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Tsallis Entropy Properties of Order Statistics and Some Stochastic Comparisons

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Abstract. Tsallis entropy and order statistics are important in engineering reliability, image and signal processing. In this paper, we try to extend the concept of Tsallis entropy using order statistics. For this purpose, we propose the Tsallis entropy of order statistics and for it we obtain upper and lower bounds and some results on stochastic comparisons.

Keywords. Hazard rate function; order statistics; stochastic comparisons; tsallis entropy.

MSC 2010: 62G30; 94A17.

1 Introduction

Ever since Shannon (1948) proposed a measure of uncertainty in a discrete distribution based on the Bolltzmann entropy, there has been a great deal of interest in the measurement of uncertainty associated with a probability distribution. There is now a huge literature devoted to the applications, generalizations and properties of Shannon measure of uncertainty. Let X be

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a random variable having an absolutely continuous cumulative distribution function (cdf), $F(t)$ and probability density function (pdf), $f(t)$. Then, the basic uncertainty measure of X is defined as

$$H(X) = -\int_{-\infty}^{+\infty} f(x)\log f(x)dx = -E[\log f(X)]. \tag{1}$$

A generalization of the Shannon entropy is Tsallis entropy (Tsallis, 1988). The Tsallis entropy of order α for random variable X is defined as

$$S_\alpha(X) = \frac{1}{\alpha-1}\left[1-\int_{-\infty}^{+\infty} f^\alpha(x)dx\right], \quad \alpha \neq 1,\ \alpha > 0. \tag{2}$$

In general, Tsallis entropy can be negative. But, by choosing an appropriate value for α, it can be nonnegative. As $\alpha \to 1$ in (2), it reduces to $H(X)$ given in (1). Also, for a nonnegative random variable X, (2) can be reduced to

$$S_\alpha(X) = \frac{1}{\alpha-1}\left[1-\frac{1}{\alpha}E_{f_{X,\alpha}}[r(X)^{\alpha-1}]\right], \tag{3}$$

where $f_{X,\alpha}(x) = \frac{-d\bar{F}^\alpha(x)}{dx} = \alpha\bar{F}^{\alpha-1}(x)f(x)$; $\alpha > 1$, $x > 0$ and $r(t) = \frac{f(t)}{\bar{F}(t)}$, is the hazard rate function of X, where $\bar{F}(t) = 1 - F(t)$. Properties of the Tsallis entropy have been investigated by several authors including Nanda and Paul (2006), Zhang (2007), Wilk and Woldarczyk (2008) and Kumar and Taneja (2011).

The aim of this paper is to study the properties of Tsallis entropy of order statistics. Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed observations from cdf, $F(t)$ and pdf, $f(t)$. The order statistics of the sample is defined by the arrangement of $X_1, X_2, \ldots, X_n$ from the smallest to the largest, denoted as $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$. Denote the pdf and the cdf of $X_{i:n}$ by $f_{i:n}(x)$ and $F_{i:n}(x)$, respectively. Then, for $1 \leqslant i \leqslant n$,

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x), \tag{4}$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx, \quad a > 0, b > 0.$$

Order statistics are used in many branches of probability and statistics including characterization of probability distributions, goodness-of-fit tests, quality control, reliability theory and many other problems. Also, in physics, order statistics is useful to construct median filters for image and signal processing. (See Arnold et al., 2008 and David and Nagaraja, 2003 for more details). Information theoretic aspects of order statistics have been studied widely. Wong and Chen (1990), Park (1995, 1996), Ebrahimi et al. (2004), Baratpour et al. (2007, 2008), Zarezadeh and Asadi (2010). Considering importance of Tsallis entropy and order statistics in image and signal processing, we try to extend the concept of Tsallis entropy using order statistics.

In recent years, stochastic orders have attracted an increasing number of authors, who used them in several areas of probability and statistics, with applications in many fields, such as reliability theory, queueing theory, survival analysis, operations research, mathematical finance, risk theory, management science and biomathematics. Indeed, stochastic orders are often invoked not only to provide useful bounds and inequalities but also to compare stochastic systems. Several authors have studied the stochastic comparisons. For example, Ebrahimi and Kirmani (1996), Raqab and Amin (1996), Kochar (1999), Abbasnejad and Argami (2011), Di Crescenzo and Longobardi (2013), Psarrakos and Navarro (2013), Gupta et al. (2014). We continue this line of researches by exploring some properties of stochastic comparisons based on Tsallis entropy of order statistics.

The paper is organized as follows: The Tsallis entropy of order statistics is studied in Section 2. In Section 3, we obtain some bounds for Tsallis entropy of order statistics. Section 4 deals with the stochastic comparisons based on the Tsallis entropy of order statistics.

2 Tsallis Entropy of Order Statistics

Tsallis entropy associated with the ith order statistics $X_{i:n}$ is given by

$$S_\alpha(X_{i:n}) = \frac{1}{\alpha - 1}\left[1 - \int_{-\infty}^{\infty} f_{i:n}^{\alpha}(x)dx\right], \tag{5}$$

where $\alpha \neq 1, \alpha > 0$ and $f_{i:n}(x)$ is pdf of ith order statistics, for $i = 1, 2, \ldots, n$ that is defined by (4). Note that for $n = 1$, (5) reduces to (2).

Now, we use the probability integral transformation $U = F(X)$, where U has standard uniform distribution. It is known that if $U_{1:n} < U_{2:n} < \cdots < U_{n:n}$ are order statistics of a random sample $\{U_1, U_2, \ldots, U_n\}$ from standard uniform distribution, then, $U_{i:n}$, $i = 1, 2, \ldots, n$, has beta distribution with parameters i and $n-i+1$. In the following lemma, we will show that the Tsallis entropy of order statistic $X_{i:n}$ can be represented in terms of Tsallis entropy of order statistics of standard uniform distribution.

***Lemma* 1.** *Let $X_1, X_2, \ldots, X_n$ be a random sample with size n from continuous cdf, $F(t)$ and pdf, $f(t)$. Let $X_{i:n}$ denotes the ith order statistics. Then the Tsallis entropy of $X_{i:n}$ can be expressed as*

$$S_\alpha(X_{i:n}) = \frac{1}{\alpha-1} - E[f^{\alpha-1}(F^{-1}(Z_i))]\left[\frac{1}{\alpha-1} - S_\alpha(U_{i:n})\right], \qquad (6)$$

where $S_\alpha(U_{i:n})$ denotes the Tsallis entropy of the ith order statistics based on a random sample of size n from uniform distribution on $(0,1)$ and Z_i has beta distribution with parameters $\alpha(i-1)+1$ and $\alpha(n-i)+1$.

Proof. By (4) and (5), and by substutiting $z = F(x)$, we have

$$\begin{aligned}
S_\alpha(X_{i:n}) =& \frac{1}{\alpha-1}\left[1 - \int_0^1 \frac{z^{\alpha(i-1)}(1-z)^{\alpha(n-i)}}{B^\alpha(i,n-i+1)} f^{\alpha-1}(F^{-1}(z))dz\right] \\
=& \frac{1}{\alpha-1}\left[1 - \frac{B\big(\alpha(i-1)+1, \alpha(n-i)+1\big)}{B^\alpha(i,n-i+1)}\right. \\
& \left. \times \int_0^1 \frac{z^{\alpha(i-1)}(1-z)^{\alpha(n-i)}}{B\left(\alpha(i-1)+1, \alpha(n-i)+1\right)} f^{\alpha-1}(F^{-1}(z))dz\right] \\
=& \frac{1}{\alpha-1}\left[1 - \frac{B\big(\alpha(i-1)+1, \alpha(n-i)+1\big)}{B^\alpha(i,n-i+1)}\right. \\
& \left. \times E[f^{\alpha-1}(F^{-1}(Z_i))]\right]. \qquad (7)
\end{aligned}$$

It is easy to see that (5), for the ith order statistics of uniform distribution (that is, the beta distribution with parameters i and $(n-i+1)$) is given by

$$S_\alpha(U_{i:n}) = \frac{1}{\alpha-1}\left[1 - \frac{B\big(\alpha(i-1)+1, \alpha(n-i)+1\big)}{B^\alpha(i,n-i+1)}\right]. \qquad (8)$$

Using (8) in (7), the result follows. □

Remark 1. In reliability engineering, $(n-i+1)$-out-of-n systems are very important kind of structures. This system functions if and only if at least

$(n-i+1)$ components out of n components function. if $X_1, X_2, \dots, X_n$ denote the independent lifetimes of the components of such system, then the lifetime of the system is equal to the ith order statistics, $X_{i:n}$. The special cases of $i=1$ and $i=n$ correspond with the series and the paraller systems, respectively. Assuming that a $(n-i+1)$-out-of-n system is put in operation at time $t=0$, then the Tsallis entropy of $X_{i:n}$ measures the uncertainty of the lifetime of the system. Hence the Tsallis entropy, as a measure of uncertainty can be important for system designers to get information about the uncertainty of the used $(n-i+1)$-out-of-n systems.

Example 3. (a) Suppose that X has uniform distribution over $[a,b]$. Then, $f^{\alpha-1}(F_X^{-1}(t)) = (\frac{1}{b-a})^{\alpha-1}$ and we have

$$E[f^{\alpha-1}(F^{-1}(Z_i))] = \left(\frac{1}{b-a}\right)^{\alpha-1}.$$

By Lemma 1, it can be easily shown that

$$S_\alpha(X_{i:n}) = S_\alpha(X_{n-i+1:n}), \qquad i = 1, 2, \dots, n. \tag{9}$$

Therefore, for first and last order statistics of a random sample of size n, (9) gives

$$S_\alpha(X_{1:n}) = S_\alpha(X_{n:n}) = \frac{1}{\alpha-1}\left[1 - (\frac{n}{b-a})^\alpha \frac{b-a}{\alpha(n-1)+1}\right].$$

On the other hand, we have

$$S_\alpha(X) = \frac{1}{\alpha-1}\left[1-(b-a)^{1-\alpha}\right],$$

hence, for $i = 1, n$, we get

$$S_\alpha(X) - S_\alpha(X_{i:n}) = \frac{(b-a)^{1-\alpha}}{\alpha-1}\left[\frac{n^\alpha}{\alpha(n-1)+1} - 1\right].$$

That is, in the uniform case, for $\alpha > 1$, the difference between Tsallis entropy of the lifetime of each component of a series system or a paraller system and Tsallis entropy of the lifetime of the system is nonnegative.
(b) Let $F(t)$ be exponential with mean $\frac{1}{\lambda}$. Then, $f^{\alpha-1}(F_X^{-1}(t)) = \lambda^{\alpha-1}(1-$

$t)^{\alpha-1}$ and we can conclude that

$$E[f^{\alpha-1}(F^{-1}(Z_i))] = \lambda^{\alpha-1}\frac{B\left(\alpha(i-1)+1, \alpha(n-i+1)\right)}{B\left(\alpha(i-1)+1, \alpha(n-i)+1\right)}.$$

Therefore, Lemma 1 for $i = 1, n$ gives

$$S_\alpha(X_{1:n}) = \frac{1}{\alpha-1} - \frac{(n\lambda)^{\alpha-1}}{\alpha(\alpha-1)}, \tag{10}$$

and

$$S_\alpha(X_{n:n}) = \frac{1}{\alpha-1}\big[1 - n^\alpha\lambda^{\alpha-1}B(\alpha(n-1)+1, \alpha)\big].$$

By replacing n by 1 in (10), we have

$$S_\alpha(X) = \frac{1}{\alpha-1}\left[1 - \frac{\lambda^{\alpha-1}}{\alpha}\right].$$

Then, we have

$$S_\alpha(X) - S_\alpha(X_{1:n}) = \frac{\lambda^{\alpha-1}}{\alpha(\alpha-1)}\left[n^{\alpha-1} - 1\right].$$

Thus, in the exponential case, the difference between Tsallis entropy of the lifetime of each component of a series system and Tsallis entropy of the lifetime of the system is nonnegative, for $\alpha > 1$.

3 Bounds for Tsallis Entropy of Order Satistics

Ebrahimi et al. (2004) obtained some bounds for Shannon entropy of order statistics. Abbasnejad and Arghami (2011) provided some bounds for Renyi entropy of order statistics. Gupta et al. (2014), derived the upper bounds for residual entropy. In this section, we derive upper and lower bounds for Tsallis entropy of order statistics.

***Theorem* 1.** *For any random variable X, with Tsallis entropy $S_\alpha(X) < \infty$, the Tsallis entropy of $X_{i:n}, i = 1, 2, \ldots, n$ is bounded as follows:*
(i) for all $\alpha > 1$ $(0 < \alpha < 1)$

$$S_\alpha(X_{i:n}) \geqslant (\leqslant)\frac{1}{\alpha-1}\big[1 - B_i\big(1 - (\alpha-1)S_\alpha(X)\big)\big],$$

where

$$B_i = \frac{1}{B^\alpha(i, n-i+1)} m_i^{\alpha(i-1)}(1-m_i)^{\alpha(n-i)},$$

where $m_i = \frac{i-1}{n-1}$ *is the mode of the beta distribution with parameters* $\alpha(i-1)+1$ *and* $\alpha(n-i)+1$*; and* $B_1 = B_n = n^\alpha$.

(ii) Let $M = f(m) < \infty$*, where* m *is the mode of the distribution of* X*. Then*

$$S_\alpha(X_{i:n}) \geqslant \frac{1}{\alpha-1} - M^{\alpha-1}\left[\frac{1}{\alpha-1} - S_\alpha(U_{i:n})\right],$$

for all $\alpha > 0$.

Proof. (i) Let g_i and m_i be the pdf and the mode of beta distribution with parameters $\alpha(i-1)+1$ and $\alpha(n-i)+1$, respectively. From (8) and since the mode of this beta distribution is $m_i = \frac{i-1}{n-1}$, we have, for $\alpha > 1$,

$$\begin{aligned}
-E[f^{\alpha-1}(F^{-1}(Z))]&\left[\frac{1}{\alpha-1} - S_\alpha(U_{i:n})\right] \\
&= -\frac{B(\alpha(i-1)+1, \alpha(n-i)+1)}{(\alpha-1)B^\alpha(i, n-i+1)} \int_0^1 g_i(z) f^{\alpha-1}(F^{-1}(z))dz \\
&\geqslant -\frac{B(\alpha(i-1)+1, \alpha(n-i)+1)}{(\alpha-1)B^\alpha(i, n-i+1)} \int_0^1 g_i(m) f^{\alpha-1}(F^{-1}(z))dz \\
&= -\frac{1}{\alpha-1} B_i \int_0^1 f^\alpha(x)dx \\
&= -\frac{1}{\alpha-1} B_i \big(1-(\alpha-1)S_\alpha(X)\big),
\end{aligned}$$

where the last equality is obtained by (2). Thus, using (6), the result follows. For $0 < \alpha < 1$ the proof is similar.

(ii) For $\alpha > 1$ ($0 < \alpha < 1$), we have

$$f^{\alpha-1}(F^{-1}(z)) \leqslant (\geqslant) M^{\alpha-1}.$$

Thus, using (6), we can conclude that

$$S_\alpha(X_{i:n}) \geqslant \frac{1}{\alpha-1} - M^{\alpha-1}\left[\frac{1}{\alpha-1} - S_\alpha(U_{i:n})\right].$$

□

When the distribution function $F(t)$ does not have a closed form, the density of order statistics and the expectation in (6), can not be easily computed. In this case, bounds in Theorem 1 are useful. Also, since the Tsallis entropy expression for some well known distribution are available, we can compute the bounds in Theorem 1, easily. When the lower bounds in both parts of Theorem 1, are available, one may use the maximum of the these lower bounds.

Note that in part(i) of Theorem 1, $B_1 = B_n = n^\alpha$ and in part (ii) of Theorem 1,

$$S_\alpha(U_{i:n}) = \frac{1}{\alpha-1}\left[1 - \frac{n^\alpha}{\alpha(n-1)+1}\right], \quad i = 1, n.$$

Thus, we have the following bounds for $S_\alpha(X_{i:n})$, $i = 1, n$:

$$S_\alpha(X_{i:n}) \geqslant (\leqslant) \frac{1}{\alpha-1}\left[1 - n^\alpha\big(1-(1-\alpha)S_\alpha(X)\big)\right], \quad \alpha > 1 \;\; (0 < \alpha < 1),$$

and for $\alpha > 0$

$$S_\alpha(X_{i:n}) \geqslant \frac{1}{\alpha-1}\left[1 - \frac{n^\alpha M^{\alpha-1}}{\alpha(n-1)+1}\right], \quad i = 1, n.$$

Considering this fact, we have the following example.

Example 4. (a) The pdf of Pareto distribution with parameters λ and β is given by

$$f(x) = \frac{\lambda\beta^\lambda}{x^{\lambda+1}}, \qquad x \geqslant \beta > 0, \quad \lambda > 0.$$

Here, $M = \lambda\beta^{-1}$ and $S_\alpha(X) = \frac{1}{\alpha-1}\left[1 - \frac{\lambda^\alpha}{\beta^{\alpha-1}(\alpha\lambda+\alpha-1)}\right]$. The distribution of $X_{1:n}$ is also Pareto with parameters $n\lambda$ and β. Thus,

$$S_\alpha(X_{1:n}) = \frac{1}{\alpha-1}\left[1 - \frac{(n\lambda)^\alpha}{\beta^{\alpha-1}(n\alpha\lambda+\alpha-1)}\right].$$

The distribution of $X_{n:n}$ is more complicated. Using (6), we have

$$S_\alpha(X_{n:n}) = \frac{1}{\alpha-1}\left[1 - \frac{(n\lambda)^\alpha}{\lambda\beta^{\alpha-1}} B(\alpha(n-1)+1, \alpha + \frac{\alpha-1}{\lambda})\right].$$

Now, we can find the following bounds for $\alpha > 1$:

$$S_\alpha(X_{i:n}) \geqslant \frac{1}{\alpha-1}\left[1 - \frac{(n\lambda)^\alpha}{\beta^{\alpha-1}(\alpha\lambda+\alpha-1)}\right], \qquad i = 1, n,$$

and

$$S_\alpha(X_{i:n}) \geqslant \frac{1}{\alpha-1}\left[1 - \frac{n^\alpha\lambda^{\alpha-1}}{\beta^{\alpha-1}\big(\alpha(n-1)+1\big)}\right], \qquad i = 1, n,$$

and for $0 < \alpha < 1$

$$\frac{1}{\alpha-1}\left[1 - \frac{n^\alpha\lambda^{\alpha-1}}{\beta^{\alpha-1}\big(\alpha(n-1)+1\big)}\right] \leqslant S_\alpha(X_{i:n}) \leqslant \frac{1}{\alpha-1}\left[1 - \frac{(n\lambda)^\alpha}{\beta^{\alpha-1}(\alpha\lambda+\alpha-1)}\right].$$

In this case, we see that for $\alpha > 1$, the lower bound in (ii) of Theorem 1 is sharper. For $0 < \alpha < 1$ the difference between the upper bound and $S_\alpha(X_{1:n})$ is

$$\frac{(n\lambda)^\alpha}{(\alpha-1)\beta^{\alpha-1}}\left[\frac{1}{(n\alpha\lambda+\alpha-1)} - \frac{1}{(\alpha\lambda+\alpha-1)}\right],$$

which is an increasing function of n. Thus, the upper bound is useful when n is not large.
(b) For the exponential destribution with mean $\frac{1}{\lambda}$, $M = \lambda$ and $S_\alpha(X) = \frac{1}{\alpha-1}\big[1 - \frac{\lambda^{\alpha-1}}{\alpha}\big]$. As noted in Example 3, we have

$$S_\alpha(X_{1:n}) = \frac{1}{\alpha-1} - \frac{(n\lambda)^{\alpha-1}}{\alpha(\alpha-1)}, \tag{11}$$

and

$$S_\alpha(X_{n:n}) = \frac{1}{\alpha-1}\big[1 - n^\alpha\lambda^{\alpha-1}B(\alpha(n-1)+1, \alpha)\big].$$

For $\alpha > 1$ the difference between $S_\alpha(X_{1:n})$ and the lower bound in (ii) is $\frac{(n\lambda)^{\alpha-1}}{\alpha-1}\big[\frac{n}{\alpha(n-1)+1} - \frac{1}{\alpha}\big]$ which is an increasing function on n. Also, the difference between $S_\alpha(X_{n:n})$ and the lower bound in (ii) is $\frac{n^\alpha\lambda^{\alpha-1}}{\alpha-1}\big[\frac{1}{\alpha(n-1)+1} - B(\alpha(n-1)+1, \alpha)\big]$ which is an increasing function on n. Thus, the lower bound is useful when n is not large.

4 Stochastic Comparisons

Let X and Y be two random variables and let the distribution function, density function and hazard rate function of X be denoted by $F(t)$, $f(t)$ and $r(t)$ and those of Y be denoted by and $G(t)$, $g(t)$ and $q(t)$, respectively. In this section, we provide some results on the Tsallis entropy of order statistics in terms of ordering properties of distributions. We need the following definitions:

1. A nonnegative random variable X is said to have increasing (decreasing) failure rate (hazard function) IFR (DFR) if $r(t) = \dfrac{f(t)}{\bar{F}(t)}$ is increasing (decreasing) in t.

2. A random variable X is said to be smaller than Y in despersion ordering (denoted by $X \overset{disp}{\leqslant} Y$) if and only if

$$F^{-1}(v) - F^{-1}(u) \leqslant G^{-1}(v) - G^{-1}(u), \qquad \forall\ 0 \leqslant u \leqslant v \leqslant 1,$$

   or equivalenty,

$$g(G^{-1}(u)) \leqslant f(F^{-1}(u)), \qquad \forall\ u \in (0,1).$$

   Here $F^{-1}(u) = \sup\{x : F(x) \leq u\}$.

3. A random variable X is said to be smaller than Y in likelihood ratio ordering (denoted by $X \overset{lr}{\leqslant} Y$) if $\frac{f_X(t)}{g_Y(t)}$ is nonincreasing in t.

4. A random variable X is said to be smaller than Y in the failure rate (hazard rate) ordering (denoted by $X \overset{fr}{\leqslant} Y$) if $r(t) \geqslant q(t)$, for all $t \geqslant 0$, where $r(t)$ and $q(t)$ are the failure rate functions of X and Y, respectively, or equivalently if $\dfrac{\bar{F}(t)}{\bar{G}(t)}$ is decreasing in t.

5. A random variable X is said to be smaller than Y in the stochastic ordering (denoted by $X \overset{st}{\leqslant} Y$) if $\bar{F}_X(t) \leqslant \bar{G}_Y(t)$ for all t.

6. A random variable X is said to be smaller than Y in entropy ordering (denoted by $X \overset{e}{\leqslant} Y$) if $H(X) \leqslant H(Y)$, where $H(X)$ and $H(Y)$ are the Shannon entropy of X and Y, respectively.

7. A random variable X is said to be smaller than Y in Tsallis entropy ordering (denoted by $X \overset{Ts}{\leqslant} Y$) if $S_\alpha(X) \leqslant S_\alpha(Y)$ for all $\alpha > 0$.

Remark 2. It is well known that $X \overset{lr}{\leqslant} Y \Rightarrow X \overset{fr}{\leqslant} Y \Rightarrow X \overset{st}{\leqslant} Y$ and $X \overset{disp}{\leqslant} Y \Rightarrow X \overset{st}{\leqslant} Y$ and $X \overset{lr}{\leqslant} Y \Rightarrow X \overset{st}{\leqslant} Y$ (Bickel and Lehmann, 1976; and Shaked and Shanthikumar, 1994).

Theorem 2. *Let X and Y be two random variables. Then, $X \overset{\text{disp}}{\leqslant} Y$ implies $X \overset{Ts}{\leqslant} Y$.*

Proof. From $X \overset{disp}{\leqslant} Y$ and using (2), we have

$$S_\alpha(X) - S_\alpha(Y) = \frac{1}{\alpha - 1}\left[\int_0^1 \left(g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u))\right) du\right] \leqslant 0,$$

for all $\alpha > 0$. □

In the special case of $\alpha \to 1$, $X \overset{disp}{\leqslant} Y$ implies $X \overset{e}{\leqslant} Y$ (Oja, 1981). By the fact that, $X \overset{disp}{\leqslant} Y$ implies that $X_{i:n} \overset{disp}{\leqslant} Y_{i:n}$ (Shaked and Shanthikumar, 1994), we have the following corollary.

Corollary 1. *Let X and Y be two random variables and denote their order statistic by $X_{i:n}$ and $Y_{i:n}$, $i = 1, 2, \ldots, n$, respectively. Then $X \overset{disp}{\leqslant} Y$ implies $X_{i:n} \overset{Ts}{\leqslant} Y_{i:n}$.*

Theorem 3. *Let X and Y be two random variables which at least one of them is DFR. Then, $X \overset{fr}{\leqslant} Y$ implies $X \overset{Ts}{\leqslant} Y$.*

Proof. From Remark 2, $X \overset{fr}{\leqslant} Y$ implies $X \overset{st}{\leqslant} Y$. But, $X \overset{st}{\leqslant} Y$ is equivalent to

$$E(\phi(X)) \leqslant (\geqslant) E(\phi(Y)),$$

for all increasing (decreasing) functions ϕ such that these expectations exist (see Shaked and Shanthikumar, 2007). First, we assume that $0 < \alpha < 1$ and X is DFR, then $r(t)^{\alpha-1}$ is increasing and we have

$$E(r(X)^{\alpha-1}) \leqslant E(r(Y)^{\alpha-1}). \tag{12}$$

On the other hand, $X \overset{fr}{\leqslant} Y$ implies that the respective hazard rate functions satisfy $r(t) \geqslant q(t)$. Hence, we get

$$E(r(Y)^{\alpha-1}) \leqslant E(q(Y)^{\alpha-1}). \tag{13}$$

Using (12) and (13) gives

$$E(r(X)^{\alpha-1}) \leqslant E(q(Y)^{\alpha-1}).$$

Thus, using (3), we obtain $X \overset{Ts}{\leqslant} Y$. For $\alpha > 1$ the proof is similar.
The proof is similar when we assume that Y is DFR. □

Using Remark 2, we have the following result.

Corollary 2. *Under the assumptions of Theorem 3, if $X \overset{lr}{\leqslant} Y$, then $X \overset{Ts}{\leqslant} Y$.*

Corollary 3. *Let X be a nonnegative random variable and denote its order statistics by $X_{i:n}, i = 1, 2, \dots, n$. Suppose X has a DFR distribution. If $X_{i:n} \overset{lr}{\leqslant} X$, then $X_{i:n} \overset{Ts}{\leqslant} X$.*

Since it is well known $X_{1:n} \overset{lr}{\leqslant} X \overset{lr}{\leqslant} X_{n:n}$ (Shaked and Shanthikumar, 1994), we have the following corollary.

Corollary 4. *Let X be a nonnegative random variable having a DFR distribution. Then, $X_{1:n} \overset{Ts}{\leqslant} X \overset{Ts}{\leqslant} X_{n:n}$.*

Theorem 4. *Let X be a nonnegative random variable and denote its order statistics by $X_{i:n}, i = 1, 2, \dots, n$. Suppose $X_{i:n}$ has a DFR distribution. Then $X_{i:n} \overset{Ts}{\leqslant} X_{j:n},\ i < j$.*

Proof. Using the result of Chan et al. (1991), we have $X_{i:n} \overset{lr}{\leqslant} X_{j:n}$. This implies that $X_{i:n} \overset{fr}{\leqslant} X_{j:n}$. Since $X_{i:n}$ has a DFR distribution Theorem 3, implies that $X_{i:n} \overset{Ts}{\leqslant} X_{j:n}$. □

Theorem 5. *Let X and Y be two nonnegative random variables and denote their order statistics by $X_{i:n}$ and $Y_{i:n}, i = 1, 2, \dots, n$, respectively. Suppose at least one of $X_{i:n}$ or $Y_{i:n}$ is DFR. Then $X_i \overset{fr}{\leqslant} Y_j$, $i, j \in \{1, \dots, n\}$ implies $X_{i:n} \overset{Ts}{\leqslant} Y_{i:n}$.*

Proof. Note that $X_i \overset{fr}{\leqslant} Y_j$ implies $X_{i:n} \overset{fr}{\leqslant} Y_{i:n}$ (Shaked and Shanthikumar, 1994). Since at least one of $X_{i:n}$ or $Y_{i:n}$ is DFR, by using Theorem 3 we have $X_{i:n} \overset{Ts}{\leqslant} Y_{i:n}$ and this completes the proof. □

***Corollary* 5.** *Under the assumptions of Theorem 5, $X_i \overset{lr}{\leqslant} Y_j$, $i, j \in \{1, \ldots, n\}$ implies $X_{i:n} \overset{Ts}{\leqslant} Y_{i:n}$.*

The following result deals with the likelihood ratio ordering of order statistics of different sample sizes.

***Theorem* 6.** *Let $X_1, X_2, \ldots, X_n, X_{\max(n,m)}$ be independent and identically distributed (iid) random variables where m and n are positive integers. Then, $X_{j:m} \overset{lr}{\leqslant} X_{i:n}$ whenever $j \leqslant i$ and $m - j \geqslant n - i$.*

Proof. See Shaked and Shanthikumar (2007). □

In particular $X_{n:n} \overset{lr}{\leqslant} X_{n+1:n+1}$ and $X_{1:n+1} \overset{lr}{\leqslant} X_{1:n}$. Using the above result, we have the following theorem.

***Theorem* 7.** *Let $X_1, X_2, \ldots, X_{n+1}$ be iid random variables with cdf $F(t)$. Suppose X has a DFR distribution.*
Then,

(i) $X_{1:n+1} \overset{Ts}{\leqslant} X_{1:n}$,

(ii) $X_{n:n} \overset{Ts}{\leqslant} X_{n+1:n+1}$.

proof. Using Theorem 6, we have $X_{1:n+1} \overset{lr}{\leqslant} X_{1:n}$. This implies that $X_{1:n+1} \overset{fr}{\leqslant} X_{1:n}$. Since X has a DFR distribution, $X_{1:n}$ and $X_{1:n+1}$ have DFR distribution (Takahasi, 1988). So, by using Theorem 3, we can conclude that $X_{1:n+1} \overset{Ts}{\leqslant} X_{1:n}$. Similarly, we can prove (ii). □

***Theorem* 8.** *Let X and Y be two continuous random variables with cdfs $F(t)$ and $G(t)$; and pdfs $f(t)$ and $g(t)$; and denote their order statistics by $X_{i:n}$ and $Y_{i:n}, i = 1, 2, \ldots, n$, respectively. Suppose*

$$W_1 = \{u \in (0,1) | \, f(F^{-1}(u)) \geqslant g(G^{-1}(u))\},$$
$$W_2 = \{u \in (0,1) | \, f(F^{-1}(u)) \leqslant g(G^{-1}(u))\},$$

$W_i \neq \emptyset$, $i = 1, 2$, $\sup_{W_2} h(u)^\alpha \leqslant \inf_{W_1} h(u)^\alpha$, *where* $h(u) = u^{i-1}(1-u)^{n-i}$, $0 < u < 1$. *If* $X \overset{Ts}{\leqslant} Y$ *then,* $X_{i:n} \overset{Ts}{\leqslant} X_{i:n}$.

Proof. From (4) and (5) and by substituting $u = F(x)$, we conclude that

$$S_\alpha(X_{i:n}) = \frac{1}{\alpha - 1}\left[1 - \int_0^1 \frac{1}{B^\alpha(i, n-i+1)} u^{\alpha(i-1)}(1-u)^{\alpha(n-i)} f^{\alpha-1}(F^{-1}(u))du\right].$$

By taking $u = G(x)$ for $S_\alpha(Y_{i:n})$, we have

$$S_\alpha(Y_{i:n}) = \frac{1}{\alpha - 1}\left[1 - \int_0^1 \frac{1}{B^\alpha(i, n-i+1)} u^{\alpha(i-1)}(1-u)^{\alpha(n-i)} g^{\alpha-1}(G^{-1}(u))du\right].$$

Thus, we find

$$S_\alpha(X_{i:n}) - S_\alpha(Y_{i:n}) = \frac{1}{(\alpha-1)B^\alpha(i, n-i+1)} D_u, \tag{14}$$

where

$$D_u = \int_0^1 h^\alpha(u)\left[g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u))\right] du.$$

Supposing $\alpha > 1$, from $X \overset{Ts}{\leqslant} Y$, we have

$$\int_0^1 \left[g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u))\right] du \leqslant 0. \tag{15}$$

Thus, from (15), we conclude that

$$\begin{aligned}
D_u &= \int_{W_1} h(u)^\alpha \left[g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u))\right] du \\
&\quad + \int_{W_2} h(u)^\alpha \left[g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u))\right] du \\
&\leqslant \inf_{W_1} h(u)^\alpha \int_{W_1} \left[g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u))\right] du \\
&\quad + \sup_{W_2} h(u)^\alpha \int_{W_2} \left[g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u))\right] du \\
&\leqslant \inf_{W_1} h(u)^\alpha \int_{W_1} \left[g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u))\right] du \\
&\quad + \inf_{W_1} h(u)^\alpha \int_{W_2} \left[g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u))\right] du
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{W_1} h(u)^{\alpha} \int_0^1 \left[g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u)) \right] du \\
&\leqslant 0.
\end{aligned}
$$

For $0 < \alpha < 1$, by noting that $\int_0^1 \left[g^{\alpha-1}(G^{-1}(u)) - f^{\alpha-1}(F^{-1}(u)) \right] du \geqslant 0$, in a similar way we can show that $D_u \leqslant 0$, that completes the proof. □

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