مجلهی پژوهشهای آماری ایران

J. Statist. Res. Iran **9** (2012): 61–85

دورهی ۹، شمارهی ۱، بهار و تابستان ۱۳۹۱، صص ۶۱–۸۵

Estimation in Simple Step-Stress Model for the Marshall-Olkin Generalized Exponential Distribution under Type-I Censoring

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Abstract. This paper considers the simple step-stress model from the Marshall-Olkin generalized exponential distribution when there is time constraint on the duration of the experiment. The maximum likelihood equations for estimating the parameters assuming a cumulative exposure model with lifetimes as the distributed Marshall-Olkin generalized exponential are derived. The likelihood equations do not lead to closed form expressions for the maximum likelihood estimators (MLEs), and they need to be solved by using an iterative procedure. We then evaluate the properties of MLEs through the mean squared error, relative absolute bias and relative error. We also derive confidence intervals for the parameters using asymptotic distributions of the MLEs and the parametric bootstrap methods. Finally, an example is presented to illustrate the discussed methods of asymptotic and bootstrap confidence intervals.

Keywords. Bootstrap method; cumulative exposure model; maximum likelihood estimation; Marshall-Olkin generalized exponential distribution; step-stress model; type-I censoring.

MSC 2010: 62F10, 62F12, 62F40, 62N01.

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1 Introduction

In most life-testing experiments, we can not continue the experiment until the last failure is observed based on cost, time and some other considerations. So, the experiment is usually terminated when either a pre-fixed censoring time t arrives (Type-I censoring scheme) or when the rth failure is observed (Type-II censoring scheme); see, for example, Harter and Balakrishnan (1996). In many situations, it may be difficult to collect data on life-time of a product under normal operating conditions as the product may have a high reliability under normal conditions. For this reason, accelerated life-testing (ALT) experiments can be used to force these products (systems or components) to fail more quickly than under normal operating conditions. Some key references in the area of accelerated testing include Nelson (1980, 1990), Meeker and Escobar (1998), and Bagdonavicius and Nikulin (2002).

A special class of the ALT is called the step-stress testing which allows the experimenter to choose one or more stress factors in a life-testing experiment. Stress factors can include humidity, temperature, vibration, voltage, load or any other factors that directly affect the life of the products. In such a life-testing experiment, n identical units are placed on an initial stress level s_0 under a m-step-stress model, and only the successive failure times are recorded. The stress levels are changed to $s_1, \ldots, s_m$ at pre-fixed times $t_1, \ldots, t_m$, respectively. The most common model used to analyse these times-to- failure data is the cumulative exposure model.

We consider here a simple step-stress model with only two stress levels. This model has been studied extensively in the literature; Sedyakin (1966) introduced the cumulative exposure model in the simple step-stress case which has been further discussed and generalized by Bagdonavicius (1978) and Nelson (1980), while Miller and Nelson (1983) and Bai et al. (1989) discussed the determination of optimal time at which to change the stress level from s_0 and s_1. Xiong (1998), Xiong and Milliken (1999), and Balakrishnan et al. (2007) have all considered inferences for the step-stress model assuming exponential lifetimes based on complete, and Type-II censored samples. Balakrishnan et al. (2009) discussed exact inference for step-stress models under exponential distribution when the available data are Type-I censored. Balakrishnan and Iliopoulos (2010) established stochastic monotonicity of the MLEs of parameters in exponential simple step-stress models. Chen and Lio (2010) obtained the maximum likelihood estimation of the parameters in the generalized exponential distribution under progressive type-I interval censoring; Also Abdel-Hamid and AL-Hussaini (2009) estimated the parameters of the step-stress accelerated life tests for the exponentiated exponential distribution with Type-I censoring.

In the next section, we introduce a Marshall-Olkin generalized exponential distribution and then, we consider a simple step-stress model with two stress levels based on the Marshall-Olkin generalized exponential distribution when there is time constraint on the duration of the experiment. The model is described with details in Section 3. we derive maximum likelihood equations for estimating the parameters. But the likelihood equations do not lead to closed form expressions for the MLE, and they need to be solved by using an iterative procedure. In Section 4, asymptotic confidence intervals of the estimators are presented. In Section 5, asymptotic variance covariance matrix of the estimators are given. Simulation studies and properties of maximum likelihood estimators are given in Section 6. we discuss the bootstrap methods in Section 7. Finally, we state some results in Section 8.

2 Marshall-Olkin Generalized Exponential Distribution

The probability distribution function (PDF) of the Marshall-Olkin generalized exponential (MOGE) distribution with the parameters p and θ is defined by

$$f(x) = \frac{pe^{-\frac{x}{\theta}}}{\theta\left\{1-(1-p)e^{-\frac{x}{\theta}}\right\}^2}, \qquad x > 0, \tag{1}$$

where $0 < p \leqslant 1$ and $\theta > 0$. The corresponding cumulative distribution function (CDF) is

$$F(x) = \frac{1-e^{-\frac{x}{\theta}}}{1-(1-p)e^{-\frac{x}{\theta}}}, \qquad x > 0. \tag{2}$$

If $p = 1$, then the MOGE distribution reduces to the exponential distribution with the scale parameter θ. This model was first proposed by Marshall and Olkin (1997) and extensively discussed by Alice and Jose (1999). Figure 1 illustrates the pdf of the MOGE distribution for some values of p and θ.

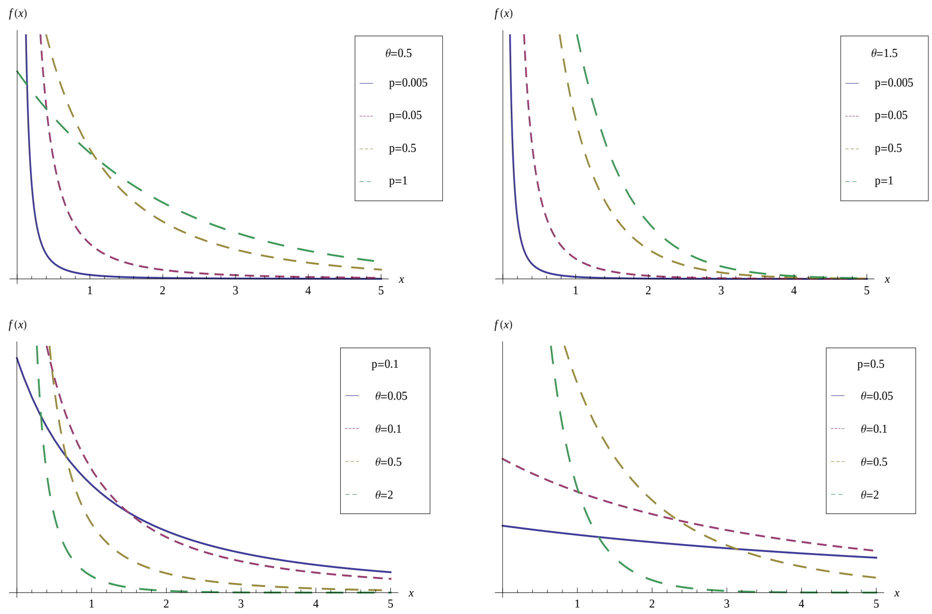

Figure 1. The pdf of the MOGE distribution for some values of p and θ.

3 Step-stress Accelerated Life Tests under Censoring

The term “Accelerated life test” applies to the type of study where failure times can be accelerated by applying higher “stress” to the component. This implies that the failure time is a function of the so called ”stress factor” and higher stress may bring quicker failure. For example, some component may fail quicker at a higher temperature however, it may have a long life at lower temperatures. At low “stress” conditions, the time required may be too large for its reliability estimation which may be tested under higher stress factors terminating the experiment in a relatively shorter time, by this process failures which under normal conditions would occur only after a long testing can be observed quicker and the size of data can be increased without a large cost and long time. This type of reliability testing is called “Accelerated life testing”. Accelerated life testing methods are also useful for obtaining information on the life of products or materials over a range of conditions, which are encountered in practice. Some information can be obtained by testing over the range of conditions of interest or over more severe conditions and then extrapolating the results over the range of interest. This type of test conditions are typically produced by testing units at high levels of temperature, voltage, pressure. vibration, cyclic rate, load etc. or some combination of them. Stress variables are used in engineering practice for many products and materials. In other fields similar problems arise when the relationship between variables could affect its life time. Therefore the models formulated are based on either past studies or theoretical development that could relate the distribution of failure time to stress or other variables. Such models are also useful in survival analysis where dependence of the life time of individuals on concomitant variables is analyzed (Sen, 1999); see for more details Nelsen (1980, 1990).

Suppose that the data come from a cumulative exposure model, and we consider a simple step-stress model with only two stress levels s_0 and s_1. The lifetime distributions at s_0 and s_1 are assumed to be the MOGE with parameters θ_1 and θ_2, respectively; and a common parameter p. The PDF and CDF are given by

$$f_k(x;p,\theta_k) = \frac{pe^{-\frac{x}{\theta_k}}}{\theta_k\{1-(1-p)e^{-\frac{x}{\theta_k}}\}^2}, \tag{3}$$

and

$$F_k(x;p,\theta_k) = \frac{1-e^{-\frac{x}{\theta_k}}}{1-(1-p)e^{-\frac{x}{\theta_k}}}, \tag{4}$$

respectively, where $x > 0$, $0 < p \leqslant 1$, $\theta_k > 0$, $k = 1, 2$. We then have the cumulative exposure distribution (CED) $G(x)$ as

$$G(x) = \begin{cases} G_1(x) = F_1(x;\theta_1), & 0 \leqslant x < t_1, \\ G_2(x) = F_2(x - t_1 + \frac{\theta_2}{\theta_1}t_1;\theta_2), & t_1 \leqslant x < \infty, \end{cases} \tag{5}$$

where $F_k(\cdot)$ is given in (4). The corresponding PDF is

$$g(x) = \begin{cases} g_1(x) = \frac{pe^{-\frac{x}{\theta_1}}}{\theta_1\{1-(1-p)e^{-\frac{x}{\theta_1}}\}^2}, & 0 \leqslant x < t_1, \\ g_2(x) = \frac{pe^{-\frac{x-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\theta_2\{1-(1-p)e^{-\frac{x-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\}^2} & t_1 \leqslant x < \infty. \end{cases} \tag{6}$$

Based on the Type-I censored data, we have n identical units under an initial stress level s_0. The stress level is changed to s_1 at time t_1, and the life-testing experiment is terminated at time t_2, where $0 < t_1 < t_2 < \infty$ are fixed in advance. Let N_1 be the number of units that fail before t_1, and N_2 be the number of units that fail before t_2 at stress level s_1, then, we will observe the following observations:

$$x_{1:n} < x_{2:n} < \cdots < x_{N_1:n} \leqslant t_1 < x_{N_1+1:n} < \cdots < x_{N_1+N_2:n} \leqslant t_2 \tag{7}$$

The likelihood of the observed failure times is then given by

$$L(p,\theta_1,\theta_2) = \begin{cases} \frac{n!}{(n-N_1)!} \left\{ \prod_{i=1}^{N_1} g_1(x_{i:n}) \right\} \{1 - G_1(t_1)\}^{n-N_1}, \\ \qquad 1 \leqslant N_1 \leqslant n, N_2 = 0, \\ \frac{n!}{(n-N_2)!} \left\{ \prod_{i=1}^{N_2} g_2(x_{i:n}) \right\} \{1 - G_2(t_2)\}^{n-N_2}, \\ \qquad 1 \leqslant N_2 \leqslant n, N_1 = 0, \\ \frac{n!}{(n-N)!} \left\{ \prod_{i=1}^{N_1} g_1(x_{i:n}) \right\} \left\{ \prod_{i=N_1+1}^{N_1+N_2} g_2(x_{i:n}) \right\} \times \{1 - G_2(t_2)\}^{n-N}, \\ \qquad 1 \leqslant N_1 \leqslant n-1, 1 \leqslant N_2 \leqslant n - N_1. \end{cases} \tag{8}$$

From (7) and (8), we observe the following:

1. If $N_1 = 0$ and $N_2 = 0$ in (7), the MLEs of p, θ_1 and θ_2 do not exist.

2. If $1 \leqslant N_1 \leqslant n$ and $N_2 = 0$ in (7), the MLE of θ_2 does not exist, since there are no failures observed after t_1.

3. If $N_1 = 0$ and $1 \leqslant N_2 \leqslant n$ in (7), no failures could be observed before t_1. Therefore, the MLE of θ_1 does not exist.

4. If $1 \leqslant N_1 \leqslant n-1$ and $1 \leqslant N_2 \leqslant n - N_1$ in (7), the MLEs of p, θ_1 and θ_2 exist.

From (5) and (6), when $1 \leqslant N_1 \leqslant n-1$ and $1 \leqslant N_2 \leqslant n-N_1$ the log-likelihood function is given by

$$\begin{aligned} l(p,\theta_1,\theta_2) &= const + n\log(p) - N_1\log(\theta_1) - N_2\log(\theta_2) - \frac{1}{\theta_1}\sum_{i=1}^{N_1} x_{i:n} \\ &- 2\sum_{i=1}^{N_1}\log\left\{1-(1-p)e^{-\frac{x_{i:n}}{\theta_1}}\right\} - \frac{1}{\theta_2}\sum_{i=N_1+1}^{N_1+N_2}\left(x_{i:n}-t_1+\frac{\theta_2}{\theta_1}t_1\right) \\ &- 2\sum_{i=N_1+1}^{N_1+N_2}\log\left\{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\} - (n-N_1-N_2)\frac{t_2-t_1}{\theta_2} \\ &- (n-N_1-N_2)\frac{t_1}{\theta_1} - (n-N_1-N_2)\log\left\{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\} \end{aligned}$$

Our objective now is to determine the maximum likelihood estimates (MLE) of the parameters p, θ_1 and θ_2, based on the observed failure times. These estimates have to be viewed as conditional MLE because we are working under the condition that $1 \leqslant N_1 \leqslant n-1$ and $1 \leqslant N_2 \leqslant n-N_1$.

The log-likelihood function can be written as

$$\begin{aligned} \frac{\partial l}{\partial p} &= \frac{n}{p} - 2\sum_{i=1}^{N_1}\frac{e^{-\frac{x_{i:n}}{\theta_1}}}{1-(1-p)e^{-\frac{x_{i:n}}{\theta_1}}} - 2\sum_{i=N_1+1}^{N_1+N_2}\frac{e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}} \\ &- (n-N_1-N_2)\frac{e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}} \qquad (9) \end{aligned}$$

$$\begin{aligned} \frac{\partial l}{\partial \theta_1} &= -\frac{N_1}{\theta_1} + \frac{1}{\theta_1^2}\sum_{i=1}^{N_1} x_{i:n} + \frac{2}{\theta_1^2}\sum_{i=1}^{N_1}\frac{(1-p)x_{i:n}e^{-\frac{x_{i:n}}{\theta_1}}}{1-(1-p)e^{-\frac{x_{i:n}}{\theta_1}}} + \frac{N_2 t_1}{\theta_1^2} \\ &+ \frac{2}{\theta_1^2}\sum_{i=N_1+1}^{N_1+N_2}\frac{(1-p)t_1e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}} + (n-N_1-N_2)\frac{t_1}{\theta_1^2} \\ &+ \frac{n-N_1-N_2}{\theta_1^2}\cdot\frac{(1-p)t_1e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}} \qquad (10) \end{aligned}$$

$$\frac{\partial l}{\partial \theta_2} = -\frac{N_2}{\theta_2} + \frac{1}{\theta_2^2}\sum_{i=N_1+1}^{N_1+N_2}(x_{i:n}-t_1) + \frac{2}{\theta_2^2}\sum_{i=N_1+1}^{N_1+N_2}\frac{(1-p)(x_{i:n}-t_1)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}$$
$$+ (n-N_1-N_2)\frac{t_2-t_1}{\theta_2^2} + \frac{n-N_1-N_2}{\theta_2^2}\cdot\frac{(1-p)(t_2-t_1)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}} \quad (11)$$

Numerical methods are applied for simultaneously solving the nonlinear equations to obtain parameters. The required numerical evaluations were implemented using the *R* Software through the package (stats 4), command `mle` with the L-BFGS-B method.

4 Interval Estimates

Because the MLE of the model parameters are not in closed form expressions, it is not possible to derive their distributions, and therefore the corresponding exact confidence intervals (CI). Hence, we will discuss here the asymptotic confidence intervals.

For large sample size n, the derivation of the asymptotic confidence intervals (ACI) for the parameters p, θ_1 and θ_2 will be based on the pivotal quantities $(\hat{p}-E(\hat{p}))/\sqrt{V(\hat{p})}$, $(\hat{\theta}_1-E(\hat{\theta}_1))/\sqrt{V(\hat{\theta}_1)}$ and $(\hat{\theta}_2-E(\hat{\theta}_2))/\sqrt{V(\hat{\theta}_2)}$, respectively. The maximum likelihood estimates, under appropriate regularity conditions (A1-A6 conditions stated in Casella and Berger (2002), Page 516) are consistent and asymptotically normally distributed. Therefore, we end up with the asymptotic two-sided $100(1-\alpha)\%$ CI of the form $\hat{p}\pm z_{\alpha/2}\sqrt{V(\hat{p})}$, $\hat{\theta}_1 \pm z_{\alpha/2}\sqrt{V(\hat{\theta}_1)}$ and $\hat{\theta}_2 \pm z_{\alpha/2}\sqrt{V(\hat{\theta}_2)}$, where z_p is the p-th upper percentile of the standard normal distribution. Here, $V(\hat{p})$, $V(\hat{\theta}_1)$ and $V(\hat{\theta}_2)$ are the diagonal elements of the inverse of the observed Fisher information matrix presented in next section.

5 Asymptotic Variances and Covariances of Estimates

The asymptotic variances and covariances of maximum likelihood estimators are given by the elements of the inverse of the Fisher information matrix

$$I_{ij}(\boldsymbol{\psi}) = E\{-\partial^2 l/\partial\psi_i\partial\psi_j\}, \qquad i,j = 1,2,3$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3)$, $\psi_1 = p$, $\psi_2 = \theta_1$ and $\psi_3 = \theta_2$.

Unfortunately, the exact mathematical expressions for the above expectations are very difficult to obtain. Therefore, the observed Fisher information matrix is given by $I_{ij}(\boldsymbol{\psi}) = \{-\partial^2 l/\partial\psi_i\partial\psi_j\}$, which is obtained by dropping the expectation operator E; for more details, see Cohen (1965).

The second partial derivatives of the maximum likelihood function are given as the following:

$$\begin{aligned}\frac{\partial^2 l}{\partial p^2} = &- \frac{n}{p^2} + 2\sum_{i=1}^{N_1} \frac{e^{\frac{-2x_{i:n}}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{x_{i:n}}{\theta_1}}\right\}^2} \\ &+ 2\sum_{i=N_1+1}^{N_1+N_2} \frac{\left(e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right)^2}{\left\{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} \\ &+ (n-N_1-N_2)\frac{\left(e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right)^2}{\left\{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} \end{aligned} \qquad (12)$$

$$\frac{\partial^2 l}{\partial p \partial \theta_1} = -\frac{2}{\theta_1^2}\sum_{i=1}^{N_1}\frac{x_{i:n}e^{-\frac{x_{i:n}}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{x_{i:n}}{\theta_1}}\right\}^2} - \frac{2t_1}{\theta_1^2}\sum_{i=N_1+1}^{N_1+N_2}\frac{e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} - \frac{(n-N_1-N_2)t_1}{\theta_1^2}\cdot\frac{e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} \tag{13}$$

$$\frac{\partial^2 l}{\partial p \partial \theta_2} = -\frac{2}{\theta_2^2}\sum_{i=N_1+1}^{N_1+N_2}\frac{(x_{i:n}-t_1)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} - (n-N_1-N_2)\frac{t_2-t_1}{\theta_2^2}\times\frac{e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} \tag{14}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial \theta_1^2} =& \frac{N_1}{\theta_1^2} - \frac{2}{\theta_1^3}\sum_{i=1}^{N_1}x_{i:n} - \frac{4}{\theta_1^3}\sum_{i=1}^{N_1}\frac{(1-p)x_{i:n}e^{-\frac{x_{i:n}}{\theta_1}}}{1-(1-p)e^{-\frac{x_{i:n}}{\theta_1}}} \\ &+ \frac{2}{\theta_1^4}\sum_{i=1}^{N_1}\frac{(1-p)x_{i:n}^2e^{-\frac{x_{i:n}}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{x_{i:n}}{\theta_1}}\right\}^2} \\ &- \frac{2N_2t_1}{\theta_1^3} - \frac{4t_1}{\theta_1^3}\sum_{i=N_1+1}^{N_1+N_2}\frac{(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}} \\ &+ \frac{2}{\theta_1^4}\sum_{i=N_1+1}^{N_1+N_2}\frac{(1-p)t_1^2e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} - (n-N_1-N_2)\frac{2t_1}{\theta_1^3} \\ &- \frac{2(n-N_1-N_2)}{\theta_1^3}\cdot\frac{(1-p)t_1e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}\end{aligned}$$

$$+ \quad \frac{n-N_1-N_2}{\theta_1^4} \cdot \frac{(1-p)t_1^2 e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} \tag{15}$$

$$\begin{aligned} \frac{\partial^2 l}{\partial\theta_1\partial\theta_2} =& \frac{2t_1}{\theta_1^2\theta_2^2}\sum_{i=N_1+1}^{N_1+N_2} \frac{(1-p)(x_{i:n}-t_1)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} \\ &+ \frac{n-N_1-N_2}{\theta_1^2\theta_2^2} \cdot \frac{(1-p)t_1(t_2-t_1)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial^2 l}{\partial\theta_2^2} =& \frac{N_2}{\theta_2^2} - \frac{2}{\theta_2^3}\sum_{i=N_1+1}^{N_1+N_2}(x_{i:n}-t_1) \\ &- \frac{4}{\theta_2^3}\sum_{i=N_1+1}^{N_1+N_2} \frac{(1-p)(x_{i:n}-t_1)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}} \\ &+ \frac{2}{\theta_2^4}\sum_{i=N_1+1}^{N_1+N_2} \frac{(1-p)(x_{i:n}-t_1)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{x_{i:n}-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} \\ &- 2(n-N_1-N_2)\frac{t_2-t_1}{\theta_2^3} \\ &- (n-N_1-N_2)\frac{2}{\theta_2^3} \cdot \frac{(1-p)(t_2-t_1)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} \\ &+ (n-N_1-N_2)\frac{(t_2-t_1)^2}{\theta_2^4} \cdot \frac{(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}}{\left\{1-(1-p)e^{-\frac{t_2-t_1}{\theta_2}-\frac{t_1}{\theta_1}}\right\}^2} \end{aligned} \tag{17}$$

Consequently, the maximum likelihood estimators of p, θ_1 and θ_2 have an asymptotic variance covariance matrix defined by inverting the Fisher information matrix and then substituting p, θ_1 and θ_2 by $\hat{p}$, $\hat{\theta}_1$ and $\hat{\theta}_2$.

6 Simulation Study

In order to obtain the MLEs of parameters and study the properties of their estimates through the mean squared errors (MSE), relative absolute biases (RAB) and relative errors (RE), we describe the algorithm to obtain the Type-I censored sample. A simulation study is performed according to the following steps:

For given values of t_1, t_2 and the parameters p, θ_1 and θ_2,

Step 1. we generate a random sample of size n from Uniform $(0, 1)$ distribution, and obtain the order statistics $(U_{1:n}, \ldots, U_{n:n})$.

Step 2. Find N_1 such that

$$U_{N_1:n} < P(X \leqslant t_1) = G_1(t_1) \leqslant U_{N_1+1:n}$$
$$\Leftrightarrow U_{N_1:n} < \frac{e^{-\frac{t_1}{\theta_1}}}{1-(1-p)e^{-\frac{t_1}{\theta_1}}} \leqslant U_{N_1+1:n}.$$

The values $x_{1:n}, x_{2:n}, \ldots, x_{N_1:n}$ construct a simulated random sample from the MOGE distribution in stress level s_0, since using the probability integral transformation theorem, the solution of the equation $U_{i:n} = G_1(x_{i:n})$ with respect to $x_{i:n}, 1 \leqslant i \leqslant N_1$, is a simulated value from the distribution with the CDF $G_1(\cdot)$. But

$$U_{i:n} = G_1(x_{i:n}) \Leftrightarrow U_{i:n} = \frac{e^{-\frac{x_{i:n}}{\theta_1}}}{1-(1-p)e^{-\frac{x_{i:n}}{\theta_1}}}$$
$$\Leftrightarrow x_{i:n} = \theta_1 \log \frac{1-(1-p)U_{i:n}}{1-U_{i:n}}, \quad i = 1, \ldots, N_1,$$

Note that $G_1(\cdot)$ is an one-to-one function, so its inverse function, $x_{i:n}$ is the unique solution of the equation.

Step 3. Next, we generate a random sample of size $m = n - N_1$ from Uniform $(0, 1)$ distribution, and obtain the order statistics $(V_{1:m}, \ldots, V_{m:m})$.

Step 4. Find N_2 such that

$$V_{N_2:m} < P(X \leqslant t_2 | X \geqslant t_1) = \frac{G_2(t_2) - G_1(t_1)}{1 - G_1(t_1)} \leqslant V_{N_2+1:m}$$
$$\Longleftrightarrow V_{N_2:m} < \frac{1 - e^{-\frac{t_2 - t_1}{\theta_2}}}{1 - (1-p)e^{-\frac{t_2 - t_1}{\theta_2} - \frac{t_1}{\theta_1}}} \leqslant V_{N_2+1:m}.$$

The values $x_{N_1+1:n}, \ldots, x_{N_1+N_2:n}$ construct a simulated random sample from the MOGE distribution in stress level s_1, since using the probability integral transformation theorem, for $1 \leqslant j \leqslant N_2$,

$$V_{j:m} = P(X \leqslant x_{N_1+j:n} | X \geqslant t_1) = \frac{G_2(x_{N_1+j:n}) - G_1(t_1)}{1 - G_1(t_1)}$$
$$\Longleftrightarrow V_{j:m} = \frac{1 - e^{-\frac{x_{N_1+j:n} - t_1}{\theta_2}}}{1 - (1-p)e^{-\frac{x_{N_1+j:n} - t_1}{\theta_2} - \frac{t_1}{\theta_1}}}$$
$$x_{N_1+j:n} = t_1 - \theta_2 \log \frac{1 - V_{j:m}}{1 - V_{j:m}(1-p)e^{-\frac{t_1}{\theta_1}}}.$$

Step 5. Based on n, N_1, N_2, t_1, t_2 and ordered observations $\{x_{1:n}, \ldots, x_{N_1:n}, x_{N_1+1:n}, \ldots, x_{N_1+N_2:n}\}$, we can obtain the MLEs $(\hat{p}, \hat{\theta_1}, \hat{\theta_2})$ by solving system of nonlinear Equation (9), (12) and (13).

Step 6. Repeat Steps $1-5$, r times representing r different samples. The value of r has been taken to be 1000.

Step 7. If $\hat{\psi_{kl}}$ is a MLE of $\psi_l, l = 1, 2, 3$ (where ψ_l is a general notation that can be replaced by p, θ_1 and θ_2 i.e. $\psi_1 \equiv p$, $\psi_2 \equiv \theta_1$ and $\psi_3 \equiv \theta_2$), based on sample $k, k = 1, \ldots, r$ then the average estimate, MSE, RAB and RE of $\hat{\psi}_l$ over the r samples are given, respectively by

$\bar{\hat{\psi}}_l = \frac{1}{r} \sum\limits_{k=1}^{r} \hat{\psi_{kl}}$,

$MSE(\hat{\psi}_l) = \frac{1}{r} \sum\limits_{k=1}^{r} (\hat{\psi_{kl}} - \psi_l)^2$,

$RAB(\hat{\psi}_l) = \frac{|\bar{\hat{\psi}}_l - \psi_l|}{\psi_l}$,

$RE(\hat{\psi}_l) = \frac{\sqrt{MSE(\hat{\psi}_l)}}{\psi_l}.$

Step 8. From Step 7 compute $\bar{\hat{p}}$, $\bar{\hat{\theta}}_1$, $\bar{\hat{\theta}}_2$, $MSE(\hat{p})$, $MSE(\hat{\theta}_1)$, $MSE(\hat{\theta}_2)$, $RAB(\hat{p})$, $RAB(\hat{\theta}_1)$, $RAB(\hat{\theta}_2)$, $RE(\hat{p})$, $RE(\hat{\theta}_1)$ and $RE(\hat{\theta}_2)$.

7 Bootstrap Confidence Intervals

In this section, we present several parametric bootstrap methods to construct CIs for p, θ_1 and θ_2, studentized-t interval, percentile interval, and adjusted percentile (BCa) interval; see Efron (1982) and Hall (1988) for more details.

The following steps are followed to obtain a bootstrap sample,

Step 1. Based on the original type-I censored sample,$\{x_{1:n}, \ldots, x_{N_1:n}, x_{N_1+1:n}, \ldots, x_{N_1+N_2:n}\}$, we obtain $\hat{p}$, $\hat{\theta}_1$ and $\hat{\theta}_2$.

Step 2. Based on n, we generate a random sample of size n from Uniform $(0,1)$ distribution, and obtain the order statistics $(U_{1:n}, \ldots, U_{n:n})$.

Step 3. For a given value of the stress change time t_1, Find N_1^* such that

$$U_{N_1^*:n} < \frac{e^{-\frac{t_1}{\hat{\theta}_1}}}{1-(1-\hat{p})e^{-\frac{t_1}{\hat{\theta}_1}}} \leqslant U_{N_1^*+1:n}.$$

For $1 \leqslant i \leqslant N_1^*$, we set

$$x_{i:n}^* = \hat{\theta}_1 \log \frac{1-(1-\hat{p})U_{i:n}}{1-U_{i:n}}.$$

Step 4. Next, we generate a random sample of size $m = n - N_1^*$ from Uniform $(0,1)$ distribution, and obtain the order statistics $(V_{1:m}, \ldots, V_{m:m})$.

Step 5. For a given value of censoring time t_2, Find N_2^* such that

$$V_{N_2^*:m} < \frac{1-e^{\frac{t_2-t_1}{\hat{\theta}_2}}}{1-(1-\hat{p})e^{-\frac{t_2-t_1}{\hat{\theta}_2}-\frac{t_1}{\hat{\theta}_1}}} \leqslant V_{N_2^*+1:m}.$$

For $1 \leqslant j \leqslant N_2^*$, we then set

$$x^*_{N_1+j:n} = t_1 - \hat{\theta}_2 \log \frac{1 - V_{j:m}}{1 - V_{j:m}(1-\hat{p})e^{-\frac{t_1}{\hat{\theta}_1}}}.$$

Step 6. Based on n, N_1^*, N_2^*, t_1, t_2 and ordered observations $\{x^*_{1:n}, \ldots, x^*_{N_1:n}, x^*_{N_1+1:n}, \ldots, x^*_{N_1+N_2:n}\}$, we can obtain the bootstrap estimates $(\hat{p}^*, \hat{\theta}_1^*, \hat{\theta}_2^*)$ by solving system of nonlinear Equation (9), (12) and (13).

Step 7. Repeat Steps $2-6$, B times and arrange all $\hat{p}^*$, $\hat{\theta}_1^*$, $\hat{\theta}_2^*$ in ascending to obtain the bootstrap sample $\{\hat{\psi}_l^{*[1]}, \hat{\psi}_l^{*[2]}, \ldots, \hat{\psi}_l^{*[B]}\}$, $l = 1, 2, 3$, where $\hat{\psi}_1^* \equiv \hat{p}^*, \hat{\psi}_2^* \equiv \hat{\theta}_1^*, \hat{\psi}_3^* \equiv \hat{\theta}_2^*$.

7.1 Studentized-t Interval

First, find the order statistics $T_l^{*[1]} < T_l^{*[2]} < \cdots < T_l^{*[B]}$, where

$$T_l^{*[j]} = \frac{\hat{\psi}_l^{*[j]} - \hat{\psi}_l}{\sqrt{\mathrm{var}(\hat{\psi}_l^{*[j]})}}, \qquad j = 1, \ldots, B, \quad l = 1, 2, 3$$

where $\hat{\psi}_1 = \hat{p}$, $\hat{\psi}_2 = \hat{\theta}_1$ and $\hat{\psi}_3 = \hat{\theta}_2$.

A two-sided $100(1-\alpha)\%$ studentized-t bootstrap confidence interval (TBCI) for ψ_l is

$$\left(\hat{\psi}_l - T_l^{*[(1-\alpha/2)B]}\sqrt{\mathrm{var}(\hat{\psi}_l)}, \hat{\psi}_l - T_l^{*[(\alpha/2)B]}\sqrt{\mathrm{var}(\hat{\psi}_l)}\right),$$

where $\mathrm{var}(\hat{\psi}_l)$ is estimated as the asymptotic variance, obtained from Section 5.

7.2 Percentile Bootstrap Confidence Interval (PBCI)

A two-sided $100(1-\alpha)\%$ percentile bootstrap confidence interval for ψ_l is

$$\left(\hat{\psi}_l^{*[(\alpha/2)B]}, \hat{\psi}_l^{*[(1-\alpha/2)B]}\right), \quad l = 1, 2, 3.$$

7.3 Adjusted Percentile (BCa) Interval

A two-sided $100(1-\alpha)\%$ BCa bootstrap confidence interval for ψ_l is

$$\left(\hat{\psi}_l^{*[\alpha_{1l}B]}, \hat{\psi}_l^{*[(1-\alpha_{2l})B]}\right), \qquad l = 1, 2, 3.$$

where

$$\alpha_{1l} = \Phi\left(\hat{z}_{0l} + \frac{\hat{z}_{0l} + z_{\alpha/2}}{1 - \hat{a}_l\big(\hat{z}_{0l} + z_{\alpha/2}\big)}\right),$$

and

$$\alpha_{2l} = \Phi\left(\hat{z}_{0l} + \frac{\hat{z}_{0l} + z_{1-\alpha/2}}{1 - \hat{a}_l\big(\hat{z}_{0l} + z_{1-\alpha/2}\big)}\right).$$

Here, $\Phi(\cdot)$ is the CDF of the standard normal distribution. The value of the bias correction $\hat{z}_{0l}$ can be computed as

$$\hat{z}_{0l} = \Phi^{-1}\left(\frac{\hat{\psi}_l^{*[j]} < \hat{\psi}_l}{B}\right), \qquad j = 1, \ldots, B, \quad l = 1, 2, 3,$$

while the acceleration a_l is estimated by

$$\hat{a}_l = \frac{\sum_{i=1}^{N_1+N_2}\big(\hat{\psi}_l^{(.)} - \hat{\psi}_l^{(i)}\big)^3}{6\left(\sum_{i=1}^{N_1+N_2}\big(\hat{\psi}_l^{(.)} - \hat{\psi}_l^{(i)}\big)^2\right)^{3/2}}, \qquad l = 1, 2, 3,$$

where $\hat{\psi}_l^{(i)}$ is the MLE of ψ_l based on the simulated Type-I censored sample with the ith observation deleted (i.e., the jackknife estimate), and

$$\hat{\psi}_l^{(\cdot)} = \frac{1}{N_1 + N_2}\sum_{i=1}^{N_1+N_2}\hat{\psi}_l^{(i)}, \qquad l = 1, 2, 3.$$

8 Numerical Results

All simulation results are summarized in Tables 1-3 , based on 1000 simulations. Tables 1-3 show the average MLEs, MSEs, RABs and REs of p, θ_1 and θ_2 for different values of t_1 and t_2. The values of the population parameters are arbitrary chosen to be $p = 0.5$, $\theta_1 = 12.18$ and $\theta_2 = 4.48$. The results are based on $n = 35$ in Table 1, $n = 50$ in Table 2 and $n = 100$ in Table 3. The values of t_2 range from 7 to 10 by Step 1 and the values of t_1 are 6, 7 and 8.

Table 1. Average estimate of p, θ_1 and θ_2 with their MSE, RAB and RE for different stress change times and different censoring values based on 1000 simulations. Population parameter values: $p = 0.5$, $\theta_1 = 12.18$, $\theta_2 = 4.48$ with $n = 35$.

t_1	t_2	$\bar{N}_1$ $\bar{N}_2$	$\bar{\hat{p}}$ $\bar{\hat{\theta}}_1$ $\bar{\hat{\theta}}_2$	MSE($\hat{p}$) MSE($\hat{\theta}_1$) MSE($\hat{\theta}_2$)	RAB($\hat{p}$) RAB($\hat{\theta}_1$) RAB($\hat{\theta}_2$)	RE($\hat{p}$) RE($\hat{\theta}_1$) RE($\hat{\theta}_2$)
6	7	19.721 4	0.553 36.402 14.170	0.170 2153.315 424.729	0.106 1.989 2.163	0.825 3.810 4.600
	8	19.521 6.924	0.551 35.362 11.337	0.164 2052.228 234.32	0.103 1.903 1.531	0.810 3.719 3.417
	9	19.702 8.82	0.568 32.992 9.875	0.165 1853.139 155.086	0.135 1.709 1.204	0.811 3.534 2.780
	10	19.671 10.442	0.587 29.183 8.191	0.157 1491.811 90.181	0.174 1.396 0.828	0.793 3.171 2.120
7	8	21.274 3.662	0.541 35.447 13.341	0.161 2042.395 428.105	0.082 1.910 1.978	0.801 3.710 4.618
	9	21.26 6.061	0.559 33.980 10.799	0.167 1903.377 210.932	0.117 1.790 1.411	0.818 3.582 3.242
	10	21.345 7.798	0.563 31.715 9.546	0.156 1717.846 149.380	0.127 1.604 1.131	0.791 3.403 2.728
8	9	22.703 3.255	0.557 32.566 11.313	0.156 1796.857 257.157	0.113 1.674 1.525	0.790 3.480 3.579
	10	22.814 5.234	0.552 32.753 11.112	0.156 1806.050 249.385	0.104 1.689 1.480	0.790 3.489 3.525

Table 2. Average estimate of p, θ_1 and θ_2 with their MSE, RAB and RE for different stress change times and different censoring values based on 1000 simulations. Population parameter values: $p = 0.5$, $\theta_1 = 12.18$, $\theta_2 = 4.48$ with $n = 50$.

t_1	t_2	$\bar{N}_1$ $\bar{N}_2$	$\bar{\hat{p}}$ $\bar{\hat{\theta}}_1$ $\bar{\hat{\theta}}_2$	MSE($\hat{p}$) MSE($\hat{\theta}_1$) MSE($\hat{\theta}_2$)	RAB($\hat{p}$) RAB($\hat{\theta}_1$) RAB($\hat{\theta}_2$)	RE($\hat{p}$) RE($\hat{\theta}_1$) RE($\hat{\theta}_2$)
6	7	28.059	0.534	0.166	0.068	0.814
		5.771	37.957	2308.284	2.116	3.945
			13.380	339.406	1.987	4.112
	8	27.944	0.559	0.159	0.118	0.797
		9.872	33.437	1878.894	1.745	3.559
			9.975	145.872	1.227	2.696
	9	28.060	0.560	0.157	0.121	0.793
		12.713	31.928	1733.627	1.621	3.418
			9.329	122.167	1.082	2.467
	10	28.270	0.593	0.147	0.186	0.767
		14.765	26.349	1250.328	1.163	2.903
			7.656	73.038	0.709	1.908
7	8	30.479	0.528	0.148	0.055	0.770
		5.057	33.322	1807.009	1.736	3.490
			12.065	287.003	1.693	3.782
	9	30.407	0.552	0.150	0.105	0.774
		8.659	31.491	1688.338	1.585	3.374
			9.675	148.085	1.160	2.716
	10	30.444	0.550	0.146	0.101	0.765
		11.187	29.962	1513.144	1.460	3.194
			8.687	97.935	0.939	2.209
8	9	32.347	0.541	0.146	0.082	0.764
		4.437	31.857	1682.816	1.616	3.368
			11.047	207.579	1.466	3.216
	10	32.453	0.543	0.144	0.086	0.758
		7.721	30.578	1571.722	1.510	3.255
			9.113	122.445	1.034	2.470

Table 3. Average estimate of p, θ_1 and θ_2 with their MSE, RAB and RE for different stress change times and different censoring values based on 1000 simulations. Population parameter values: $p = 0.5$, $\theta_1 = 12.18$, $\theta_2 = 4.48$ with $n = 100$.

t_1	t_2	$\bar{N}_1$ / $\bar{N}_2$	$\bar{\hat{p}}$ / $\bar{\hat{\theta}}_1$ / $\bar{\hat{\theta}}_2$	MSE($\hat{p}$) / MSE($\hat{\theta}_1$) / MSE($\hat{\theta}_2$)	RAB($\hat{p}$) / RAB($\hat{\theta}_1$) / RAB($\hat{\theta}_2$)	RE($\hat{p}$) / RE($\hat{\theta}_1$) / RE($\hat{\theta}_2$)
6	7	56.068	0.554	0.139	0.107	0.747
		11.521	29.047	1432.920	1.385	3.108
			9.556	130.639	1.133	2.551
	8	56.008	0.553	0.137	0.105	0.740
		19.806	28.445	1360.074	1.335	3.028
			8.466	87.415	0.890	2.087
	9	55.683	0.525	0.123	0.050	0.702
		25.542	27.474	1206.766	1.256	2.852
			8.014	70.589	0.789	1.875
	10	56.232	0.564	0.126	0.127	0.709
		29.601	24.083	962.639	0.977	2.547
			6.992	45.530	0.561	1.506
7	8	60.682	0.529	0.125	0.058	0.707
		10.211	27.969	1289.749	1.296	2.949
			8.865	104.429	0.979	2.281
	9	60.692	0.543	0.121	0.086	0.695
		17.224	25.289	1050.936	1.076	2.662
			7.760	68.393	0.732	1.846
	10	60.878	0.534	0.113	0.069	0.672
		22.327	24.017	913.168	0.972	2.481
			7.043	43.645	0.572	1.475
8	9	65.057	0.516	0.109	0.031	0.662
		8.767	25.703	1070.588	1.110	2.686
			8.591	92.071	0.918	2.142
	10	65.103	0.573	0.119	0.147	0.689
		15.157	22.368	826.355	0.836	2.360
			6.979	49.926	0.558	1.577

8.1 Real Data

In this section, we fit the MOGE model to one real data set. Data set is given by Gupta and Kundu (2003) on the failure times of the air conditioning system of the air plane 7912. The data set is as follows:

$$1, 3, 5, 7, 11, 11, 11, 12, 14, 14, 14, 16, 16, 20, 21, 23, 42, 47, 52, 62, 71, 71, 87, 90,$$

$$95, 120, 120, 225, 246, 261.$$

Here we fit the MOGE model to the real data set and show that the MOGE distribution is more flexible for analyzing of the data than the exponentiated exponential distribution (EE) and exponential distribution (E).

In order to compare the models, we usually used two criteria: Akaike Information Criterion (AIC) and BIC (Bayesian Information Criterion) which are defined as follows:

$$AIC = -2\log \hat{L} + 2k, \quad BIC = -2\log \hat{L} + k\log(n),$$

where k is the number of free parameters in the model and n is the sample size. For fitting a data set, the best model is a model with the smallest value of AIC and BIC statistics.

We can also perform formal goodness-of-fit tests in order to verify which distribution fits better to these data. We apply Kolmogorov-Smirnov (K-S) statistics and the p-value from the chi-square goodness of fit test, where the lower values of K-S statistic and the upper value of p-value for models indicate that these models could be chosen as the best model to fit the data.

The K-S statistic and the corresponding p-value evaluations were implemented using the R software through the command `ks.test`. MLEs of the model parameters and the values of the K-S statistics, P-value, AIC and BIC statistics are listed in Table 4.

Table 4. MLEs of the model parameters for Gupta and Kundu data, K-S statistics, P-value and the measures AIC and BIC.

Model	Parameters	K-S	P-value	AIC	BIC
NGE	$\hat{\theta} = 0.10$ $\hat{p} = 0.389$	0.1268	0.7207	306.841	309.644
EE	$\hat{\alpha} = 0.810$ $\hat{\lambda} = 0.014$	0.1719	0.3382	308.401	311.204
E	$\hat{\theta} = 0.017$	0.2129	0.1319	307.2594	308.661

Hence, it is clear that the new generalized exponential distribution fits quite well to this data set and is better than the exponentiated exponential distribution with the parameters α and λ and the exponential distribution with the parameter θ.

After analyzing the data, we found that the failure distribution of the air-conditioning system for the air plane is well by the new generalized exponential distribution. Now, we suppose this data of size $n = 30$ and simple step-stress model under type-I censoring with $t_1 = 35$ and $t_2 = 90$ had occurred on this data. We then computed the MLEs of p, θ_1 and θ_2 and the estimates of their standard deviations. The MLE of parameters p, θ_1 and θ_2 are obtained as $\hat{p} = 0.702$, $\hat{\theta_1} = 56.003$ and $\hat{\theta_2} = 81.909$. Also, standard deviations p, θ_1 and θ_2 are 0.985, 62.689 and 58.044, respectively.

8.2 Illustrative Example

In this subsection, we present an example to illustrate the estimation procedure, the asymptotic confidence interval and bootstrap CI methods for the parameters p, θ_1 and θ_2. In this example, we simulate a sample of size $n = 35$, using the algorithm presented in Section 5, based on population parameter values $p = 0.5$, $\theta_1 = 12.18$, $\theta_2 = 4.48$. The two stress levels used in the simulation are $\theta_1 = 12.18$ and $\theta_2 = 4.48$. The stress change time t_1 and the censoring time t_2 are chosen to be equal 7 and 9, respectively. Under type-I censoring, the simulated failure time data are presented in Table 5, while the MLE, MSE, RAB and RE of the parameters are presented in Table 6. Using the asymptotic confidence intervals (ACI) and bootstrap CIs presented in Sections 4 and 7, Table 7 shows 90%, 95% and 99% ACIs and bootstrap CIs for the parameters p, θ_1 and θ_2.

It can be seen from the simulated data, presented in Table 5, that there are 21 and 5 failure times in the intervals $(0, 7]$ and $(7, 9]$, respectively. From the results of Table 6, the confidence interval length at $1-\alpha = 0.90$ is smaller than the confidence interval length at $1 - \alpha = 0.95$ and $1 - \alpha = 0.99$. Also, the confidence interval length at $1 - \alpha = 0.95$ is smaller than the confidence interval length at $1 - \alpha = 0.99$.

Table 5. Simulated data.

Stress Level $\theta_1 = 12.18$ Failure times in the interval $(0, 7]$	Stress Level $\theta_2 = 4.48$ Failure times in the interval $(7, 9]$
0.0534, 0.6067, 0.7266, 0.7848, 1.1430, 1.1931, 1.2286, 1.6487, 1.9272, 2.0842, 2.0949, 2.3394, 2.5292, 2.9790, 4.0447, 4.2743, 4.8495, 4.8899, 5.5299, 5.6333, 6.6981,	7.3320, 7.4011, 7.8178, 8.3481, 8.7218,

Table 6 MLEs of p, θ_1 and θ_2 with their MSE, RAB and RE. Population parameter values: $p = 0.5$, $\theta_1 = 12.18$ and $\theta_2 = 4.48$ with $n = 35$.

t_1	t_2	N_1 N_2	$\hat{p}$ $\hat{\theta}_1$ $\hat{\theta}_2$	MSE($\hat{p}$) MSE($\hat{\theta}_1$) MSE($\hat{\theta}_2$)	RAB($\hat{p}$) RAB($\hat{\theta}_1$) RAB($\hat{\theta}_2$)	RE($\hat{p}$) RE($\hat{\theta}_1$) RE($\hat{\theta}_2$)
7	9	21	0.516	0.0002	0.032	0.032
		5	12.142	0.001	0.003	0.003
			5.917	2.066	0.321	0.321

Table 7. Asymptotic confidence intervals and bootstrap CIs of p, θ_1 and θ_2 based on $B = 1000$ replications.

parameters	Methods	90%	95%	99%
p	ACI	(0,1)	(0,1)	(0,1)
	TBCI	(0.0071,1)	(0,1)	(0, 1)
	PBCI	(0.0423,1)	(0.038,1)	(0.0307, 1)
	BCa	(0.0421,1)	(0.0379,1)	(0.0301, 1)
θ_1	ACI	(0,40.4356)	(0,45.8559)	(0,56.4494)
	TBCI	(5.5222,43.0002)	(4.4028, 50.1454)	(2.8538,69.9666)
	PBCI	(5.7641,100)	(5.4154,100)	(4.4354,100)
	BCa	(5.7126,100)	(5.1652,100)	(4.4109,100)
θ_2	ACI	(0,15.4026)	(0,17.2197)	(0,20.7712)
	TBCI	(2.6601,18.4992)	(2.2923,23.9044)	(1.8570,37.7206)
	PBCI	(2.7852,50.2819)	(2.3384,67.5619)	(1.7659,100)
	BCa	(1.8491,27.2998)	(1.5657,35.5511)	(0, 58.2719)

9 Conclusions

In this paper, we considered a simple step-stress model with two stress levels from the MOGE distribution when there was time constraint on the duration of the experiment. We also obtained maximum likelihood equations for estimating the distribution parameters. In addition, asymptotic variance and covariance of the estimators were given. We also evaluated the properties of maximum likelihood estimation through the mean squared error, relative absolute bias and relative error. Furthermore, asymptotic confidence intervals of the estimators derived. We have also proposed several different procedures for constructing bootstrap confidence intervals. Finally, some simulation results are presented.

From results of Tables 1-3, we observe that:

1. For fixed values of t_1, by increasing t_2 the value of $\bar{N}_2$, the average of failures observed after t_1 before termination, increases and MSE and RAB of $\hat{\theta}_1$ and $\hat{\theta}_2$ decrease. The MSE of $\hat{p}$ decreases and RAB of $\hat{p}$ increases except for some cases. Also RE of $\hat{p}$, $\hat{\theta}_1$ and $\hat{\theta}_2$ decrease.

2. For fixed values of t_2, large values of t_1 would provide more data under low stress and less data under high stress, so the number of failures $\bar{N}_1$, the average number of failures before t_1, increases and $\bar{N}_2$ decreases. The MSE and RE of $\hat{p}$ decrease and also, RAB of $\hat{p}$ decreases except for some cases. The MSE, RAB and RE of $\hat{\theta_2}$ increase.

3. For fixed values of t_1 and t_2, as n increases the MSE, RAB and RE of $\hat{\theta}_1$ and $\hat{\theta}_2$ decrease. Also, the MSE and RE of $\hat{p}$ decrease and the RAB of $\hat{p}$ decreases except for some cases.

Acknowledgements

Authors are grateful to the referees of the journal for their suggestions and for their help in writing the paper in acceptable form.

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