^{j} i = \sum_{j=1}^{k-s}\frac{j(j+1)}{2} = \frac{(k-s)(k-s+1)(k-s+2)}{6}$$
$$= \frac{(4k-n)(4k-n+1)(4k-n+2)}{6},$$

where $s = n-3k$.

Conjectures

In view of the argument given in this note, we make the following conjectures.

Using Lemma 7, we note that if $k \leqslant n \leqslant 2k$ then the number of minimal path sets of size 1 is $2k-n = \binom{2k-n}{1}$.

Using the first equation given in the proof of Theorem 9, we note that if $2k \leqslant n \leqslant 3k+1$ then the number of minimal path sets of size 2 is

$$\frac{(3k-n)(3k-n+1)}{2} = \binom{3k-n+1}{2}.$$

And by Remark 8, we note that if $3k \leqslant n \leqslant 4k+1$ then the number of minimal path sets of size 3 is

$$\frac{(4k-n)(4k-n+1)(4k-n+2)}{6} = \binom{4k-n+2}{3}.$$

Now let $p_n^{\bar{r}_k(n)}(k)$ denote the number of minimal path sets with minimum size $\bar{r}_k(n) = \lfloor n/k \rfloor$, in a con$|k|n$:F system.

Conjecture 1. $p_n^{\bar{r}_k(n)}(k) = \binom{(\bar{r}_k(n)+1)k-n+\bar{r}_k(n)-1}{\bar{r}_k(n)}$.

Note that the number of minimal path sets with minimum size and the number of path sets with minimum size in a con$|k|n$:F system are the same. Therefore, we can use the results of Seth and Sadegh (1999) to compute $p_n^{\bar{r}_k(n)}(k)$ but the expression as given in Conjecture 1 is simpler.

Conjecture 2. We have shown in section 2 and section 3 that the recursive relations for $p_n^C(k)$ and $p_n^L(k)$, for the cases $k=2$ and $k=3$ are the same but with different initial conditions. Hence, we conjecture that this property may also hold for a general con$|k|n$:F system, that is, for $k \geqslant 4$.

We may add that direct computation for determination of $p_n(k)$, the number of minimal path sets of a con$|k|n$:F system is a difficult task.

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