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پژوهشکده‌ی آمار

A Two-parameter Generalized Skew-Cauchy Distribution

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Abstract. In this paper, we discuss a new generalization of univariate skew-Cauchy distribution with two parameters, we denoted this by $\mathrm{GSC}\,(\lambda_1, \lambda_2)$, that it has more flexible than the skew-Cauchy distribution (denoted by $\mathrm{SC}\,(\lambda)$), introduced by Behboodian et al. (2006). Furthermore, we establish some useful properties of this distribution and by two numerical example, show that $\mathrm{GSC}\,(\lambda_1, \lambda_2)$ can fits the data better than $\mathrm{SC}\,(\lambda)$.

Keywords. Generalized skew-Cauchy; generalized skew-normal; skew-Cauchy and skew-normal distributions.

1 Introduction

Azzalini (1985, 1986) introduced the standard skew-normal distribution as a generalization of the normal distribution. A random variable Z_λ has a standard skew-normal distribution with parameter $\lambda \in \mathbb{R}$, denoted by $\mathrm{SN}\,(\lambda)$, if its pdf is

$$f\,(z;\lambda) = 2\phi\,(z)\,\Phi\,(\lambda z) \qquad z \in \mathbb{R},$$

where $\phi\,(\cdot)$ and $\Phi\,(\cdot)$ are the standard normal pdf and cdf, respectively. This distribution has been studied and generalized by some researchers. Jamalizadeh et al. (2008) discussed a new class of skew-normal distribution with two parameters. A random variable Z_{λ_1,λ_2} has a two-parameter generalized skew-normal distribution with parameters $\lambda_1, \lambda_2 \in \mathbb{R}$, denoted by

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GSN(λ_1, λ_2), if its pdf is

$$f(z, \lambda_1, \lambda_2) = \frac{2\pi}{\cos^{-1}\left(-\frac{\lambda_1\lambda_2}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right)} \phi(z)\,\Phi(\lambda_1 z)\,\Phi(\lambda_2 z) \qquad z \in \mathbb{R},$$

also, they established some simple and useful properties of this distribution.

If $X \sim \mathrm{N}(0,1)$ be independent of Z_λ, it is easy to show that $\frac{Z_\lambda}{X} \sim C(0,1)$ for $\lambda \in \mathbb{R}$. However $W_\lambda = \frac{Z_\lambda}{|X|}$ when $\lambda \neq 0$ is not $C(0,1)$. Behboodian et al. (2006) refer to it as a *skew-Cauchy distribution* with parameter $\lambda \in \mathbb{R}$ and denoted it by $W_\lambda \sim \mathrm{SC}(\lambda)$. They derived the density of W_λ as follows

$$f(w; \lambda) = \frac{1}{\pi(1+w^2)}\left(1 + \frac{\lambda w}{\sqrt{1+(1+\lambda^2)w^2}}\right) \qquad w \in \mathbb{R},$$

and discussed some simple and important characteristics of this distribution. Let $X \sim \mathrm{N}(0,1)$ be independent of Z_{λ_1,λ_2}. In this paper, we consider the distribution of $W_{\lambda_1,\lambda_2} = \frac{Z_{\lambda_1,\lambda_2}}{|X|}$, refer to it as a *two-parameter generalized skew-Cauchy distribution* and denote this by GSC(λ_1, λ_2).

This paper is organized as follows. In the next section we derive the density of W_{λ_1,λ_2} and present some simple properties of this distribution. In Section 3 we discuss about the moments of GSC(λ_1, λ_2). Some important properties of GSC(λ_1, λ_2) are given in Section 4, and in Section 5, two numerical examples to compare GSC and SC are provided.

2 Two-parameter Generalized Skew-Cauchy Distribution

In this section, we derive the density of W_{λ_1,λ_2} and establish some simple properties of this distribution.

Definition 1. A random variable W_{λ_1,λ_2} has a two-parameter generalized skew-Cauchy distribution with parameters $\lambda_1, \lambda_2 \in \mathbb{R}$, if $W_{\lambda_1,\lambda_2} \stackrel{d}{=} \frac{Z_{\lambda_1,\lambda_2}}{|X|}$, where $Z_{\lambda_1,\lambda_2} \sim$ GSN(λ_1, λ_2) and $X \sim \mathrm{N}(0,1)$ are independent.

To obtain the density of W_{λ_1,λ_2}, let $g(w; \lambda_1, \lambda_2)$ and $G(w; \lambda_1, \lambda_2)$ denote the pdf and cdf of W_{λ_1,λ_2}, respectively. Then

$$G(w; \lambda_1, \lambda_2) = P\left(Z_{\lambda_1,\lambda_2} \leqslant w\,|X|\right) = E\left\{\Phi\left(w\,|X|\,; \lambda_1, \lambda_2\right)\right\},$$

where $\Phi(\cdot;\lambda_1,\lambda_2)$ is the cdf of $Z_{\lambda_1,\lambda_2} \sim \mathrm{GSN}(\lambda_1,\lambda_2)$. We have

$$G(w;\lambda_1,\lambda_2) = 2\int_0^\infty \Phi(wx;\lambda_1,\lambda_2)\,\phi(x)\,dx,$$

which, by differentiation, we obtain

$$g(w;\lambda_1,\lambda_2) = \frac{2}{\cos^{-1}\left(-\frac{\lambda_1\lambda_2}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right)}\int_0^\infty x e^{-\frac{1}{2}x^2(1+w^2)}\Phi(\lambda_1 wx)\,\Phi(\lambda_2 wx)\,dx.$$

If

$$g_1(w;\lambda_1,\lambda_2) = \frac{\lambda_1 w}{\sqrt{1+\left(1+\lambda_1^2\right)w^2}}\left\{\frac{1}{4}+\frac{1}{2\pi}\tan^{-1}\left(\frac{\lambda_2 w}{\sqrt{1+\left(1+\lambda_1^2\right)w^2}}\right)\right\},$$

$$g_2(w;\lambda_1,\lambda_2) = \frac{\lambda_2 w}{\sqrt{1+\left(1+\lambda_2^2\right)w^2}}\left\{\frac{1}{4}+\frac{1}{2\pi}\tan^{-1}\left(\frac{\lambda_1 w}{\sqrt{1+\left(1+\lambda_2^2\right)w^2}}\right)\right\}$$

and $a = \frac{2\pi}{\cos^{-1}\left(-\frac{\lambda_1\lambda_2}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right)}$, then by integration by parts, we obtain the density of W_{λ_1,λ_2} as

$$g(w;\lambda_1,\lambda_2) = \frac{a}{\pi(1+w^2)}\left\{\frac{1}{4}+g_1(w;\lambda_1,\lambda_2)+g_2(w;\lambda_1,\lambda_2)\right\}. \tag{1}$$

If $\lambda_1=\lambda_2=\lambda$ and $b=\frac{\pi}{\tan^{-1}\left(\sqrt{1+2\lambda^2}\right)}$ then the above density reduce to

$$g(w;\lambda) = \frac{b}{\pi(1+w^2)}\left[\frac{1}{4}+\frac{\lambda w}{\sqrt{1+(1+\lambda^2)w^2}} \times\left\{\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{\lambda w}{\sqrt{1+(1+\lambda^2)w^2}}\right)\right\}\right]. \tag{2}$$

If the pdf of a variable is (1), we denote this by $\mathrm{GSC}(\lambda_1,\lambda_2)$, and if the pdf of a variable is (2), then we denote this by $\mathrm{GSC}(\lambda)$. Figure 1 illustrates several of the possible shapes obtained from $\mathrm{GSC}(\lambda_1,\lambda_2)$ under various choices of (λ_1,λ_2). Some simple properties of $\mathrm{GSC}(\lambda_1,\lambda_2)$ is presented as follows.

Theorem 1. 1. $\mathrm{GSC}(0,0)=C(0,1)$.

2. $\text{GSC}(\lambda_1, 0) = \text{SC}(\lambda_1)$ *and* $\text{GSC}(0, \lambda_2) = \text{SC}(\lambda_2)$.

3. $\text{GSC}(\lambda_1, \lambda_2) = \text{GSC}(\lambda_2, \lambda_1)$.

4. $W_{\lambda_1,\lambda_2} \sim \text{GSC}(\lambda_1, \lambda_2) \Leftrightarrow -W_{\lambda_1,\lambda_2} \overset{d}{=} W_{-\lambda_1,-\lambda_2} \sim \text{GSC}(-\lambda_1, -\lambda_2)$.

5. *If* $X, X_1, X_2, X_3 \overset{iid}{\sim} \text{N}(0,1)$ *and* $X_{1:3} \leqslant X_{2:3} \leqslant X_{3:3}$ *be the corresponding order statistics, then*

$$\frac{X_{1:3}}{|X|} \sim \text{GSC}(-1,-1), \quad \frac{X_{2:3}}{|X|} \sim \text{GSC}(1,-1), \quad \frac{X_{3:3}}{|X|} \sim \text{GSC}(1,1).$$

Proof. The parts 1, 2, 3 and 4 are easily obtained from Definition 1 and the density of W_{λ_1,λ_2}. For part 5, we know that

$$X_{1:3} \sim \text{GSN}(-1,-1), \qquad X_{2:3} \sim \text{GSN}(1,-1), \quad X_{3:3} \sim \text{GSN}(1,1),$$

(see Jamalizadeh et al., 2008), thus by Definition 1 the proof is completed. □

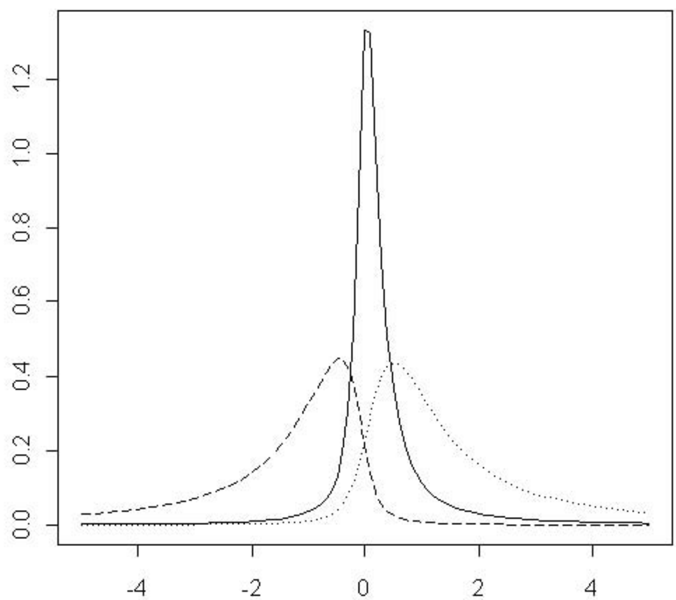

Figure 1. Example of the $\text{GSC}(\lambda_1, \lambda_2)$ density for $(\lambda_1, \lambda_2) = (-3, 5)$ (solid line), $(\lambda_1, \lambda_2) = (1, 2)$ (dotted line), $(\lambda_1, \lambda_2) = (-1, -3)$ (dashed line).

3 Moments

In this section we discuss about the moments of GSC(λ_1, λ_2). We show that the odd moments of two-parameter generalized skew-Cauchy distribution are divergent. Suppose that $W_{\lambda_1,\lambda_2} \sim \text{GSC}(\lambda_1, \lambda_2)$. By the Definition 1, we have

$$E\left(W^m_{\lambda_1,\lambda_2}\right) = E\left(\frac{Z^m_{\lambda_1,\lambda_2}}{|X|^m}\right),$$

where $m = 1, 2, \dots$. Since Z_{λ_1,λ_2} and X are independent and $Y = |X|^2 \sim \chi^2_1$, then

$$E\left(W^m_{\lambda_1,\lambda_2}\right) = E\left(Z^m_{\lambda_1,\lambda_2}\right) E\left(Y^{-\frac{m}{2}}\right).$$

Let $a = \frac{2\pi}{\cos^{-1}\left(-\frac{\lambda_1\lambda_2}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right)}$. Now, we calculate $E\left(Z^m_{\lambda_1,\lambda_2}\right)$ and $E\left(Y^{-\frac{m}{2}}\right)$.

$$\begin{aligned}
E\left(Z^m_{\lambda_1,\lambda_2}\right) &= a\int_{-\infty}^{+\infty} z^{m-1} z\phi(z)\,\Phi(\lambda_1 z)\,\Phi(\lambda_2 z)\,dz \\
&= a(m-1)\int_{-\infty}^{+\infty} z^{m-2}\phi(z)\,\Phi(\lambda_1 z)\,\Phi(\lambda_2 z)\,dz \\
&\quad + \frac{a\lambda_1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} z^{m-1}\phi\left(\sqrt{1+\lambda_1^2}z\right)\Phi(\lambda_2 z)\,dz \\
&\quad + \frac{a\lambda_2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} z^{m-1}\phi\left(\sqrt{1+\lambda_2^2}z\right)\Phi(\lambda_1 z)\,dz \\
&= a(m-1)E\left(Z^{m-2}_{\lambda_1,\lambda_2}\right) + \frac{a\lambda_1\left(1+\lambda_1^2\right)^{-\frac{m}{2}}}{2\sqrt{2\pi}} \\
&\quad \times E\left(Y_1^{m-1}\right) + \frac{a\lambda_2\left(1+\lambda_2^2\right)^{-\frac{m}{2}}}{2\sqrt{2\pi}} E\left(Y_2^{m-1}\right),
\end{aligned}$$

where $Y_1 \sim \text{SN}\left(\frac{\lambda_1}{\sqrt{1+\lambda_1^2}}\right)$ and $Y_2 \sim \text{SN}\left(\frac{\lambda_2}{\sqrt{1+\lambda_2^2}}\right)$.
The moments generating function of $S \sim \text{SN}(\lambda)$ is

$$M_S(t) = 2e^{\frac{t^2}{2}}\Phi\left(\frac{\lambda t}{\sqrt{1+\lambda^2}}\right),$$

and also, we have

$$E\left(Y^{-\frac{m}{2}}\right) = \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} y^{-\frac{m}{2}} e^{-\frac{y}{2}} y^{-\frac{1}{2}}\,dy = \frac{\Gamma\left(\frac{1-m}{2}\right)}{\sqrt{\pi}2^{\frac{m}{2}}}.$$

Since the gamma function is divergent for all nonpositive integers, thus $E\left(Y^{-\frac{m}{2}}\right)$ is divergent for $m = 1, 3, \ldots$. Therefore, if $m = 2, 4, \ldots$, then

$$E\left(W_{\lambda_1,\lambda_2}^{m}\right) = \frac{a\Gamma\left(\frac{1-m}{2}\right)}{\sqrt{\pi}2^{\frac{m}{2}}}\left\{(m-1)\,E\left(Z_{\lambda_1,\lambda_2}^{m-2}\right)\right.$$

$$\left. + \frac{\lambda_1\left(1+\lambda_1^2\right)^{-\frac{m}{2}}}{2\sqrt{2\pi}}E\left(Y_1^{m-1}\right) + \frac{\lambda_2\left(1+\lambda_2^2\right)^{-\frac{m}{2}}}{2\sqrt{2\pi}}E\left(Y_2^{m-1}\right)\right\}.$$

For a special case

$$E\left(W_{\lambda_1,\lambda_2}^{2}\right) = \frac{a\Gamma\left(-\frac{1}{2}\right)}{2\sqrt{\pi}}\left\{1 + \frac{\lambda_1}{2\sqrt{2\pi}\left(1+\lambda_1^2\right)}E\left(Y_1\right) + \frac{\lambda_2}{2\sqrt{2\pi}\left(1+\lambda_2^2\right)}E\left(Y_2\right)\right\}$$

$$= \frac{a\Gamma\left(-\frac{1}{2}\right)}{2\sqrt{\pi}}\left\{1 + \frac{\lambda_1^2}{2\pi\left(1+\lambda_1^2\right)\sqrt{1+2\lambda_1^2}} + \frac{\lambda_2^2}{2\pi\left(1+\lambda_2^2\right)\sqrt{1+2\lambda_2^2}}\right\}.$$

The value of gamma function for some special cases that may has been used to calculating the moments as follows.

$$\Gamma(-0.5) = -3.544908$$
$$\Gamma(-1.5) = 2.363272$$
$$\Gamma(-2.5) = -0.9453087$$
$$\Gamma(-3.5) = 0.2700882$$

4 Some Important Properties

***Theorem* 2.** *Suppose that* $X, U_1, U_2, U_3 \overset{iid}{\sim} \mathrm{N}(0,1)$ *and also,* $Y_1 = \frac{U_1}{|X|}$, $Y_2 = \frac{U_2}{|X|}$ *and* $Y_3 = \frac{U_3}{|X|}$. *Then*

$$W_{\lambda_1,\lambda_2} \overset{d}{=} Y_1 |\left(Y_2 < \lambda_1 Y_1, Y_3 < \lambda_2 Y_1\right) \sim \mathrm{GSC}(\lambda_1, \lambda_2).$$

Proof. It is clearly that

$$W_{\lambda_1,\lambda_2} \overset{d}{=} \left.\frac{U_1}{|X|}\right| \left(U_2 < \lambda_1 U_1, U_3 < \lambda_2 U_1\right).$$

Let $U \overset{d}{=} U_1 |\left(U_2 < \lambda_1 U_1, U_3 < \lambda_2 U_1\right)$. We know that

$$P\left(U_2 < \lambda_1 U_1, U_3 < \lambda_2 U_1\right) = \frac{\cos^{-1}\left(-\frac{\lambda_1\lambda_2}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right)}{2\pi},$$

(see Jamalizadeh et al., 2008), and also

$$f_U(u) = \frac{P(U_2 < \lambda_1 U_1, U_3 < \lambda_2 U_1 | U_1 = u)\,\phi(u)}{P(U_2 < \lambda_1 U_1, U_3 < \lambda_2 U_1)}$$
$$= \frac{2\pi}{\cos^{-1}\left(-\frac{\lambda_1\lambda_2}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right)}\phi(u)\,\Phi(\lambda_1 u)\,\Phi(\lambda_2 u),$$

then $U \sim \mathrm{GSN}(\lambda_1, \lambda_2)$ and therefore, by Definition 1 the proof is completed. □

Corollary 1. Suppose that $X, U_1, U_2, U_3 \overset{iid}{\sim} \mathrm{N}(0,1)$ and also, $Y_1 = \frac{U_1}{|X|}$, $Y_2 = \frac{U_2}{|X|}$ and $Y_3 = \frac{U_3}{|X|}$. Then

$$Y_1|\,(Y_2 < \lambda Y_1, Y_3 < \lambda Y_1) \sim \mathrm{GSC}(\lambda).$$

Theorem 3. *Suppose that* $(U_1, U_2, U_3) \sim \mathrm{N}_3\left(\mathbf{0}, \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}\right)$*, with* $\rho_{23} = \rho_{12}\rho_{13}$*, and also* $X \sim \mathrm{N}(0,1)$ *be independent of* (U_1, U_2, U_3)*. Then*

$$W_{\lambda_1,\lambda_2} \overset{d}{=} \left.\frac{U_1}{|X|}\right|(\min(U_2, U_3) > 0) \sim \mathrm{GSC}(\lambda_1, \lambda_2),$$

where $\lambda_1 = \frac{\rho_{12}}{\sqrt{1-\rho_{12}^2}}$ *and* $\lambda_2 = \frac{\rho_{13}}{\sqrt{1-\rho_{13}^2}}$.

Proof. Since $U_1|\,(\min(U_2, U_3) > 0) \sim \mathrm{GSN}(\lambda_1, \lambda_2)$ (see Jamalizadeh et al., 2008), then by Definition 1 the proof is completed. □

We need to the next definition and lemma to present the next theorem.

Definition 2. We say that $\mathbf{V} = (V_1, V_2, V_3)$ has a standard trivariate Cauchy distribution if its pdf is

$$f(\mathbf{v}; \Sigma) = \frac{1}{\pi^2 |\Sigma|^{\frac{1}{2}} (1 + \mathbf{v}\Sigma^{-1}\mathbf{v})^2},$$

where $\mathbf{v}' = (v_1, v_2, v_3)$ and $\Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}$, (see Fang et al., 1990).

We denote this distribution by $\mathbf{V} \sim C_3(\mathbf{0}, \Sigma)$. It can be shown that for $i = 1, 2, 3$, we have $V_i \sim C(0, 1)$.

***Lemma* 1.** *Suppose that* $(U_1, U_2, U_3) \sim \mathrm{N}_3\left(\mathbf{0}, \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}\right)$ *and* $X \sim \mathrm{N}(0, 1)$ *are independent. Then*

$$\left(\frac{U_1}{|X|}, \frac{U_2}{|X|}, \frac{U_3}{|X|}\right) \sim C_3(\mathbf{0}, \Sigma).$$

Proof. Suppose that $\mathbf{U} = (U_1, U_2, U_3) \sim \mathrm{N}_3(\mathbf{0}, \Sigma)$ and $\mathbf{V} = (V_1, V_2, V_3) \overset{d}{=} \left(\frac{U_1}{|X|}, \frac{U_2}{|X|}, \frac{U_3}{|X|}\right)$, then

$$\begin{aligned} F_{\mathbf{V}}(\mathbf{v}) &= P(\mathbf{V} \leqslant \mathbf{v}) = P\left(\frac{U_1}{|X|} \leqslant v_1, \frac{U_2}{|X|} \leqslant v_2, \frac{U_3}{|X|} \leqslant v_3\right) \\ &= P(\mathbf{U} \leqslant \mathbf{v}|X|) = E\{\Phi_3(\mathbf{v}|X|; \Sigma)\} \\ &= 2\int_0^\infty \Phi_3(\mathbf{v}x; \Sigma)\,\phi(x)dx, \end{aligned}$$

where $\Phi_3(\cdot; \Sigma)$ is the cdf of $\mathrm{N}_3(\mathbf{0}, \Sigma)$. Upon differentiating this expression of $F_{\mathbf{V}}(\mathbf{v})$, we obtain

$$f_{\mathbf{V}}(\mathbf{v}) = \frac{\partial}{\partial v_1 \partial v_2 \partial v_3} F_{\mathbf{V}}(\mathbf{v}) = 2\int_0^\infty x^3 \frac{e^{-\frac{1}{2}x^2\left(\mathbf{v}\Sigma^{-1}\mathbf{v}\right)}}{(2\pi)^{\frac{3}{2}} |\Sigma|^{\frac{1}{2}}} \cdot \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx,$$

then, by integration by parts, we have

$$\begin{aligned} f_{\mathbf{V}}(\mathbf{v}) &= \frac{2}{(2\pi)^2 |\Sigma|^{\frac{1}{2}}} \left(-\frac{x^2 e^{-\frac{1}{2}x^2\left(1+\mathbf{v}\Sigma^{-1}\mathbf{v}\right)}}{1 + \mathbf{v}\Sigma^{-1}\mathbf{v}} - \frac{2e^{-\frac{1}{2}x^2\left(1+\mathbf{v}\Sigma^{-1}\mathbf{v}\right)}}{\left(1 + \mathbf{v}\Sigma^{-1}\mathbf{v}\right)^2}\right)_0^\infty \\ &= \frac{1}{\pi^2 |\Sigma|^{\frac{1}{2}} \left(1 + \mathbf{v}\Sigma^{-1}\mathbf{v}\right)^2}. \end{aligned}$$

□

Table 1. MLEs for the lifespan of rats (*ad libitum diet*) under GSC and SC models.

	SC $(\boldsymbol{\lambda})$	GSC $(\boldsymbol{\lambda})$	GSC $(\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2})$
$\widehat{\boldsymbol{\lambda}}$	0.2902438	0.09042141	—
$\widehat{\boldsymbol{\lambda}}_1$	—	—	−1.656760
$\widehat{\boldsymbol{\lambda}}_2$	—	—	2.153499
Log-likelihood	−137.655963	−137.999645	−119.832504

***Theorem* 4.** *Suppose that* $(V_1, V_2, V_3) \sim C_3(\mathbf{0},\Sigma)$ *and* $\Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}$ *with* $\rho_{23} = \rho_{12}\rho_{13}$. *Then*

$$V_1 | (\min(V_2, V_3) > 0) \sim \mathrm{GSC}(\lambda_1, \lambda_2),$$

where $\lambda_1 = \frac{\rho_{12}}{\sqrt{1-\rho_{12}^2}}$ *and* $\lambda_2 = \frac{\rho_{13}}{\sqrt{1-\rho_{13}^2}}$.

Proof. As in Lemma 1, suppose that $(V_1, V_2, V_3) \stackrel{d}{=} \left(\frac{U_1}{|X|}, \frac{U_2}{|X|}, \frac{U_3}{|X|}\right)$, where $(U_1, U_2, U_3) \sim \mathrm{N}_3\left(\mathbf{0}, \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}\right)$, with $\rho_{23} = \rho_{12}\rho_{13}$, and $X \sim \mathrm{N}(0, 1)$ are independent. Then the proof is completed by theorem 2. □

5 Data Illustration

In this section we consider two data sets to compare $\mathrm{SC}(\lambda)$ and $\mathrm{GSC}(\lambda_1, \lambda_2)$.

Example 1. This example considers the standardized data concerning the lifespan of rats that they were under an *ad libitum* diet (that is, "free eating"), given in Landau and Everitt (2003). We want to compare $\mathrm{SC}(\lambda)$ and $\mathrm{GSC}(\lambda_1, \lambda_2)$, by fitting them for these standardized data. We estimate parameters by numerically maximizing the likelihood function. The obtained numerical results are presented in Table 1. Based on log-likelihood, $\mathrm{GSC}(\widehat{\lambda}_1, \widehat{\lambda}_2)$ fits the data better than $\mathrm{SC}(\widehat{\lambda})$. Figure 2 illustrates the histogram of the data with the fitted densities.

Example 2. In this example, we consider the standardized roller data set, available for downloading at http://lib.stat.cmu.edu/jasadata/laslett and

Table 2. MLEs for the roller data set under GSC and SC models

	SC (λ)	**GSC (λ)**	**GSC (λ_1, λ_2)**
$\widehat{\lambda}$	0.1649845	0.0590169	—
$\widehat{\lambda}_1$	—	—	1.580540
$\widehat{\lambda}_2$	—	—	−1.337220
Log-likelihood	−1871.571888	−1872.913647	−1731.895581

alternatively analyzed by Gomez et al. (2006). The data set consists of 1150 heights measured at 1 micron intervals along the drum of a roller (i.e. parallel to the axis of the roller). For this standardized data set the obtained numerical results are presented in Table 2. Based on log-likelihood, GSC($\widehat{\lambda}_1, \widehat{\lambda}_2$) fits the data better than SC($\widehat{\lambda}$). This point is further illustrated in Figure 3, where a histogram of the data is plotted together with the fitted densities.

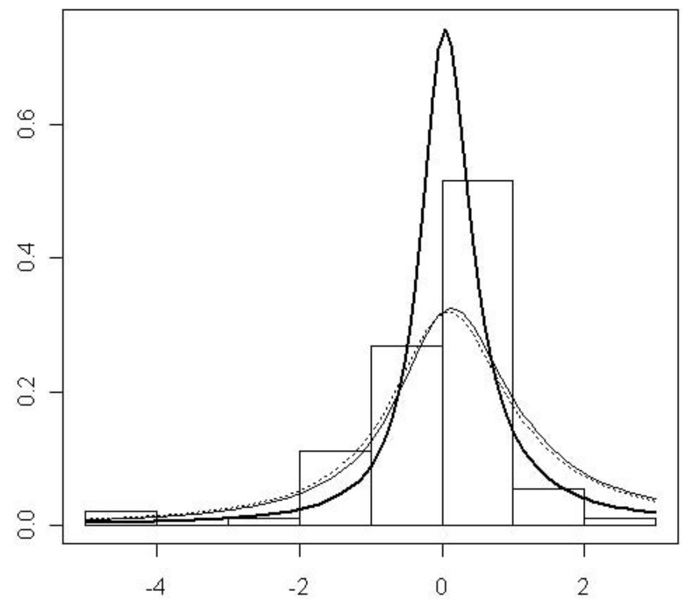

Figure 2. Histogram of the lifespan of rats (ad libitum diet). The lines represent distributions fitted using MLE: GSC($\widehat{\lambda}_1$,$\widehat{\lambda}_2$) (bold solid line), GSC($\widehat{\lambda}$) (dotted line), SC($\widehat{\lambda}$) (solid line).

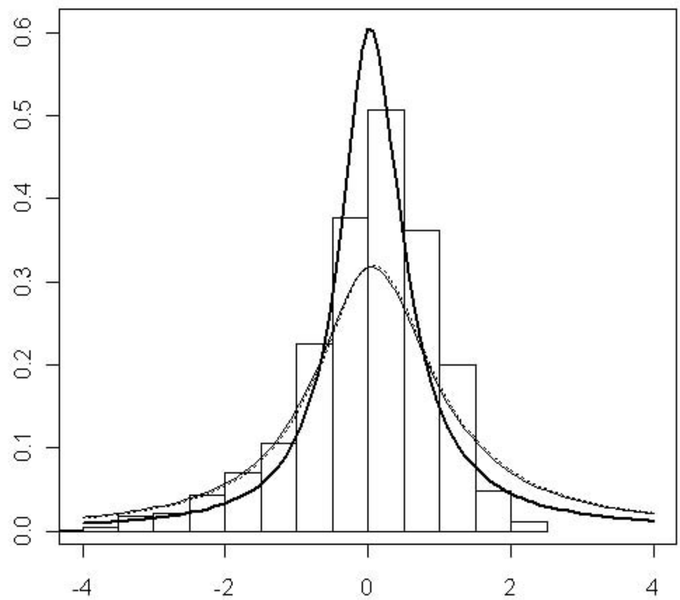

Figure 3. Histogram of the standardized roller data. The lines represent distributions fitted using MLE: GSC($\widehat{\lambda}_1$,$\widehat{\lambda}_2$) (bold solid line), GSC($\widehat{\lambda}$) (dotted line), SC($\widehat{\lambda}$) (solid line).

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