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Recurrence Relations for Moment Generating Functions of Generalized Order Statistics Based on Doubly Truncated Class of Distributions

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Abstract. In this paper, we derived recurrence relations for joint moment generating functions of nonadjacent generalized order statistics (GOS) of random samples drawn from doubly truncated class of continuous distributions. Recurrence relations for joint moments of nonadjacent GOS (ordinary order statistics (OOS) and k-upper records (k-RVs) as special cases) are obtained. Single and product moment generating functions (moments) of nonadjacent GOS are derived. Doubly truncated new modified Weibull (Weibull, Extreme-value, exponential and Rayleigh), three Burr type XII (Lomax) and inverse Weibull distributions, among others, arise as special cases of this doubly truncated class. Two applications are introduced, the first is the characterizations for members of the class based on recurrence relations for moments of GOS, OOS and k-RVs. As the second application we found Tables of single and product moments of OOS from doubly truncated Lomax distribution.

Keywords. Generalized order statistics; recurrence relations; moment generating functions; order statistics; k-records; characterizations; truncated distributions.

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1 Introduction

Udo Kamps (1995) has introduced GOS as random variables having certain joint density function, which includes as a special case the joint density functions of many models of ordered random variables such as ordinary order statistics (OOS) (David, 1981; and Arnold, Balakrishnan and Nagaraja, 1992), sequential order statistics (SOS) (Cramer and Kamps, 1996), record values, k-record values (k-RVs), and Pfeifer's records (Nevzorov, 1987; and Ahsanullah, 1995), Progressive Type-II censoring order statistics (PCOS) (Balakrishnan and Asgharzadeh, 2005; and Sarhan, Ammar and Abuammoh, 2008). The structural similarities of these models are based on the similarity of their joint density function. Therefore, all of these models are contained in the model of GOS.

Our aim In this paper is derive recurrence relations for joint moment generating functions of nonadjacent GOS of random samples drawn from doubly truncated class of continuous distributions. Recurrence relations for joint moments of nonadjacent GOS (OOS and upper k-RVs as special cases) are obtained. Single and product moment generating functions (moments) of nonadjacent GOS are derived.

Recently, new and more general research based on generalized order statistics have been made by several authors. Statistical inference and characterizations based on properties generalized order statistics have been established by several authors. Such authors are Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000) Habibullah and Ahsanullah (2000), Pawlas and Szynal (2001), Ahmad and fawzy (2003), Saran and Pandey (2003), Athar and Islam (2004), AL-Hussaini et al. (2005), Ahmad (2006, 2008), Ahmad and Abu-Shal (2006), among others. A large number of publications are concerned with recurrence relations of moments of OOS and RVs, details survey of the OOS are given in Balakrishnan, Malik and Ahmed (1988), Balakrishnan and Sultan (1998), Balakrishnan Ahsanullah (1995), Ahmad (2001) and AL-Hussaini et al (2004).

Our aim in this paper is to drive recurrence relations for joint moment generating functions of nonadjacent generalized order statistics of random samples drawn from doubly truncated class of continuous distributions. Joint moments of nonadjacent GOS are obtained from these relations. Specialization to single, product moment generating functions and moments of nonadjacent GOS, OOS and RVs are obtained. Single and product moment generating functions and moments of nonadjacent GOS, OOS and RVs are derived as special cases of this class.

Let X be a random variable (rv) has a class with distribution functions (df) which given by

$$F(x) = 1 - \{ah(x) + b\}^c, \qquad \alpha \leqslant x \leqslant \beta, \tag{1}$$

and the corresponding probability density function (pdf) is given by

$$f(x) = -ach'(x)\{ah(x) + b\}^{c-1}, \qquad \alpha \leqslant x \leqslant \beta, \tag{2}$$

where $a, b, c \neq 0$ are constants such that $F(\alpha) = 0,\ F(\beta) = 1$ and $h(x)$ is a monotonic differentiable arbitrary function of x in $[\alpha, \beta]$ such that $f(x)$ is a pdf.

The corresponding survival function (sf) is given by

$$\overline{F}(x) = 1 - F(x) = \{ah(x) + b\}^c, \qquad \alpha \leqslant x \leqslant \beta.$$

so, from (2), we can obtain

$$\overline{F}(x) = \frac{\{ah(x) + b\}f(x)}{-ach'(x)} \tag{3}$$

The doubly truncated pdf $f_d(x)$, df $F_d(x)$, and sf $\overline{F}_d(x)$, are given in general forms, respectively, by

$$\begin{aligned} f_d(x) &= Df(x), \qquad P_1 \leqslant x \leqslant Q_1, \\ F_d(x) &= DF(x) - P_2, \\ \overline{F}_d(x) &= Q_2 + D\overline{F}(x), \end{aligned} \tag{4}$$

where

$$D = \frac{1}{Q - P}, \quad Q = F(Q_1), \quad P = F(P_1), \quad Q_2 = D(Q-1), \quad \text{and} \quad P_2 = DP.$$

Substituting (3) in (4), we obtain

$$\overline{F}_d(x) = Q_2 - \frac{\{ah(x) + b\}f_d(x)}{ach'(x)}. \tag{5}$$

2 Recurrence Relation for Moment Generating Functions of GOS from Doubly Truncated Distributions

Suppose that $X_{r;n,m,k}$ is the rth GOS, so if $X_{1;n,m,k}, X_{2;n,m,k}, \ldots, X_{n;n,m,k}$ are n GOS from the *df* (5), where $n > 1$, $m \geqslant -1$, $k \geqslant 1$ are real numbers. Then the joint doubly truncated pdf of $X_{r_1;n,m,k}, \ldots, X_{r_\ell;n,m,k}$ is given for $1 \leqslant r_1 \leqslant \cdots \leqslant r_\ell \leqslant n$, $r_0 = 0$, $r_{\ell+1} = n+1$ (see, Ahmad and Abu-Shal, 2006), by

$$\begin{aligned} f_{r_1,\ldots,r_\ell}(x_1,\ldots,x_\ell) = C(\ell, r_\ell)\Bigg[\prod_{i=0}^{\ell-1}\{\overline{F}(x_i)\}^m[h_m\{F(x_{i+1})\} \\ - h_m\{F(x_i)\}]^{r_{i+1}-r_i-1}\Bigg] \times \left\{\prod_{i=1}^{\ell} f(x_i)\right\}\{\overline{F}(x_l)\}^{\gamma_{r_l}-1}, \end{aligned} \tag{6}$$

for $F^{-1}(0+) < x_1 \leqslant \cdots \leqslant x_n < F^{-1}(1)$, where

$$C(\ell, r_\ell) = \frac{C_{r_\ell - 1}}{\prod_{i=0}^{\ell-1}(r_{i+1} - r_i - 1)!}, \quad C_{r_\ell - 1} = \prod_{i=0}^{r_\ell}\gamma_i, \quad \gamma_i = k + (n-i)(m+1), \tag{7}$$

and for $0 < t < 1$

$$h_m(t) = \begin{cases} -\frac{(1-t)^{m+1}}{(m+1)} & m \neq -1 \\ -\ln(1-t) & m = -1. \end{cases}$$

The joint moment generating function of $X^{j_1}_{r_1;n,m,k}, \ldots, X^{j_l}_{r_l;n,m,k}$, for $j_1, \ldots, j_l \geqslant 1$, is given by

$$\begin{aligned} M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k}(t_1,\ldots,t_\ell) &= E\left\{\exp\left(\sum_{i=1}^{\ell} t_i X^{j_i}_{r_i,n,m,k}\right)\right\} \\ &= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\cdots\int_{x_{\ell-1}}^{\infty}\exp\left(\sum_{i=1}^{\ell} t_i x_i^{j_i}\right) \\ &\quad \times f_{r_1,\ldots,r_\ell}(x_1,\ldots,x_\ell)dx_\ell \ldots dx_1, \end{aligned} \tag{8}$$

and the joint moments are given by

$$\mu^{(j_1,\dots,j_\ell)}_{r_1,\dots,r_\ell,n,m,k}(t_1,\dots,t_\ell) = E\left(\prod_{i=1}^{\ell} X^{j_i}_{r_i,n,m,k}\right)$$
$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\dots\int_{x_{\ell-1}}^{\infty}\prod_{i=1}^{\ell} x_i^{j_i} f_{r_1,\dots,r_\ell}(x_1,\dots,x_\ell)dx_\ell\dots dx_1.$$

Now we can a recurrence relation for joint moment generating functions of nonadjacent GOS for the class (5) by the following theorem.

***Theorem* 1.** *Let $X_{r_1;n,m,k},\dots,X_{r_\ell;n,m,k}$ be n GOS formed from a random sample of size n drawn form the df (5). Suppose for $j_1,\dots,j_\ell \geqslant 1$, $1 \leqslant r_1 \leqslant \dots \leqslant r_\ell \leqslant n$, that the expectation*

$$E\left\{X^{j_\ell-1}_{r_\ell,n,m,k}\exp\left(\sum_{i=1}^{\ell} t_i X^{j_i}_{r_i,n,m,k}\right)\cdot\frac{ah(X_{r_\ell,n,m,k})+b}{h'(X_{r_\ell,n,m,k})}\right\},$$

is finite, then for $r_{\ell-1} < r^*_\ell < r_\ell$, $m \geqslant -1$, and $k \geqslant 1$ the following recurrence relation

$$M^{(j_1,\dots,j_\ell)}_{r_1,\dots,r_\ell,n,m,k}(t_1,\dots,t_\ell) - M^{(j_1,\dots,j_\ell)}_{r_1,\dots,r^*_\ell,n,m,k}(t_1,\dots,t_\ell) =$$
$$\xi Q_2\left\{M^{(j_1,\dots,j_\ell)}_{r_1,\dots,r_\ell,n-1,m,k+m}(t_1,\dots,t_\ell) - M^{(j_1,\dots,j_\ell)}_{r_1,\dots,r^*_\ell,n-1,m,k+m}(t_1,\dots,t_\ell)\right\}$$
$$-\frac{t_\ell j_\ell}{ac\gamma_{r_\ell}}E\left\{\Psi_{j_\ell}(X_{r_\ell,n,m,k};t_\ell)\exp\left(\sum_{i=1}^{\ell-1} t_i X^{j_i}_{r_i,n,m,k}\right)\right\} \qquad (9)$$

where

$$\Psi_{j_\ell}(x_{r_\ell,n,m,k};t_\ell) = x^{j_\ell-1}_{r_\ell,n,m,k}\exp\left(t_\ell x^{j_\ell}_{r_\ell,n,m,k}\right)\cdot\frac{ah(x_{r_\ell,n,m,k})+b}{h'(x_{r_\ell,n,m,k})} \qquad (10)$$

$$\xi = \xi(r_\ell,n,m,k) = \prod_{i=1}^{r_\ell-1}\frac{\gamma_i}{\gamma_i^*}, \quad \gamma_i^* = \gamma_i - 1 = (k+m)+(n-1-i)(m+1)$$
$$\textit{and} \quad \gamma_r + m = \gamma_{r-1} - 1.$$

Proof. By replacing f and $\overline{F}$ in (6) by f_d and $\overline{F_d}$ respectively, we can write (8) as

$$\begin{aligned}
M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k}(t_1,\ldots,t_\ell) &= C(\ell,r_\ell)\int_{P_1}^{Q_1}\int_{x_1}^{Q_1}\cdots\int_{x_{\ell-1}}^{Q_1}\exp\left(\sum_{i=1}^{\ell}t_ix_i^{j_i}\right)\\
&\quad\times\left\{\prod_{i=1}^{\ell}f_d(x_i)\right\}\cdot\left[\prod_{i=0}^{\ell-1}\{\overline{F}_d(x_i)\}^m[h_m\{F_d(x_{i+1})\}\right.\\
&\quad\left.-h_m\{F_d(x_i)\}]^{r_{i+1}-r_i-1}\right]\{\overline{F}_d(x_\ell)\}^{\gamma_{r_\ell}-1}dx_\ell\ldots dx_1\\
&= C(\ell,r_\ell)\int_{P_1}^{Q_1}\int_{x_1}^{Q_1}\ldots.\int_{x_{\ell-1}}^{Q_1}\exp\left(\sum_{i=1}^{\ell}t_ix_i^{j_i}\right)\\
&\quad\times\left\{\prod_{i=1}^{\ell-1}f_d(x_i)\right\}\cdot\left[\prod_{i=0}^{\ell-2}\{\overline{F}_d(x_i)\}^m[h_m\{F_d(x_{i+1})\}\right.\\
&\quad\left.-h_m\{F_d(x_i)\}]^{r_{i+1}-r_i-1}\right]\{\overline{F}_d(x_\ell)\}^m\\
&\quad\times I(x_{\ell-1})dx_{\ell-1}\ldots dx_1,
\end{aligned}\tag{11}$$

where

$$\begin{aligned}
I(x_{\ell-1}) &= \int_{x_{\ell-1}}^{Q_1}\exp\left(t_\ell x_\ell^{j_\ell}\right)[h_m\{F_d(x_\ell)\}-h_m\{F_d(x_{\ell-1})\}]^{r_\ell-r_{\ell-1}-1}\\
&\quad\times\{\overline{F}_d(x_\ell)\}^{\gamma_{r_\ell}-1}\{f_d(x_\ell)\}dx_\ell\\
&= \int_{x_{\ell-1}}^{Q_1}\exp\left(t_\ell x_\ell^{j_\ell}\right)[h_m\{F_d(x_\ell)\}-h_m\{F_d(x_{\ell-1})\}]^{r_\ell-r_{\ell-1}-1}\\
&\quad d\{-\overline{F}_d(x_\ell)\}^{\frac{\gamma_{r_\ell}}{\gamma_{r_\ell}}},
\end{aligned}$$

making use of integration by parts we can write $I(x_{\ell-1})$ in the form

$$\begin{aligned}
I(x_{\ell-1}) &= \frac{t_\ell j_\ell}{ac\gamma_{r_\ell}}\int_{x_{\ell-1}}^{Q_1}x_\ell^{j_{\ell-1}}\exp\left(t_\ell x_\ell^{j_\ell}\right)[h_m\{F_d(x_\ell)\}-h_m\{F_d(x_{\ell-1})\}]^{r_\ell-r_{\ell-1}-1}\\
&\quad\times\{\overline{F}_d(x_\ell)\}^{\gamma_{r_\ell}}dx_\ell+\frac{r_\ell-r_{\ell-1}-1}{\gamma_{r_\ell}}\int_{x_{\ell-1}}^{Q_1}\exp\left(t_\ell x_\ell^{j_\ell}\right)[h_m\{F_d(x_\ell)\}
\end{aligned}$$

$$- h_m\{F_d(x_{\ell-1})\}]^{r_\ell - r_{\ell-1}-2}\{\overline{F}_d(x_\ell)\}^{\gamma_{r_{\ell-1}}-1}\{f_d(x_\ell)\}dx_\ell,$$

substituting $I(x_{\ell-1})$ in (11), we obtain

$$\begin{aligned} M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k}(t_1,\ldots,t_\ell) = {} & \frac{t_\ell j_\ell C(\ell,r_\ell)}{\gamma_{r_\ell}}\int_{P_1}^{Q_1}\int_{x_1}^{Q_1}\cdots\int_{x_{\ell-1}}^{Q_1} x_\ell^{j_\ell-1}\exp\left(\sum_{i=1}^{\ell} t_i x_i^{j_i}\right) \\ & \times \left[\prod_{i=0}^{\ell-1}\{\overline{F}_d(x_i)\}^m[h_m\{F_d(x_{i+1})\} - h_m\{F_d(x_i)\}]^{r_{i+1}-r_i-1}\right] \\ & \times \{\overline{F}_d(x_\ell)\}^{\gamma_{r_\ell}}\left\{\prod_{i=1}^{\ell-1} f_d(x_i)\right\} dx_\ell \ldots .dx_1 + \frac{(r_\ell - r_{\ell-1}-1)C(\ell,r_\ell)}{\gamma_{r_\ell}} \\ & \times \int_{P_1}^{Q_1}\int_{x_1}^{Q_1}\cdots\int_{x_{\ell-1}}^{Q_1}\exp\left(\sum_{i=1}^{\ell} t_i x_i^{j_i}\right)\left\{\prod_{i=0}^{\ell} f_d(x_i)\right\} \\ & \times \{\overline{F}_d(x_\ell)\}^{\gamma_{r_{\ell-1}}-1}\left[\prod_{i=0}^{\ell-1}\{\overline{F}_d(x_i)\}^m[h_m\{F_d(x_{i+1})\}\right. \\ & \left. - h_m\{F_d(x_i)\}\right]^{r_{i+1}-r_i-1} dx_\ell \ldots dx_1. \end{aligned} \tag{12}$$

from (7) we can show that

$$\frac{(r_\ell - r_{\ell-1}-1)C(\ell,r_\ell)}{\gamma_{r_\ell}} = \frac{C_{r_\ell^*-1}}{\prod_{i=1}^{\ell-2}(r_i - r_{i-1}-1)!(r_\ell^* - r_{\ell-1}-1)},$$

where

$$C^*(\ell,r_\ell) = \frac{\prod_{i=0}^{r_\ell}\gamma_i^*}{\prod_{i=0}^{\ell-1}(r_{i+1}-r_i-1)!}, \qquad \gamma_i^* = \gamma_i - 1.$$

so (12) can be written as

$$\begin{aligned} & M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k}(t_1,\ldots,t_\ell) - M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell^*,n,m,k}(t_1,\ldots,t_\ell) = \frac{t_\ell j_\ell C(\ell,r_\ell)}{\gamma_{r_\ell}} \\ & \times \int_{P_1}^{Q_1}\int_{x_1}^{Q_1}\cdots\int_{x_{\ell-1}}^{Q_1} x_\ell^{j_{\ell-1}}\exp\left(\sum_{i=1}^{\ell} t_i x_i^{j_i}\right)\left\{\prod_{i=1}^{\ell-1} f_d(x_i)\right\}\{\overline{F}_d(x_\ell)\}^{\gamma_{r_\ell}} \end{aligned}$$

$$\times \left[\prod_{i=0}^{\ell-1}\{\overline{F}_d(x_i)\}^m[h_m\{F_d(x_{i+1})\} - h_m\{F_d(x_i)\}]^{r_{i+1}-r_i-1}\right] dx_\ell \ldots dx_1, \quad (13)$$

Making use of (5) we can write (13) in the form

$$\begin{aligned}
M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k}(t_1,\ldots,t_\ell) - M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r^*_\ell,n,m,k}(t_1,\ldots,t_\ell) &= \frac{t_\ell j_\ell Q_2 C(\ell, r_\ell)}{\gamma_{r_\ell}} \\
&\times \int_{P_1}^{Q_1}\int_{x_1}^{Q_1}\cdots\int_{x_{\ell-1}}^{Q_1} x_\ell^{j_\ell-1}\exp\left(\sum_{i=1}^{\ell} t_i x_i^{j_i}\right)\left\{\prod_{i=1}^{\ell-1} f_d(x_i)\right\} \\
&\times \left[\prod_{i=0}^{\ell-1}\{\overline{F}_d(x_i)\}^m[h_m\{F_d(x_{i+1})\} - h_m\{F_d(x_i)\}]^{r_{i+1}-r_i-1}\right] \\
&\times \{\overline{F}_d(x_\ell)\}^{\gamma_{r_\ell}-1} dx_\ell \ldots dx_1 - \frac{t_\ell j_\ell C(\ell, r_\ell)}{ac\gamma_{r_\ell}} \\
&\times \int_{P_1}^{Q_1}\int_{x_\ell}^{Q_1}\cdots\int_{x_{\ell-1}}^{Q_1} x_\ell^{j_\ell-1}\exp\left(\sum_{i=1}^{\ell} t_i x_i^{j_i}\right)\frac{ah(x_\ell)+b}{h'(x_\ell)} \\
&\times \left\{\prod_{i=1}^{\ell} f_d(x_i)\right\}\{\overline{F}_d(x_\ell)\}^{\gamma_{r_\ell}-1}\left[\prod_{i=0}^{\ell-1}\{\overline{F}_d(x_i)\}^m][h_m\{F_d(x_{i+1})\}\right. \\
&\left. - h_m\{F_d(x_i)\}]^{r_{i+1}-r_i-1}\right] dx_\ell \ldots dx_1, \qquad (14)
\end{aligned}$$

from (8) in the last term we can write (14) in the form

$$\begin{aligned}
M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k}(t_1,\ldots,t_\ell) - M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r^*_\ell,n,m,k}(t_1,\ldots,t_\ell) &= \frac{t_\ell j_\ell Q_2 C(\ell, r_\ell)}{\gamma_{r_\ell}} \\
&\times \int_{P_1}^{Q_1}\int_{x_1}^{Q_1}\cdots\int_{x_{\ell-1}}^{Q_1} x_\ell^{j_\ell-1}\exp\left(\sum_{i=1}^{\ell} t_i x_i^{j_i}\right)\left\{\prod_{i=1}^{\ell-1} f_d(x_i)\right\} \\
&\times \left[\prod_{i=0}^{\ell-1}\{\overline{F}_d(x_i)\}^m[h_m\{F_d(x_{i+1})\} - h_m\{F_d(x_i)\}]^{r_{i+1}-r_i-1}\right] \\
&\times \{\overline{F}_d(x_\ell)\}^{\gamma_{r_\ell}-1} dx_\ell \ldots dx_1 - \frac{t_\ell j_\ell}{ac\gamma_{r_\ell}} E\Bigg[X^{j_\ell-1}_{r_\ell,n,m,k}
\end{aligned}$$

$$\times \exp\left(\sum_{i=1}^{\ell} t_i X_{r_i,n,m,k}^{j_i}\right) \frac{ah(X_{r_\ell,n,m,k})+b}{h'(X_{r_\ell,n,m,k})}\Bigg], \tag{15}$$

then by replacing n by $n-1$ and k by $k+m$ in (13), and substituting in the first term in the right hand side in (15), the recurrence relation (9) can be obtained.

Theorem 2. *The recurrence relation (9) is satisfied, if and only if$X$ has the df (5).*

Proof. If X has *df* (5), then recurrence relation (9) is satisfied from Theorem 1. On the other hand, if the recurrence relation (9) is satisfied, then by using (6) and (13) in (9), we obtain

$$\int_{P_1}^{Q_1}\int_{x_1}^{Q_1}\cdots\int_{x_{\ell-1}}^{Q_1} x_\ell^{j_{\ell-1}} \exp\left(\sum_{i=1}^{\ell} t_i x_i^{j_i}\right)\left[\prod_{i=0}^{\ell-1}\{\overline{F}_d(x_i)\}^m [h_m\{F_d(x_{i+1})\}\right.$$
$$\left. - h_m\{F_d(x_i)\}]^{r_{i+1}-r_i-1}\right]\cdot\left\{\prod_{i=1}^{\ell-1} f_d(x_i)\right\}\{\overline{F}_d(x_\ell)\}^{\gamma_{r_\ell}-1}$$
$$\left[\overline{F}_d(x_\ell) - Q_2 + \frac{\{ah(x_\ell)+b\}}{ach'(x_\ell)} f_d(x_\ell)\right] dx_\ell \ldots dx_1 = 0.$$

Applying the extension of Müntz-Szàsz theorem, see Hwang and Lin (1984), we have

$$\overline{F}_d(x_\ell) - Q_2 + \frac{\{ah(x_\ell)+b\}}{ach'(x_\ell)} f_d(x_\ell) = 0.$$

This leads to

$$\overline{F}_d(x_\ell) = Q_2 - \frac{\{ah(x_\ell)+b\}}{ach'(x_\ell)} f_d(x_\ell).$$

whose solution is the *df* (5).

2.1 Remarks

1. If $r_\ell = r_{\ell-1}+1$, then

$$I(x_{\ell-1}) = \int_{x_{\ell-1}}^{Q_1} \exp(t_\ell x_\ell^{j_\ell})\{\overline{F}_d(x_\ell)\}^{\gamma_{r_\ell}-1}\{f_d(x_\ell)\}dx_\ell,$$

by substituting $I(x_{\ell-1})$ in (9), and by the same way in the above proof we can obtain

$$M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k}(t_1,\ldots,t_\ell) = \xi Q_2 M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n-1,m,k+m}(t_1,\ldots,t_\ell) - \frac{t_\ell j_\ell}{ac\gamma_{r_\ell}} \times E\left\{\Psi_{j_\ell}(x_{r_\ell,n,m,k};t_\ell)\exp\left(\sum_{i=1}^{\ell-1} t_i X^{j_i}_{r_i,n,m,k}\right)\right\}, \quad (16)$$

2. By differentiating (9), with respect to $t_1,\ldots,t_\ell$ and putting $t_1=\cdots=t_\ell=0$, we obtain the following recurrence relation for the joint moments of nonadjacent GOS

$$\mu^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k} - \mu^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r^*_\ell,n,m,k} = -\frac{j_\ell}{ac\gamma_{r_\ell}} E\{\Psi_{j_\ell,\ldots,j_1}(X_{r_\ell,n,m,k},\ldots, X_{r_1,n,m,k})\} + \xi Q_2 \Phi^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n-1,m,k+m}, \quad (17)$$

where

$$\Phi^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n-1,m,k+m} = \mu^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n-1,m,k+m} - \mu^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r^*_\ell,n-1,m,k+m}, \quad (18)$$

and

$$\Psi_{j_\ell,\ldots,j_1}(x_{r_\ell,n,m,k},\ldots,x_{r_1,n,m,k}) = x^{j_\ell-1}_{r_\ell,n,m,k}\left(\prod_{i=1}^{\ell-1} x^{j_i}_{r_i,n,m,k}\right)\frac{ah(x_{r_\ell,n,m,k})+b}{h'(x_{r_\ell,n,m,k})}. \quad (19)$$

3. In the case of OOS $[m=0,\ k=1,\ \gamma_i=n-i+1$ and $\xi=\frac{n}{n-r_\ell+1}]$, relation (9) reduces to

$$M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell:n}(t_1,\ldots,t_\ell) - M^{(j_1,\ldots,j_l)}_{r_1,\ldots,r^*_\ell:n}(t_1,\ldots,t_\ell) = \frac{nQ_2}{n-r_\ell+1} \times\left\{M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell:n-1}(t_1,\ldots,t_\ell) - M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r^*_\ell:n-1}(t_1,\ldots,t_\ell)\right\} - \frac{t_\ell j_\ell}{ac(n-r_\ell+1)} E\left\{\Psi_{j_\ell}(X_{r_\ell:n};t_\ell)\exp\left(\sum_{i=1}^{\ell-1} t_i X^{j_i}_{r_i:n}\right)\right\}, \quad (20)$$

and relation (17) reduces to

$$\mu^{(j_1,\dots,j_\ell)}_{r_1,\dots,r_\ell:n} - \mu^{(j_1,\dots,j_\ell)}_{r_1,\dots,r^*_\ell:n} = -\frac{j_\ell}{ac(n-r_\ell+1)} E\{\Psi_{j_\ell,\dots,j_1}(X_{r_\ell:n},\dots,x_{r_1:n})\} + \frac{nQ_2}{n-r_\ell+1}\Phi^{(j_1,\dots,j_\ell)}_{r_1,\dots,r_\ell,n-1}, \tag{21}$$

where

$$X_{r_1,\dots,r_\ell;n,0,1} \equiv X_{r_1,\dots,r_\ell:n}.$$

(The recurrence relation (21) agrees with AL-Hussaini et al. (2004) when $h(x) = e^{-\lambda(x)}$, $a = 1$, $b = 0$ and $c = 1$).

4. In the case of krv [$m = -1$, $k \geqslant 1$, $\gamma_i = k$ and $(1-k^{-1})^{1-r_\ell}$], relation (9) reduces to

$$\begin{aligned} M^{(j_1,\dots,j_\ell)}_{(r_1,\dots,r_\ell;k)}(t_1,\dots,t_\ell) - M^{(j_1,\dots,j_\ell)}_{(r_1,\dots,r^*_\ell;k)}(t_1,\dots,t_\ell) &= Q_2(1-k^{-1})^{1-r_\ell} \\ &\times \left\{ M^{(j_1,\dots,j_\ell)}_{(r_1,\dots,r_\ell;k-1)}(t_1,\dots,t_\ell) - M^{(j_1,\dots,j_\ell)}_{(r_1,\dots,r^*_\ell;k-1)}(t_1,\dots,t_\ell) \right\} \\ &- \frac{t_\ell j_\ell}{ack} E\left\{ \Psi_{j_\ell}(X_{(r_\ell,k)};t_\ell) \exp\left(\sum_{i=1}^{\ell-1} t_i X^{j_i}_{(r_i,k)} \right) \right\}, \end{aligned} \tag{22}$$

and relation (17) reduces to

$$\mu^{(j_1,\dots,j_\ell)}_{(r_1,\dots,r_\ell,k)} - \mu^{(j_1,\dots,j_\ell)}_{(r_1,\dots,r^*_\ell;k)} = -\frac{j_\ell}{ack} E[\Psi_{j_\ell,\dots,j_1}\{X_{(r_\ell,k)},\dots,X_{(r_1,k)}\}] + Q_2(1-k^{-1})^{1-r_\ell}\Phi^{(j_1,\dots,j_\ell)}_{(r_1,\dots,r_\ell,k-1)}, \tag{23}$$

where

$$X_{r_1,\dots,r_\ell,n,-1,k} \equiv X_{(r_1,\dots,r_\ell,k)}.$$

The recurrence relations for RVs was obtain if we put $k = 1$ in (23).

5. If we put $j_1 = \cdots = j_{\ell-1} = 0$, $j_\ell = j$, $r_\ell = r$, $t_1 = \cdots = t_{\ell-1} = 0$ and $t_\ell = t$, in (9) and (15) , respectively, we have

$$M^{(j)}_{r,n,m,k}(t) - M^{(j)}_{r-1,n,m,k}(t) = \xi Q_2 \left\{ M^{(j)}_{r,n-1,m,k+m}(t) - M^{(j)}_{r-1,n-1,m,k+m}(t) \right\} - \frac{tj}{ac\gamma_r} E\{\Psi_j(X_{r,n,m,k};t)\}, \tag{24}$$

and

$$\mu^{(j)}_{r,n,m,k} - \mu^{(j)}_{r-1,n,m,k} = -\frac{j}{ac\gamma_r} E\{\Psi_j(X_{r,n,m,k})\} + \xi Q_2 \Phi^{(j)}_{r,n-1,m,k+m}. \tag{25}$$

(Recurrence relation (25) agrees with (2.5) of Mahmoud and Al-Nagar (2006) if $\phi(x) = x^j$)
and when $h(x) = \exp\{-\lambda(x)\}$, $a = 1$, $b = 0$ and $c = 1$, this result agrees with one of Ahmad and Fawzy (2003) results.

6. If we put $j_1 = \cdots = j_{\ell-2} = 0$, $j_{\ell-1} = e$, $j_\ell = \varepsilon$, $r_{\ell-1} = r$, $r_\ell = s$ $t_1 = \cdots = t_{\ell-2} = 0$ $t_{\ell-1} = t_1$ and $t_\ell = t_2$, in (9) and (17), we obtain

$$\begin{aligned} M^{(e,\varepsilon)}_{r,s,n,m,k}(t_1,t_2) - M^{(e,\varepsilon)}_{r,s-1,n,m,k}(t_1,t_2) = \xi Q_2 \Big\{ & M^{(e,\varepsilon)}_{r,s,n-1,m,k+m}(t_1,t_2) \\ & - M^{(e,\varepsilon)}_{r,s-1,n-1,m,k+m}(t_1,t_2) \Big\} - \frac{t_2 \varepsilon}{ac\gamma_s} \\ & \times E\{\Psi_\varepsilon(X_{s,n,m,k}; t_2) \exp(t_1 X^e_{r,n,m,k})\}. \end{aligned} \tag{26}$$

$$\begin{aligned} \mu^{(e,\varepsilon)}_{r,s,n,m,k} - \mu^{(e,\varepsilon)}_{r,s-1,n,m,k} = & -\frac{\varepsilon}{ac\gamma_s} E\{\Psi_{e,\varepsilon}(X_{s,n,m,k}, X_{r,n,m,k})\} \\ & + \xi Q_2 \Phi^{(e,\varepsilon)}_{r,s,n-1,m,k+m}. \end{aligned} \tag{27}$$

Recurrence relation (27) agrees with (3.4) of Mahmoud and Al-Nagar (2006) if $\phi(x_r x_s) = x_r^e x_s^\varepsilon$.

7. In the case OOS, relations (25) and (27), reduce to

$$\mu^{(j)}_{r:n} - \mu^{(j)}_{r-1:n} = -\frac{j}{ac(n-r+1)} E\{\Psi_j(X_{r:n})\} + \frac{nQ_2}{(n-r+1)} \Phi^{(j)}_{r:n-1}, \tag{28}$$

and

$$\begin{aligned} \mu^{(e,\varepsilon)}_{r,s:n} - \mu^{(e,\varepsilon)}_{r,s-1:n} = & -\frac{\varepsilon}{ac(n-s+1)} E\{\Psi_{e,\varepsilon}(X_{s:n}, X_{r:n})\} \\ & + \frac{nQ_2}{n-s+1} \Phi^{(e,\varepsilon)}_{r,s:n-1}. \end{aligned} \tag{29}$$

8. In the case of k-RVs (25) and (27), reduce to

$$\mu^{(j)}_{(r,k)} - \mu^{(j)}_{(r-1,k)} = -\frac{j}{ack} E\{\Psi_j(X_{(r,k)})\} + Q_2(1-k^{-1})^{1-r}\Phi^{(j)}_{(r,k-1)}, \quad (30)$$

and

$$\mu^{(e,\varepsilon)}_{(r,s,k)} - \mu^{(e,\varepsilon)}_{(r,s-1,k)} = -\frac{\varepsilon}{ack} E\{\Psi_{e,\varepsilon}(X_{(s,k)}, X_{(r,k)})\} + Q_2(1-k^{-1})^{1-r}\Phi^{(e,\varepsilon)}_{(r,s,k-1)}. \quad (31)$$

2.2 Special Cases

1) The left-truncated case can be obtained when $Q_2 = 0$, so relations (9) and (17), reduce to

$$M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k}(t_1,\ldots,t_\ell) - M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r^*_\ell,n,m,k}(t_1,\ldots,t_\ell) = -\frac{t_\ell j_\ell}{ac\gamma_{r_\ell}} E\left\{\Psi_{j_\ell}(x_{r_\ell,n,m,k};t_\ell)\exp\left(\sum_{i=1}^{\ell-1} t_i X^{j_i}_{r_i,n,m,k}\right)\right\}, \quad (32)$$

and

$$\mu^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k} - \mu^{(j_1,\ldots,j_\ell)}_{r_\ell,\ldots,r^*_\ell,n,m,k} = -\frac{j_\ell}{ac\gamma_{r_\ell}} E\{\Psi_{j_\ell,\ldots,j_1}(X_{r_\ell,n,m,k},\ldots, X_{r_1,n,m,k})\}. \quad (33)$$

2) The right-truncated case can be obtained $Q_2 = \frac{Q-1}{Q}$, so relations (9) and (17), reduce to

$$M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k}(t_1,\ldots,t_\ell) - M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r^*_\ell,n,m,k}(t_1,\ldots,t_\ell) = \xi\frac{Q-1}{Q} \left\{M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n-1,m,k+m}(t_1,\ldots,t_\ell) - M^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r^*_\ell,n-1,m,k+m}(t_1,\ldots,t_\ell)\right\} - \frac{t_\ell j_\ell}{ac\gamma_{r_\ell}} E\left\{\Psi_{j_\ell}(X_{r_\ell,n,m,k};t_\ell)\exp\left(\sum_{i=1}^{\ell-1} t_i X^{j_i}_{r_i,n,m,k}\right)\right\}, \quad (34)$$

and

$$\mu^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n,m,k} - \mu^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r^*_\ell,n,m,k} = -\frac{j_\ell}{ac\gamma_{r_\ell}} E\{\Psi_{j_\ell,\ldots,j_1}(X_{r_\ell,n,m,k},\ldots, X_{r_1,n,m,k})\} + \xi\frac{Q-1}{Q}\Phi^{(j_1,\ldots,j_\ell)}_{r_1,\ldots,r_\ell,n-1,m,k+m}. \quad (35)$$

3 Applications

In this section, two applications of the previous results are introduced, the first one is the characterizations of members of the class based on recurrence relations for moments of GOS, OOS and k-RVs for doubly truncated modified Weibull (Weibull, Extreme-value, exponential), doubly truncated inverse Weibull distributions and doubly truncated three-parameters Burr type XII (Lomax and Rayleigh). In the second application we will find the Tables of single and product moments of OOS arising from doubly truncated Lomax distribution.

1. Doubly truncated new modified Weibull distribution

$a = 1, b = 0, c = 1$ and $h(x) = \exp\{-\theta x^p \exp(\lambda x)\}$, then the recurrence relations (17), (25) and (27), reduce respectively, to

$$\mu^{(j_1,\dots,j_\ell)}_{r_1,\dots,r_\ell,n,m,k} = \mu^{(j_1,\dots,j_\ell)}_{r_1,\dots,r^*_\ell,n,m,k} + \frac{j_\ell}{\gamma_{r_\ell}\theta} E\left\{ \frac{X^{j_\ell - p}_{r_1,\dots,r_\ell,n,m,k} \exp(-\lambda X_{r_1,\dots,r_\ell,n,m,k})}{p + \lambda X_{r_1,\dots,r_l,n,m,k}} \prod_{i=1}^{\ell-1} X^{j_i}_{r_i,n,m,k} \right\} + \xi Q_2 \Phi_{r_1,\dots,r_\ell,n-1,m,k+m}, \tag{36}$$

$$\mu^{(j)}_{r,n,m,k} = \mu^{(j)}_{r-1,n,m,k} + \frac{j}{\gamma_r\theta} E\left\{ \frac{X^{j-p}_{r,n,m,k} \exp(-\lambda X_{r,n,m,k})}{p + \lambda X_{r,n,m,k}} \right\} + \xi Q_2 \Phi_{r,n-1,m,k+m}, \tag{37}$$

and

$$\mu^{(e,\varepsilon)}_{r,s,n,m,k} = \mu^{(e,\varepsilon)}_{r,s-1,n,m,k} + \frac{\varepsilon}{\gamma_s\theta} E\left\{ \frac{X^{e}_{r,n,m,k} X^{\varepsilon-p}_{s,n,m,k} \exp(-\lambda X_{s,n,m,k})}{p + \lambda X_{s,n,m,k}} \right\} + \xi Q_2 \Phi_{r,s,n-1,m,k+m}. \tag{38}$$

The recurrence relations (21), (28) and (29), reduce to

$$\theta(n - r_\ell + 1)\mu^{(j_1,\dots,j_\ell)}_{r_1,\dots,r_\ell:n} = \theta(n - r_\ell + 1)\mu^{(j_1,\dots,j_\ell)}_{r_1,\dots,r^*_\ell:n} + j_\ell E\left\{ \frac{X^{j_\ell-p}_{r_1,\dots,r_\ell:n} \exp(-\lambda X_{r_1,\dots,r_\ell:n})}{p + \lambda X_{r_1,\dots,r_\ell:n}} \prod_{i=1}^{\ell-1} X^{j_i}_{r_i:n} \right\} + \frac{n}{(n - r_\ell + 1)} Q_2 \Phi_{r_1,\dots,r_\ell,n-1}, \tag{39}$$

$$\mu_{r:n}^{(j)} = \mu_{r-1:n}^{(j)} + \frac{j}{\theta(n-r+1)} E \left\} \frac{X_{r:n}^{j-p} \exp(-\lambda X_{r:n})}{p+\lambda X_{r:n}} \right\} + \frac{n\theta}{\theta(n-r+1)} Q_2 \Phi_{r:n-1}, \tag{40}$$

and

$$\mu_{r,s:n}^{(e,\varepsilon)} = \mu_{r,s-1:n}^{(e,\varepsilon)} + \frac{\varepsilon}{\theta(n-s+1)} E \left\{ \frac{X_{r:n}^{e} x_{s:n}^{\varepsilon-p} \exp(-\lambda x_{s:n})}{p+\lambda x_{s:n}} \right\} + \frac{n\theta}{\theta(n-s+1)} Q_2 \Phi_{r,s:n-1}. \tag{41}$$

The recurrence relations (23), (30) and (31), reduce to

$$\mu_{(r_1,\ldots,r_l,k)}^{(j_1,\ldots,j_\ell)} = \mu_{(r_1,\ldots,r_\ell^*;k)}^{(j_1,\ldots,j_\ell)} + \frac{j_\ell}{\theta k} E \left[\frac{x_{(r_1,\ldots,r_\ell;k)}^{j_\ell-p} \exp\{-\lambda X_{(r_1,\ldots,r_\ell;k)}\}}{(p+\lambda X_{(r_1,\ldots,r_\ell;k)})} \prod_{i=1}^{\ell-1} X_{(r_i;k)}^{j_i} \right] + Q_2(1-k^{-1})^{1-r_\ell} \Phi_{r_1,\ldots,r_\ell,k-1}, \tag{42}$$

$$\mu_{(r,k)}^{(j)} = \mu_{(r-1,k)}^{(j)} + \frac{j}{\theta k} E \left[\frac{X_{(r;k)}^{j-p} \exp\{-\lambda X_{(r;k)}\}}{p+\lambda X_{(r;k)}} \right] + Q_2(1-k^{-1})^{1-r} \Phi_{r,k-1}, \tag{43}$$

and

$$\theta k \mu_{(r,s,k)}^{(e,\varepsilon)} = \theta k \mu_{(r,s-1,k)}^{(e,\varepsilon)} + \varepsilon E \left[\frac{X_{(r,s;k)}^{e} X_{(r,s;k)}^{\varepsilon-p} \exp\{-\lambda X_{(r,s;k)}\}}{p+\lambda X_{(r,s;k)}} \right] + \theta k Q_2 (1-k^{-1})^{1-r} \Phi_{r,s,k-1}. \tag{44}$$

I) If we put $\lambda = 0$, we obtain the recurrence relations for Weibull distribution.

II) If we put $p = 0$, we obtain the recurrence relations for Extreme-value distribution.

2. Doubly truncated inverse Weibull distribution
$a = -1, b = 1, c = 1$ and $h(x) = \exp[-\theta x^{-p}]$, then the recurrence relations (17), (25) and (27), reduce respectively, to

$$\mu_{r_1,\ldots,r_\ell,n,m,k}^{(j_1,\ldots,j_\ell)} - \mu_{r_1,\ldots,r_\ell^*,n,m,k}^{(j_1,\ldots,j_\ell)} = \frac{j_\ell}{\gamma_{r_\ell} \theta p} E \Bigg[X_{r_1,\ldots,r_\ell,n,m,k}^{j_\ell+p} \{\exp(\theta X_{r_1,\ldots,r_\ell,n,m,k}^{-p}) - 1\}$$

$$\left.\prod_{i=1}^{\ell-1} X_{r_i,n,m,k}^{j_i}\right] + \xi Q_2 \Phi_{r_1,\ldots,r_\ell,n-1,m,k+m}, \tag{45}$$

$$\begin{aligned}\mu_{r,n,m,k}^{(j)} - \mu_{r-1,n,m,k}^{(j)} = \frac{j}{\gamma_r \theta p} E\left[X_{r,n,m,k}^{j+p}\{\exp(\theta X_{r,n,m,k}^{-p}) - 1\}\right] \\ + \xi Q_2 \Phi_{r,n-1,m,k+m},\end{aligned} \tag{46}$$

and

$$\begin{aligned}\mu_{r,s,n,m,k}^{(e,\varepsilon)} - \mu_{r,s-1,n,m,k}^{(e,\varepsilon)} = \frac{\varepsilon}{\gamma_s \theta p} E\left[X_{r,n,m,k}^{e} X_{s,n,m,k}^{\varepsilon+p}\{\exp(\theta X_{s,n,m,k}^{-p}) - 1\}\right] \\ + \xi Q_2 \Phi_{r,s,n-1,m,k+m}.\end{aligned} \tag{47}$$

The recurrence relations (21), (28) and (29), reduce to

$$\begin{aligned}\mu_{r_1,\ldots,r_\ell:n}^{(j_1,\ldots,j_\ell)} - \mu_{r_1,\ldots,r_\ell^*:n}^{(j_1,\ldots,j_\ell)} = \frac{j_\ell}{\theta p(n - r_\ell + 1)} E\Bigg[X_{r_1,\ldots,r_\ell:n}^{j_\ell+p}\{\exp(\theta X_{r_1,\ldots,r_\ell:n}^{-p}) - 1\} \\ \left.\prod_{i=1}^{\ell-1} X_{r_i:n}^{j_i}\right] + \frac{n}{n - r_\ell + 1} Q_2 \Phi_{r_1,\ldots,r_\ell,n-1},\end{aligned} \tag{48}$$

$$\begin{aligned}\mu_{r:n}^{(j)} - \mu_{r-1:n}^{(j)} = \frac{j}{\theta p(n - r + 1)} E\left[X_{r:n}^{j+p}\{\exp(\theta X_{r:n}^{-p}) - 1\}\right] \\ + \frac{n}{(n - r + 1)} Q_2 \Phi_{r:n-1},\end{aligned} \tag{49}$$

and

$$\begin{aligned}\mu_{r,s:n}^{(e,\varepsilon)} = \mu_{r,s-1:n}^{(e,\varepsilon)} + \frac{\varepsilon}{\theta p(n - s + 1)} E\left[X_{r:n}^{e} x_{s:n}^{\varepsilon+p}\{\exp(X_{s:n}^{-p}) - 1\}\right] \\ + \frac{n}{n - s + 1} Q_2 \Phi_{r,s:n-1}.\end{aligned} \tag{50}$$

The recurrence relations (23), (30) and (31), reduce to

$$\begin{aligned}\mu_{(r_1,\ldots,r_l,k)}^{(j_1,\ldots,j_\ell)} = \mu_{(r_1,\ldots,r_\ell^*;k)}^{(j_1,\ldots,j_\ell)} + \frac{j_\ell}{\theta p k} E\left[x_{(r_1,\ldots,r_\ell;k)}^{j_\ell+p}[\exp\{\theta X_{(r_1,\ldots,r_\ell;k)}^{-p}\} - 1] \prod_{i=1}^{\ell-1} X_{(r_i;k)}^{j_i}\right] \\ + Q_2(1 - k^{-1})^{1-r_\ell} \Phi_{r_1,\ldots,r_\ell,k-1},\end{aligned} \tag{51}$$

$$\mu_{(r,k)}^{(j)} = \mu_{(r-1,k)}^{(j)} + \frac{j}{\theta pk} E\left[X_{(r;k)}^{j+p}[\exp\{\theta X_{(r;k)}^{-p}\} - 1]\right] + Q_2(1-k^{-1})^{1-r}\Phi_{r,k-1}, \tag{52}$$

and

$$\mu_{(r,s,k)}^{(e,\varepsilon)} = \mu_{(r,s-1,k)}^{(e,\varepsilon)} + \frac{\varepsilon}{\theta pk} E\left[X_{(r,s;k)}^{e} X_{(r,s;k)}^{\varepsilon+p}[\exp\{\theta X_{(r,s;k)}^{-p}\} - 1]\right] + Q_2(1-k^{-1})^{1-r}\Phi_{r,s,k-1}. \tag{53}$$

3. Doubly truncated Burr Type XII distribution.
$a = \frac{1}{\sigma}, b = 1, c = -\theta$ and $h(x) = x^{\eta}$, then the recurrence relations (17), (25) and (27), reduce respectively, to

$$(\eta\gamma_{r_\ell}\theta - j_\ell)\mu_{r_1,\ldots,r_\ell,n,m,k}^{(j_1,\ldots,j_\ell)} - \eta\gamma_{r_\ell}\theta\mu_{r_1,\ldots,r_\ell^*,n,m,k}^{(j_1,\ldots,j_\ell)} = \sigma j_\ell \mu_{r_1,\ldots,r_\ell,n,m,k}^{(j_1,\ldots,j_\ell-\eta)} + \eta\theta\gamma_{r_\ell}\xi Q_2 \Phi_{r_1,\ldots,r_\ell,n-1,m,k+m}^{(j_1,\ldots,j_\ell)}, \tag{54}$$

$$(\eta\gamma_r\theta - j)\mu_{r,n,m,k}^{(j)} - \eta\gamma_r\theta\mu_{r-1,n,m,k}^{(j)} = \sigma j \mu_{r,n,m,k}^{(j-\eta)} + \eta\theta\xi\gamma_r Q_2 \Phi_{r,n-1,m,k+m}^{(j)} \tag{55}$$

and

$$(\eta\theta\gamma_s - \varepsilon)\mu_{r,s,n,m,k}^{(e,\varepsilon)} = \eta\gamma_s\theta\mu_{r,s-1,n,m,k}^{(e,\varepsilon)} + \sigma\varepsilon\mu_{r,s,n,m,k}^{(e,\varepsilon-\eta)} + \eta\theta\gamma_s\xi Q_2 \Phi_{r,s,n-1,m,k+m}^{(e,\varepsilon)}. \tag{56}$$

The recurrence relations (21), (28) and (29), reduce respectively, to

$$\{\eta\theta(n - r_\ell + 1) - j_\ell\}\mu_{r_1,\ldots,r_\ell:n}^{(j_1,\ldots,j_\ell)} - \eta\theta(n - r_\ell + 1)\mu_{r_1,\ldots,r_\ell^*:n}^{(j_1,\ldots,j_\ell)} = \sigma j_\ell \mu_{r_1,\ldots,r_\ell:n}^{(j_1,\ldots,j_\ell-\eta)} + \eta\theta n Q_2 \Phi_{r_1,\ldots,r_\ell:n-1}^{(j_1,\ldots,j_\ell)}, \tag{57}$$

$$\{\eta\theta(n-r+1) - j\}\mu_{r:n}^{(j)} - \eta\theta(n-r+1)\mu_{r-1:n}^{(j)} = \sigma j \mu_{r:n}^{(j-\eta)} + \eta\theta n Q_2 \Phi_{r:n-1}^{(j)} \tag{58}$$

and

$$\{\eta\theta(n - s + 1) - \varepsilon\})\mu_{r,s:n}^{(e,\varepsilon)} - \eta\theta(n - s + 1)\mu_{r,s-1:n}^{(e,\varepsilon)} = \sigma\varepsilon\mu_{r,s:n}^{(e,\varepsilon-\eta)} + \eta\theta n Q_2 \Phi_{r,s:n-1}^{(e,\varepsilon)} \tag{59}$$

The recurrence relations (23), (30) and (31), reduce respectively, to

$$(\eta\theta k - j_\ell)\mu^{(j_1,\ldots,j_\ell)}_{(r_1,\ldots,r_\ell,k)} - \eta\theta k\mu^{(j_1,\ldots,j_\ell)}_{(r_1,\ldots,r^*_\ell;k)} = \sigma j_\ell \mu^{(j_1,\ldots,j_\ell-\eta)}_{(r_1,\ldots,r_\ell,k)} + \eta\theta k Q_2(1-k^{-1})^{1-r_\ell} \times \Phi^{(j_1,\ldots,j_\ell)}_{(r_1,\ldots,r_\ell,k-1)}, \tag{60}$$

$$(\eta\theta k - j)\mu^{(j)}_{(r,k)} - \eta\theta k\mu^{(j)}_{(r-1,k)} = \sigma j\mu^{(j-\eta)}_{(r,k)} + \eta\theta k Q_2(1-k^{-1})^{1-r}\Phi^{(j)}_{(r,k-1)} \tag{61}$$

and

$$(\eta\theta k - \varepsilon)\mu^{(e,\varepsilon)}_{(r,s,k)} - \eta\theta k\mu^{(e,\varepsilon)}_{(r,s-1,k)} = \sigma\varepsilon\mu^{(e,\varepsilon-\eta)}_{(r,s,k)} + \eta\theta k Q_2(1-k^{-1})^{1-r} \times \Phi^{(e,\varepsilon)}_{(r,s,k-1)}. \tag{62}$$

When $\eta = 1$ in (54), (55), (56), (57), (58), (59), (60), (61) and (62), the recurrence relations for the Lomax distribution are obtained.

3.1 Another Application

In this application we compute numerically the single moments of OOS from doubly truncated Lomax distribution.

The recurrence relations of single and product moments of OOS from doubly truncated Lomax distribution can be obtained as

$$[\theta(n-r+1)-1]\mu_{r:n} = \theta(n-r+1)\mu_{r-1:n} + \theta n Q_2[\mu_{r:n-1} - \mu_{r-1:n-1}] + \sigma \tag{63}$$

$$[\theta(n-s+1)-1]\mu_{r,s:n} = \theta(n-s+1)\mu_{r,s-1:n} + \theta n Q_2[\mu_{r,s:n-1} - \mu_{r,s-1:n-1}] + \sigma\mu_{r:n}. \tag{64}$$

Suppose that : $P = 0.25$, $Q = 0.75$ and then $Q_2 = (Q-1)/(Q-P) = -0.5$. We consider the following three cases of the parameter values:

i) If $\theta = 3$ and $\sigma = 1$, then the *cdf* of Lomax distribution is $F(x) = 1 - (1+x)^{-3}$, so we can obtain $P_1$and Q_1 from $P = F(P_1)$ and $Q = F(Q_1)$ it is:

$$P_1 = 0.1006 \quad \text{and} \quad Q_1 = 0.5874.$$

ii) If $\theta = 3$ and $\sigma = 2$, then the *cdf* of Lomax distribution is $F(x) = 1 - (1 + \frac{x}{2})^{-3}$, so we can find P_1 and Q_1 as:

$$P_1 = 0.2012 \quad \text{and} \quad Q_1 = 1.1748.$$

iii) If $\theta = 3$ and $\sigma = 3$, then the *cdf* of Lomax distribution is $F(x) = 1 - (1 + \frac{x}{3})^{-3}$, so we can obtain P_1 and Q_1 as:

$$P_1 = 0.301927 \quad \text{and} \quad Q_1 = 1.7622.$$

and we will use the relations.

$$E(X^j_{0:n}) = P^j_1, \quad n \geqslant 0 \quad \text{and} \quad E(X^j_{n:n-1}) = Q^j_1, \quad n \geqslant 1.$$

(see Khan and Khan, 1987).

Table 1. The single moments of OOS from doubly truncated Lomax distribution.

r	n	$\mu_{r:n}$		
		$\theta = 3,\ \sigma = 1$	$\theta = 3,\ \sigma = 2$	$\theta = 3\ ,\sigma = 3$
1	1	0.2859	0.5718	0.8576
1	2	0.2096	0.4192	0.6288
2	2	0.3622	0.7243	1.0865
1	3	0.1769	0.3538	0.5307
2	3	0.2750	0.5500	0.8250
3	3	0.4057	0.8115	1.2172
1	4	0.1591	0.3182	0.4772
2	4	0.2304	0.4608	0.6912
3	4	0.3196	0.6392	0.9588
4	4	0.4344	0.8689	1.3033
1	5	0.1479	0.2959	0.4438
2	5	0.2037	0.4073	0.6110
3	5	0.2705	0.5410	0.8115
4	5	0.3524	0.7047	1.0570
5	5	0.4549	0.9099	1.3649

Table 2. The product moments of OOS from doubly truncated Lomax distribution.

r	s	n	$\mu_{r,s:n}$		
			$\theta = 3,\ \sigma = 1$	$\theta = 3,\ \sigma = 2$	$\theta = 3,\ \sigma = 3$
1	2	2	0.0817	0.3269	0.7356
1	2	3	0.0527	0.2108	0.4743
1	3	3	0.0743	0.2975	0.6694
2	3	3	0.1180	0.4724	1.0631
1	2	4	0.0393	0.1575	0.3544
1	3	4	0.0530	0.2122	0.4775
1	4	4	0.0705	0.2820	0.6345
2	3	4	0.0790	0.3159	0.7109
2	4	4	0.1034	0.4137	0.9309
3	4	4	0.1449	0.5801	1.3052

Note: It should be noted from Tables 1 and 2 that

i) The values of different moments are decreasing with increasing the sample size n .

ii) The following relation

$$\sum_{i=1}^{n} \mu_{i:n} = nE(X) = n\mu_{1:1} \tag{65}$$

is satisfied for all values of single moments in Table 1.

iii) The following relation

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mu_{i,j:n} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{i:n} X_{j:n}) = \frac{n(n-1)}{2} \mu_{1:1}^{2} \tag{66}$$

is holds for all values of product moments in Table 2.

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