© Statistical Research and Training Center

مجلّه‌ی پژوهش‌های آماری ایران

J. Statist. Res. Iran **11** (2014): 147–176

دوره‌ی ۱۱، شماره‌ی ۲، پاییز و زمستان ۱۳۹۳، صص ۱۴۷–۱۷۶

پژوهشکده‌ی آمار

Tracking Interval for Doubly Censored Data with Application of Plasma Droplet Spread Samples

H. Panahi† and A. Sayyareh‡,*

† Islamic Azad University, Lahijan Branch

‡ Razi University

Abstract. Doubly censoring scheme, which includes left as well as right censored observations, is frequently observed in practical studies. In this paper we introduce a new interval say tracking interval for comparing the two rival models when the data are doubly censored. We obtain the asymptotic properties of maximum likelihood estimator under doubly censored data and drive a statistic for testing the null hypothesis that the proposed non-nested models are equally close to the true model against the alternative hypothesis that one model is closer when we are faced with an experimental situation. Monte Carlo simulations are performed to observe the behavior of the theoretical results, and the proposed methodology is illustrated with data from spreading of the micro plasma droplets. We also perform the statistical analysis of these data using the probability models including Weibull, Burr type XII, Burr type III and inverse Weibull distributions. One important result of this study is that the Burr type XII distribution, in contrast to inverse Weibull distribution, may describe more closely to Weibull distribution for spread factor data under doubly censored sample.

Keywords. Doubly censoring; Fisher information; Kullback-Leibler risk; model selection; plasma droplets; tracking interval; Vuong's test.

MSC 2010: 62N01, 62N02, 62N03.

* Corresponding author

1 Introduction

The idea of model selection is beginning with a set of data and rival models to choice the best one. The decision making on this set is an important question in statistical inference. Some tests and criteria are designed to answer to this question that which of the rival models is the best one, or at least, which of them are equivalent to select as the best. Since then, several articles have been published on model selection based on complete data, for example, Cox (1961, 1962) modified the classical hypothesis testing to test the non nested hypothesis, Akaike (1973) introduced the Akaike Information Criterion (AIC) to select the best model under parsimony, Vuong (1989) tested the equivalence of two models, In Vuong viewpoint, the best model is the model which maximizes the relevant part of the Kullback-Leibler (KL) risk, see Kullback and Leibler (1951). The null hypothesis of Vuong test is the expectation under the true model of the log-likelihood ratio of the two rival models are equal to zero, which means that, two proposed models are equivalent. This expectation however is unknown. But Vuong test works, because the decision making procedure by Vuong test does not depend on this unknown quantity. Recently, Commanges et al. (2008) have considered the normalized difference of AIC as an estimate of a difference of KL risks between two models. Sayyareh et al. (2011) and Sayyareh (2012a) compare some tests and model selection criteria for parametric and linear models.
However, in many experimental studies, the experimenter may not always obtain complete information on failure times for all experimental units. For example, units may break accidentally in an industrial experiment. There are a lot of situations in which the removal of units prior to failure is pre-planned. Data obtained from such experiments are called *censored data*. Different types of censoring arise based on how the data are collected from the experiment. The scheme of doubly censored sampling is an important method for obtaining data in experiments. Doubly censored data are commonly observed in clinical or biomedical studies, where the first few observations and the last few observations are unavailable from a sequence of observations. Examples of doubly censored data are abundant. For example, Ren and Peer (2000) analyzed doubly censored data in a study of the effectiveness of screening mammograms, Cai and Cheng (2004) studied the HIV data under doubly censoring scheme, Jones and Rocke (2002) considered an effective treatment of genetic disorder and Khan et. al. (2011) analyzed doubly censored data in a study of the laryngeal cancer. While much research has been done on the

doubly censoring scheme, lack attention has been paid to the comparison of two non-nested models under doubly censored plasma droplets data. Thus, we applied this censoring in study of micro plasma droplets spread factor data. Plasma spray coating is a process by which the high temperature of a plasma is employed to melt powders of metallic or non-metallic materials and spray them onto a substrate, forming a dense deposit. The process is commonly used to apply protective coatings on components to shield them from wear, corrosion, and high temperatures. In plasma spraying, spread factor of the micro plasma droplets is a fundamental character in deposit formation. Physical properties and mechanical characteristics of a plasma spray deposit primarily depend on this crucial stage.

Development of model selection, which can predict spreading of micro plasma droplets, can potentially reduce the cost of the development of new plasma coatings considerably. Ideally, fitted distribution functions will allow us to predict coating thickness and properties to meet the requirements of individual applications, without having to do extensive experimentation. The models will also allow us to improve and optimize the design of existing spraying guns. A literature survey carried out by the authors indicated lack of published data on the distribution modeling of micro plasma droplet spreading in coating processes.

The main aim of this paper is twofold. First we focus on the behavior of the two rival models under doubly censored sample. In other words, we want to decide whether or not the two rival models are two equivalent models. For this purpose, we propose the tracking interval which should contain the difference of risks with a given probability. When rival models are non-nested, we introduce a test statistic that converges in distribution to the normal distribution and use it to construct the tracking interval. This interval helps us to evaluate proposed models in comparison with each other. In other words, if the calculated distance includes zero, it can be concluded that based on the predetermined confidence, both proposed models are equivalent. This interval could be useful in a wide variety of applications. For example, Commenges et al. (2008) considered the tracking interval between two models in two applications. The first is a study of the relationship between body-mass index and depression in elderly people. The second is the choice between models of HIV dynamics, where one model makes the distinction between activated CD4+T lymphocytes and the other does not. Panahi and Sayyareh (2014, 2015) used the tracking interval for comparison of two rival models of micro-droplet splashing data under different censoring schemes.

The second aim of this paper is to analyze the spread factor of plasma droplets data under doubly censoring scheme. It is clear that the spread factor is always positive and therefore, it is reasonable to analyze the spread factor data using the probability distribution, which has support only on the positive real axis. Thus, we have considered different two-parameter distributions such as Weibull, inverse Weibull, Burr type XII and Burr type III distributions. One of the important problems in engineering experiments namely the prediction interval for future observation. Therefore, we construct the prediction interval for future observation and the ratio of two future consecutive data, based on the doubly censored sample.
The rest of this paper is organized as follows. In Section 2, as preliminary, we briefly mention about the doubly censored sample, the theory about models and KL divergence. In Section 3, we bring the main results which we need to construct the tracking interval for the difference of the expected KL divergence of two non-nested models under doubly censored sample. Simulation results and real data analysis are provided in Section 4, and finally we conclude the paper in Section 5.

2 Preliminary

2.1 Doubly Censored Sample

The most common type of censoring is right censoring, in which the experimental time is larger than the observed right censoring time. In some cases, however, data are subject to left, as well as, right censoring. When left censoring occurs, the only information available to a statistician is that the experimental time is less than or equal to the observed left censoring time. Data with both right and left censored observations are known as doubly censored data. A doubly censoring scheme is a multiple censoring schemes. The main aim of this paper is to focus on the non nested models under doubly censoring scheme. Suppose that, $Y_1 \leqslant \ldots \leqslant Y_n$ are the ordered sample of $X_1, \ldots, X_n$. Thus, doubly censoring scheme can be described as follows:
Consider the experiment in which n identical units are placed on test simultaneously. The first $r_1 - 1$ experiment times may be left-censored due to negligence or problems at the beginning of the experiment, and the experiment terminates as soon as the r_2th unit failed. Then, the likelihood function for doubly censored sample $y_{r_1}, \ldots, y_{r_2}$ with $r_1 - 1$ observations censored on

the left and $n - r_2$ observations censored on the right is given by

$$l_n^f(\alpha) \propto \{F^\alpha(y_{r_1})\}^{(r_1-1)}\{1 - F^\alpha(y_{r_2})\}^{(n-r_2)} \prod_{i=r_1}^{r_2} f^\alpha(y_i) \qquad (1)$$

where, $F^\alpha(\cdot)$ is the distribution function. Note that the complete sample case $(r_1 - 1 = n - r_2 = 0)$, the right type II censored sample case $(r_1 - 1 = 0, n - r_2 > 0)$ and the left type II censored sample case $(r_1 - 1 > 0, n - r_2 = 0)$ are special cases of doubly censored samples.

2.2 Statistical Models and Kullback-Leibler (KL) Divergence

Consider a sample of independently identically distributed (*i.i.d.*) random variables, $X_1, \ldots, X_n$ having probability density function $h(x) \equiv h$. Let us consider two rival models:

$$F^\alpha = \{f^\alpha(\cdot),\ \alpha \in \mathrm{M} \subseteq R^p\} = (f) \quad \text{and} \quad G^\beta = \left\{g^\beta(\cdot),\ \beta \in \mathrm{B} \subseteq R^q\right\} = (g).$$

Definition 1. (i) (f) and (g) are non overlapping if $(f) \bigcap (g) = \phi$; (ii) (f) is nested in (g) if $(f) \subset (g)$; (iii) (f) is well- specified if there is a value $\alpha_0 \in \mathrm{M}$ such that $f^{\alpha_0}(\cdot) = h$; otherwise it is misspecified.
Let $X_1, \ldots, X_n$ be a random sample from h and $f^\alpha(\cdot)$ as a proposed model, then quasi log-likelihood function is given by

$$L_{n_c}^f(\alpha) = \sum_{i=1}^n \log f^\alpha(x_i),$$

where, $L_{n_c}^f(\alpha)$ is the quasi log-likelihood function for complete sample. Under the following condition, $\hat{\alpha}_n$ is a quasi maximum likelihood estimator (QMLE):

$$L_n^f(\hat{\alpha}_n) = \mathrm{sup}_{\alpha \in M} L_{n_c}^f(\alpha),$$

If the model is well-specified then $\alpha_0 = \alpha_*$, where $\alpha_* = \mathrm{arg}\ \max_{\alpha \in M} E_h\{L_n^f(\alpha)\}$, and refer to as the pseudo-true value of the α. The KL information in favor of h against f^α is defined as

$$\mathrm{KL}(h, f^\alpha) = E_h\left\{\log \frac{h(X)}{f^\alpha(X)}\right\} = \int_{-\infty}^{\infty} h(x) \log \frac{h(x)}{f^\alpha(x)} dx.$$

We have $\mathrm{KL}(h, f^\alpha) \geqslant 0$ and $\mathrm{KL}(h, f^\alpha) = 0$, imply that $h = f^\alpha$, that is $\alpha = \alpha_0$. The KL divergence is often intuitively interpreted as a distance between the two probability measures, but this is not mathematically a distance; in particular, the KL divergence is not symmetric. It may be felt that this is a drawback. But, this feature may also have a deep meaning in some model selection problem when there is no symmetry between the true and the proposed models. $\text{KL}(h, f^{\alpha})$ has a minimum at α_*. If $\alpha_0 \neq \alpha_*$ the model is mis-specified and $\text{KL}(h, f^{\alpha}) > 0$. The KL information of f^{α} against g^{β} is defined as $\text{KL}(f^{\alpha}, g^{\beta}) = E_h\big[\log\{f^{\alpha}(X)/g^{\beta}(X)\}\big]$. We say that (f) is closer to h than (g) if $\text{KL}(h,\ f^{\alpha_*}) < \text{KL}(h,\ g^{\beta_*})$. We cannot estimate $\text{KL}(h, f^{\alpha_*})$ because the entropy of h, $E_h\left\{\log h(X)\right\}$, cannot be correctly estimated. However, we can estimate the difference of risks $\Delta_d(f^{\alpha_*}, g^{\beta_*}) = E\text{KL}(h, f^{\alpha_*}) - E\text{KL}(h, g^{\beta_*})$, a quantitative measure of the difference of misspecification by $[-n^{-1}\{L_n^f(\hat{\alpha}_n) - L_n^g(\hat{\beta}_n)\}]$. This result may not be completely satisfactory in practice if n is not very large because the distribution we will use is $f^{\hat{\alpha}_n}$ rather than f^{α_*}. Thus it is more relevant to consider the risk $E_h\left[\log\{h(X)/f^{\hat{\alpha}_n}(X)\}\right]$ that we call the expected KL risk and that we denote by $E\text{KL}(h, f^{\hat{\alpha}_n})$.

3 Main Results

Suppose that, $Y_1 \leqslant \ldots \leqslant Y_n$ are the ordered sample of $X_1, \ldots, X_n$. Thus, based on (1), the quasi log-likelihood function of the doubly censored sample is:

$$L_n^f(\alpha) \propto \sum_{i=r_1}^{r_2} \log f^{\alpha}(y_i) + (r_1 - 1)\log F^{\alpha}(y_{r_1}) + (n - r_2)\log \overline{F}^{\alpha}(y_{r_2})$$

where, $\overline{F}^{\alpha}(\cdot) = 1 - F^{\alpha}(\cdot)$ is the survival function. Therefore, the differences of the quasi log-likelihood functions of the two rival models can be obtained as:

$$\begin{aligned} L_n^{\frac{f}{g}}(\hat{\alpha}_n, \hat{\beta}_n) &= L_n^f(\hat{\alpha}_n) - L_n^g(\hat{\beta}_n) \\ &\propto \sum_{i=r_1}^{r_2} \log \frac{f^{\hat{\alpha}_n}(y_i)}{g^{\hat{\beta}_n}(y_i)} + (r_1 - 1)\log \frac{F^{\hat{\alpha}_n}(y_{r_1})}{G^{\hat{\beta}_n}(y_{r_1})} \\ &\quad + (n - r_2)\log \frac{\overline{F}^{\hat{\alpha}_n}(y_{r_2})}{\overline{G}^{\hat{\beta}_n}(y_{r_2})}. \end{aligned}$$

Let $Y_1, Y_2, \ldots, Y_n$ are ordered observations of n independent units taken from a true distribution. Under the doubly censoring scheme, since the experiment is truncated at $Y_{r_1}(=\zeta_{1n})$ and $Y_{r_2}(=\zeta_{2n})$, by pdf $h^*(\cdot) \equiv h^*$. It is assumed that $\frac{r_1}{n} \xrightarrow{P} p_1 \in (0,1)$ and $\frac{r_2}{n} \xrightarrow{P} p_2 \in (0,1)$ such that $Y_{r_1}(=\zeta_{1n})$ and $Y_{r_2}(=\zeta_{2n})$ converges in probability to ζ_1 and ζ_2, the p_1th and p_2th percentiles of true distribution respectively. Thus, based on Sayyareh (2012b),

$$\frac{1}{n}\sum_{i=r_1}^{r_2} \log \frac{f^{\hat{\alpha}_n}(y_i)}{g^{\hat{\beta}_n}(y_i)} \xrightarrow{P} (p_2 - p_1) E_{h*} \left\{ \log \frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)} \right\},$$

$$\frac{1}{n}(r_1 - 1) \log \frac{F^{\hat{\alpha}_n}(y_{r_1})}{G^{\hat{\beta}_n}(y_{r_1})} \xrightarrow{P} (p_1) \log \frac{F^{\alpha_*}(\zeta_1)}{G^{\beta_*}(\zeta_1)}$$

and

$$\frac{1}{n}(n - r_2) \log \frac{\overline{F}^{\hat{\alpha}_n}(y_{r_2})}{\overline{G}^{\hat{\beta}_n}(y_{r_2})} \xrightarrow{P} (1 - p_2) \log \frac{\overline{F}^{\alpha_*}(\zeta_2)}{\overline{G}^{\beta_*}(\zeta_2)}.$$

Then the difference quasi log-likelihood function of two rival models is converges in probability as below:

$$\frac{1}{n} L_n^{\frac{f}{g}}\left(\hat{\alpha}_n, \hat{\beta}_n\right) \xrightarrow{P} (p_2 - p_1) E_{h^*} \left\{ \log \frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)} \right\} + p_1 \log \frac{F^{\alpha_*}(\zeta_1)}{G^{\beta_*}(\zeta_1)} + (1 - p_2) \log \frac{\overline{F}^{\alpha_*}(\zeta_2)}{\overline{G}^{\beta_*}(\zeta_2)}$$

where

$$\alpha_* = \arg\max_{\alpha \in \mathrm{M}} \left\{ (p_2 - p_1) E_{h^*}\left[\log f^{\alpha}(Y)\right] + p_1 \log F^{\alpha}(\zeta_1) + (1 - p_2) \log \overline{F}^{\alpha}(\zeta_2) \right\}$$

and

$$\beta_* = \arg\max_{\beta \in \mathrm{B}} \left\{ (p_2 - p_1) E_{h^*}\left[\log g^{\beta}(Y)\right] + p_1 \log G^{\beta}(\zeta_1) + (1 - p_2) \log \overline{G}^{\beta}(\zeta_2) \right\},$$

α_* and β_* are pseudo-true values of α and β, respectively. Also quasi maximum likelihood estimator of α say $\hat{\alpha}_n$, is the solution of

$$\frac{\partial}{\partial \alpha} L_n^f(\alpha) = \sum_{i=r_1}^{r_2} \frac{\partial}{\partial \alpha} \log f^{\alpha}(y_i) + (r_1 - 1) \frac{\partial}{\partial \alpha} \log F^{\alpha}(y_{r_1}) + (n - r_2) \frac{\partial}{\partial \alpha} \log \overline{F}^{\alpha}(y_{r_2}) = 0.$$

The minimum assumptions, $\Re$, for compact neighborhood M of α_* are:
$\Re_1$: The parameter space M is an open interval in R.
$\Re_2$: $\frac{\partial}{\partial\alpha} f^{\alpha}(x)$ is a strictly monotone function on M for each x and also, $\dfrac{\partial}{\partial\alpha} f^{\alpha}(x)$ and $\dfrac{\partial^2}{\partial\alpha^2} f^{\alpha}(x)$ all exist for every α.
$\Re_3$: For all $\alpha \in M$ the partial derivative $\frac{\partial}{\partial\alpha} f^{\alpha}(x)$, is integrable on R, the partial derivative $\frac{\partial}{\partial\alpha} F^{\alpha}(x)$, exists for $x \in \chi$ and satisfies

$$\frac{\partial}{\partial\alpha} F^{\alpha}(x) = \int_{-\infty}^{x} \frac{\partial}{\partial\alpha} f^{\alpha}(u) du.$$

$\Re_4$: For every α, we have $\left|\frac{\partial f^{\alpha}(x)}{\partial\alpha}\right| \leqslant \mathrm{K}_1$ and $\left|\frac{\partial^2 f^{\alpha}(x)}{\partial\alpha^2}\right| \leqslant K_2$ and $\left|\frac{\partial^3 f^{\alpha}(x)}{\partial\alpha^3}\right| \leqslant K_3$, where, $\int K_i d\mu(x) < \infty;\ i = 1, 2, 3$.
$\Re_5$: For every α, $\dfrac{1}{\overline{F}^{\alpha}(x)}$ and $\dfrac{1}{F^{\alpha}(x)}$ are bounded by $\upsilon_1(x)$ and $\upsilon_2(x)$ respectively, where, $E\{\upsilon_1(X)\} \leqslant N_1$ and $E\{\upsilon_2(X)\} \leqslant N_2$; N_1 and N_2 are positive constants.
$\Re_6$: For every α, we have, $\gamma = \int (\frac{\partial}{\partial\alpha} \log f^{\alpha}(x))^2 f^{\alpha}(x) d\mu(x) < \infty$.

Theorem 1. *Assume that $f^{\alpha}(\cdot)$ is well specified model satisfied the suitable conditions ($\Re_1$ -$\Re_6$). Then as $n \to \infty$, the asymptotic distribution of maximum likelihood estimator, $\sqrt{n}(\hat{\alpha}_n - \alpha_0)$ is, $N(0, J_{f_d}^{-1})$, where,*

$$J_{f_d} = \gamma + p_1\nu_1 + (1 - p_2)\nu_2. \tag{2}$$

Proof. The proof of Theorem 1 is provided in the Appendix A. □

Theorem 2. *If the proposed model is misspecified and $f^{\alpha_*} \neq g^{\beta_*}$, then*

$$\sqrt{n}\left\{\frac{1}{n} L_n^{\frac{f}{g}}(\hat{\alpha}_n, \hat{\beta}_n) - (p_2 - p_1)E_{h^*}(\tau_1) - p_1\tau_2 - (1 - p_2)\tau_3\right\} \xrightarrow{D} N\left(0, \omega_{*d}^2\right),$$

where, $\tau_1 = \log \dfrac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}$, $\tau_2 = \log \dfrac{F^{\alpha_*}(\zeta_1)}{G^{\beta_*}(\zeta_1)}$, $\tau_3 = \log \dfrac{\overline{F}^{\alpha_*}(\zeta_2)}{\overline{G}^{\beta_*}(\zeta_2)}$ and

$$\omega_{*d}^2 = Var_h\left\{\log \frac{f^{\alpha_*}(W)}{g^{\beta_*}(W)}\right\} + p_1 Var_{h_1^*}\left\{\log \frac{f^{\alpha_*}(Z)}{g^{\beta_*}(Z)}\right\}$$
$$+ (1 - p_2) Var_{h_2^*}\left\{\log \frac{f^{\alpha_*}(U)}{g^{\beta_*}(U)}\right\},$$

$w = (w_1, \ldots, w_n)$ = the complete data, $z = (z_1, \ldots, z_{r_1 - 1})$ and $u = (u_1, \ldots, u_{n-r_2})$ = the complete data of size $r_1 - 1$ and $n - r_2$, from the right and left truncated population.

Proof. see the Appendix B. □

Hence, we investigate the following statistics:

$$\begin{aligned}
\hat{\omega}_{nd}^2 = & \frac{1}{n}\sum_{i=1}^{n}\left\{\log\frac{f^{\hat{\alpha}_n}(w_i)}{g^{\hat{\beta}_n}(w_i)}\right\}^2 - \left[\frac{1}{n}\sum_{i=1}^{n}\left\{\log\frac{f^{\hat{\alpha}_n}(w_i)}{g^{\hat{\beta}_n}(w_i)}\right\}\right]^2 \\
& + \left(\frac{r_1-1}{n}\right)\left[\frac{1}{r_1-1}\sum_{i=1}^{r_1-1}\left(\log\frac{f^{\hat{\alpha}_n}(z_i)}{g^{\hat{\beta}_n}(z_i)}\right)^2\right. \\
& \left. - \left\{\frac{1}{r_1-1}\sum_{i=1}^{r_1-1}\left(\log\frac{f^{\hat{\alpha}_n}(z_i)}{g^{\hat{\beta}_n}(z_i)}\right)\right\}^2\right] \\
& + (\frac{n-r_2}{n})\left[\frac{1}{n-r_2}\sum_{i=1}^{n-r_2}\left(\log\frac{f^{\hat{\alpha}_n}(u_i)}{g^{\hat{\beta}_n}(u_i)}\right)^2\right. \\
& \left. - \left\{\frac{1}{n-r_2}\sum_{i=1}^{n-r_2}\left(\log\frac{f^{\hat{\alpha}_n}(u_i)}{g^{\hat{\beta}_n}(u_i)}\right)\right\}^2\right]
\end{aligned}$$

Thus, based on Vuong (1989), we consider the hypotheses as

$$H_0 : E_{h*}\left[\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}\right] = 0 \Rightarrow \left[\log\frac{\overline{F}^{\alpha_*}(\zeta_2)}{\overline{G}^{\beta_*}(\zeta_2)} = 0 \text{ and } \log\frac{F^{\alpha_*}(\zeta_1)}{G^{\beta_*}(\zeta_1)} = 0\right]$$

$$H_f : E_{h*}\left[\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}\right] > 0 \Rightarrow \left[\log\frac{\overline{F}^{\alpha_*}(\zeta_2)}{\overline{G}^{\beta_*}(\zeta_2)} > 0 \text{ and } \log\frac{F^{\alpha_*}(\zeta_1)}{G^{\beta_*}(\zeta_1)} > 0\right]$$

$$H_g : E_{h*}\left[\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}\right] < 0 \Rightarrow \left[\log\frac{\overline{F}^{\alpha_*}(\zeta_2)}{\overline{G}^{\beta_*}(\zeta_2)} < 0 \text{ and } \log\frac{F^{\alpha_*}(\zeta_1)}{G^{\beta_*}(\zeta_1)} < 0\right]$$

Thus, from (Vuong; Theorem 5.1),

$$\text{under} \qquad H_0 : \Im = \frac{L_n^{\frac{f}{g}}\left(\hat{\alpha}_n, \hat{\beta}_n\right)}{\sqrt{n}\hat{\omega}_{nd}} \xrightarrow{D} N(0,1)$$

$$\text{under} \qquad H_f : \Im = \frac{L_n^{\frac{f}{g}}\left(\hat{\alpha}_n, \hat{\beta}_n\right)}{\sqrt{n}\hat{\omega}_{nd}} \xrightarrow{a.s} +\infty$$

$$\text{under} \qquad H_g : \Im = \frac{L_n^{\frac{f}{g}}\left(\hat{\alpha}_n, \hat{\beta}_n\right)}{\sqrt{n}\hat{\omega}_{nd}} \xrightarrow{a.s} -\infty.$$

If the value of the statistic $\Im$ is higher than $Z_{1-\alpha}$ then one rejects the null hypothesis that the model are equivalent in favor of F^{α} being better than G^{β}. If $\Im$ is smaller than $-Z_{1-\alpha}$ then one rejects the null hypothesis in favor of G^{β} being better than F^{α}, finally if $|\Im| < Z_{1-\alpha}$ then one cannot discriminate between the two rival models based on the given data. Also, $Z_{1-\alpha}$ is $(1-\alpha)$th quantile of standard normal distribution.

3.1 Tracking Interval for a Difference of KL Divergences

In this section, we propose the tracking interval for

$$\Delta_{nd}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) = E\mathrm{KL}(h, f^{\hat{\alpha}_n}) - E\mathrm{KL}(h, g^{\hat{\beta}_n}),$$

which should contain the difference of risks with a given probability. This is not a usual confidence interval because $\Delta_{nd}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$ changes with n. Although it converges toward $\Delta_d(f^{\alpha_*}, g^{\beta_*})$, we wish to approach $\Delta_{nd}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$ for values of n for which the Akaike correction is not negligible.
We can say that the expected KL risk, $E\mathrm{KL}(h, f^{\hat{\alpha}_n})$, is the sum of the misspecification risk $\mathrm{KL}(h, f^{\alpha_*})$ plus the statistical risk $\frac{1}{2\mathrm{n}}\mathrm{Tr}(I_f J_f^{-1})$ as (Linhart and Zucchini, 1986):

$$E\mathrm{KL}\ (h, f^{\hat{\alpha}_n}) = \mathrm{KL}\ (h, f^{\alpha_*}) + \frac{1}{2\mathrm{n}}\ \mathrm{Tr}\ (I_{f_d} J_{f_d}^{-1}) + o(n^{-1}), \tag{3}$$

where, $J_{f_d} = -E_{h^*}\left(\frac{\partial^2 \log f^{\alpha}(Y)}{\partial\alpha\, \partial\alpha'}\right)\Big|_{\alpha_*}$ and $I_{f_d} = E_{h^*}\left(\frac{\partial \log f^{\alpha}(Y)}{\partial\alpha}.\frac{\partial \log f^{\alpha}(Y)}{\partial\alpha'}\right)\Big|_{\alpha_*}$.
Note that if (f) is well specified, we have $\mathrm{KL}(h, f^{\alpha_*}) = 0$ and $E\mathrm{KL}\ (h, f^{\hat{\alpha}_n}) =$

$\frac{p}{2n} + o(n^{-1})$. Also based on Sayyareh (2012b), we have

$$E\text{KL}(h, f^{\hat{\alpha}_n}) = -E_h\left\{\frac{1}{n}L_n^f(\hat{\alpha}_n)\right\} + F(h) + \frac{1}{n}\text{Tr}(I_{f_d}J_{f_d}^{-1}) + o_p(n^{-1}). \quad (4)$$

Here we have essentially estimated $E_h\{\log\ f^{\alpha_*}(X)\}$ by $E_h\{\frac{1}{n}L_n^f(\hat{\alpha}_n)\}$, but because of the overestimation bias, the factor $1/2$ in the last term disappears. Akaike criterion follows from (4) by multiplying by $2n$, deleting the constant term, $F(h)$, which we cannot estimate that, and replacing the expected value of the normalized version of maximized likelihood function by its empirical version. Thus, we can estimate the difference of risks $\Delta_{nd}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})$ as:

$$\Delta_{nd}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) = E_h\left[-\frac{1}{n}\left\{L_n^{f/g}(\hat{\alpha}_n, \hat{\beta}_n) - Tr(I_{f_d}J_{f_d}^{-1}) + Tr(I_{g_d}J_{g_d}^{-1})\right\}\right]$$

Now, using the Akaike approximation, $\text{Tr}(I_{f_d}J_{f_d}^{-1}) \approx p$, the simple estimator of $\Delta_{nd}\left(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}\right)$ is

$$D_{nd}\left(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}\right) = -n^{-1}\left\{L_n^{\frac{f}{g}}(\hat{\alpha}_n, \hat{\beta}_n) - (p-q)\right\}$$

where, p and q are the number of parameters in two rival models.
Now, we emphasis on the case where $f^{\alpha_*} \neq g^{\beta_*}$. Thus using Theorem 2, we have

$$n^{\frac{1}{2}}\left\{D_{nd}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) - \Delta_{nd}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n})\right\} \xrightarrow{D} N(0, \omega_{*d}^2).$$

From this, the tracking interval for $\Delta_{nd}\left(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}\right)$ is given by

$$\left[D_{nd}\left(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}\right) - n^{-\frac{1}{2}}z_{\frac{\alpha}{2}}\hat{\omega}_{nd}, D_{nd}\left(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}\right) + n^{-\frac{1}{2}}z_{\frac{\alpha}{2}}\hat{\omega}_{nd}\right]. \quad (5)$$

This interval has the property as

$$P_h\left[\ \text{A}_\text{n} < \Delta_{nd}\left(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}\right) < \text{B}_\text{n}\right] \to 1-\alpha$$

where, $\text{A}_n = D_{nd}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) - n^{-\frac{1}{2}}z_{\frac{\alpha}{2}}\hat{\omega}_{nd}$; $\text{B}_n = D_{nd}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) + n^{-\frac{1}{2}}z_{\frac{\alpha}{2}}\hat{\omega}_{nd}$ and P_h represents the probability with density h.

4 Simulations and Real Data Analysis

4.1 Simulations

In this section we present some numerical experiments, mainly to observe how the two rival models behave for different sample sizes, different parameter values of true distribution and for different censoring schemes. For this purpose, we consider *i.i.d.* sample of size n of Weibull density $(f^{(\alpha,\beta)}_{Weibull} = \alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}; x > 0)$ say h. Because of the application of Burr distribution (Burr, 1942) in the study of biological, engineering, industrial, reliability and life testing, and several industrial and economic experiments (see for example, Panahi and Sayyareh, 2013; Rastogi and Tripathi, 2012; and Raqab and Kundu, 2005), the rival models are considered as $f^{(p,b)}_{BurrXII} = pbx^{b-1}\left(1+x^{b}\right)^{-p-1}$ and $g^{(c,k)}_{BurrIII} = ckx^{-k-1}\left(1+x^{-k}\right)^{-c-1}$, which are misspecified models. First we generate 10^4 Monte-Carlo data sets of sample size n from $Weibull\ (\alpha,\beta)$ distribution and estimate the unknown parameters of $f^{(p,b)}_{BurrXII}$ and $g^{(c,k)}_{BurrIII}$ using the maximum likelihood method under doubly censoring. Then we construct a 0.95 tracking interval from (5). We choose different sample sizes, namely n=20, 50,100 and 200, whereas (α,β) for different sample sizes are taken as $(\alpha,\beta) = (0.5,1)$, (1,1), (2,1). For doubly censoring schemes, we present the results when $r_1 = [n * p_1(= 0.05)]$; $r_2 = [n * p_2(= 0.96)]$ and $r_1 = [n * p_1(= 0.16)]$; $r_2 = [n * p_2(= 0.70)]$, the integer parts of $n * p_i$; $i = 1,2$, respectively. The results are reported in Tables 1 and 2. Note that we consider W, BXII and BIII instead of Weibull, Burr XII and Burr III for simplicity.

Some of the points are quite clear from the Tables 1 and 2. It is observed that for n=20, 50 and 100, the tracking intervals contain zero, which indicates that the BXII and BIII are equal or observationally equal. But for n=200, when the censored data increase, both limits of the tracking intervals are negative. So, the BXII density is better than the BIII density to estimate the true model ($Weibull\ (\alpha,\beta)$). Note that, we say one model is better than the other one when the tracking interval does not contain zero. In other words, both limits of the tracking intervals are negative or positive. Also, the average tracking interval lengths decrease as sample size increases for fixed (α,β). For example, from Table 1 and $n = 20$, the average length of the tracking interval for $(\alpha = 0.5, \beta = 1)$ is 1.07003, this reduces to 0.378789 for $n = 50$, 0.267308 for $n = 100$ and 0.195206 for $n = 200$. Furthermore, we see that for fixed n and (α,β) when censored observations increase, the tracking interval lengths increase.

Table 1. Choice between BXII and BIII models using tracking interval when $r_1 = [n * p_1(= 0.16)]$; $r_2 = [n * p_2(= 0.70)]$

(α, β)	$(0.5, 1)$	$(1, 1)$	$(2, 1)$
$(n = 20)$	(-0.58363, 0.48639)	(-0.59153, 0.49305)	(-0.60420, 0.50522)
	1.07003	1.08458	1.10943
$(n = 50)$	(-0.284478, 0.094310)	(-0.282334, 0.091930)	(-0.287928, 0.098036)
	0.378789	0.374265	0.385964
$(n = 100)$	(-0.244766, 0.022542)	(-0.242922, 0.022358)	(-0.243322, 0.023218)
	0.267308	0.265281	0.266541
$(n = 200)$	(-0.215278, -0.020071)	(-0.21663, -0.02023)	(-0.215560, -0.019817)
	0.195206	0.19640	0.195742

Table 2. Choice between BXII and BIII models using tracking interval when $r_1 = [n * p_1(= 0.05)]$; $r_2 = [n * p_2(= 0.96)]$

(α, β)	$(0.5, 1)$	$(1, 1)$	$(2, 1)$
$(n = 20)$	(-0.035038, 0.044976)	(-0.034334, 0.045122)	(-0.040738, 0.048479)
	0.080014	0.079457	0.089218
$(n = 50)$	(-0.042450, 0.028837)	(-0.042655, 0.028574)	(-0.042607, 0.028515)
	0.071288	0.071230	0.071123
$(n = 100)$	(-0.029802, 0.033489)	(-0.029481, 0.034172)	(-0.029671, 0.033630)
	0.063292	0.063653	0.063301
$(n = 200)$	(-0.037674, 0.017800)	(-0.037647, 0.017755)	(-0.037566, 0.017791)
	0.055474	0.055402	0.055357

4.2 Real Data Analysis

Data set 1: In this example, we present the data analysis of the micro plasma spread factor data reported by Kang and Ng (2006) for illustrative purposes. The authors are thankful to Dr. Asadi (2008 and 2012) for providing the data, which represent spread factor of plasma droplets. The spread factor, ς, is defined as the ratio of splat diameter to droplet diameter, *i.e.*

$$\varsigma = \frac{d_e}{D_p},$$

where, d_e is the equivalent diameter of the elliptical splat area and D_p is the droplet diameter before impacting on the substrate. The elliptical splat area

is converted to an equivalent splat of circular shape so that its equivalent diameter d_e can be derived. The spread factor data measured by means of Scanning Electron Microscopy (SEM) and high resolution surface profilometry and reported in 0 (Data set I) and 30 (Data set II) substrate inclination angles respectively. For datasets 1 and 2, we have divided each data point by 4.48 and 4.54 respectively.
First we want to check whether the Weibull distribution fits the datasets or not, and that we have used the complete datasets. For this purpose, we present the q-q plots of datasets I (0°) and II (30°) in Figures 1 and 2 respectively. These plots show a strong relationship supporting the appropriateness of the Weibull distribution. For datasets I and II, we also fit Weibull, Burr XII, Burr III and inverse Weibull and report the estimated parameter values, Kolmogorov distances and the AIC values in Tables 3 and 4 respectively. We plot the empirical and the fitted cumulative distribution functions for different distributions and for both the datasets in Figures 3 and 4 respectively.

Table 3. Estimated parameters, K-S distances and AIC values for different distribution functions of Dataset I.

Distribution	Estimated Parameters	K-S	AIC
W	$\alpha = 9.47927,\ \beta = 0.59586$	0.1046	-66.46386
BXII	$p = 1.08586,\ b = 12.02986$	0.1738	-54.98781
BIII	$c = 0.94491,\ k = 12.58930$	0.1648	-54.82070
IW	$\theta = 6.70588,\ \lambda = 0.57373$	0.1863	-41.26251

Table 4. Estimated parameters, K-S distances and AIC values for different distribution functions of Dataset II.

Distribution	Estimated Parameters	K-S	AIC
W	$\alpha = 9.47280,\ \beta = 0.61149$	0.092	-43.01254
BXII	$p = 1.08011,\ b = 13.2313$	0.150	-39.52303
BIII	$c = 0.95207,\ k = 13.7940$	0.1415	-39.42007
IW	$\theta = 7.87697,\ \lambda = 0.56435$	0.1615	-34.09201

From Tables 3 and 4, it is clear that, Weibull with estimated parameters $\alpha \cong 9.5$ and $\beta \cong 0.6$ is the best fitted model based on the minimum AIC criterion or the minimum Kolmogorov distance. We also want to observe how the two rival models behave for these data using tracking interval.

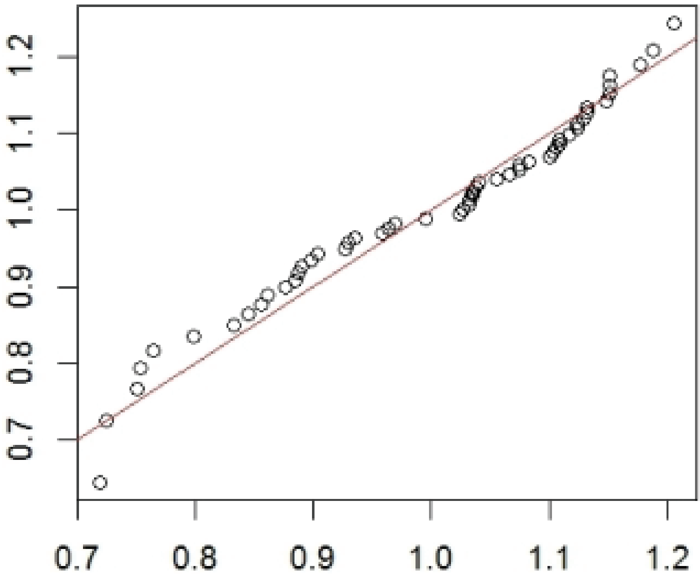

Figure 1. The q–q plot of dataset I.

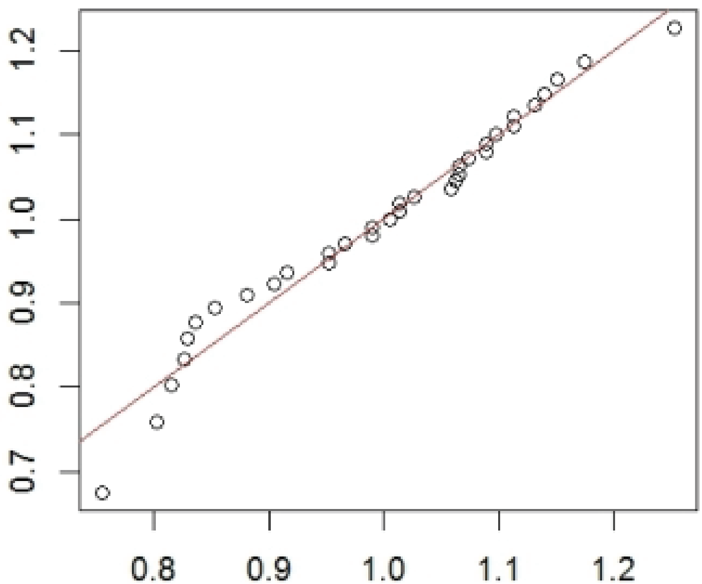

Figure 2. The q–q plot of dataset II.

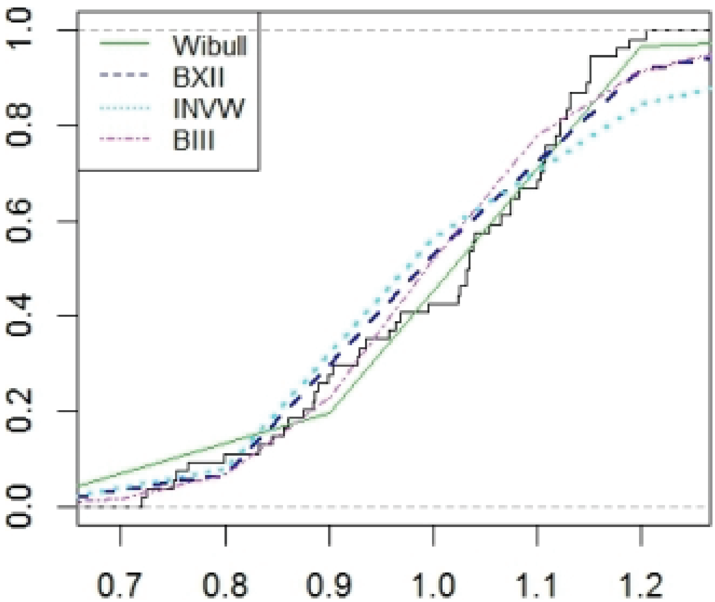

Figure 3. Empirical survival function and the fitted survival functions for dataset I.

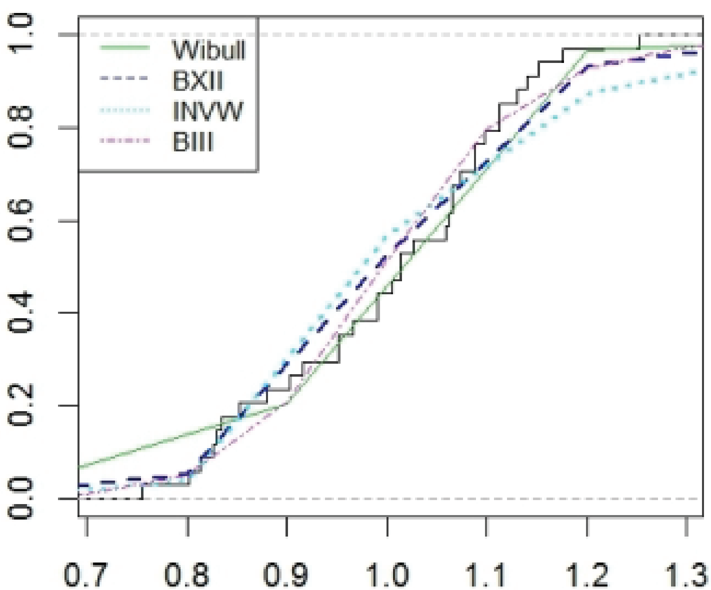

Figure 4. Empirical survival function and the fitted survival functions for dataset II.

We consider the following four different cases of rival models and censoring schemes:

Case 1: (Burr XII and inverse Weibull) and ($r_1 = 3$, $r_2 = 50$).

Case 2: (Burr XII and inverse Weibull) and ($r_1 = 7$, $r_2 = 34$).

Case 3: (Burr XII and Burr III) and ($r_1 = 3$, $r_2 = 50$).

Case 4: (Burr XII and Burr III) and ($r_1 = 7$, $r_2 = 34$).

Note that, the true model (h) is Weibull distribution. For dataset I and Cases 1, 2, 3 and 4, the tracking intervals are (-0.183724, -0.074251), (-0.139226, -0.056710), (-0.063805, 0.034970) and (-0.230943, 0.042245) respectively. It is observed that for Cases 1 and 2, both limits of the tracking intervals are negative, which indicates that the Burr XII is better than the inverse Weibull to estimate the true model. For Cases 3 and 4, the tracking intervals contain zero, as expected. So the Burr XII and the Burr III distributions are equivalent to consider as an estimate for the true mode. Also, for dataset II, we have the following four different cases of rival models and censoring schemes:

Case 1: (Burr XII and inverse Weibull) and ($r_1 = 2$, $r_2 = 32$).

Case 2: (Burr XII and inverse Weibull) and ($r_1 = 5$, $r_2 = 25$).

Case 3: (Burr XII and Burr III) and ($r_1 = 2$, $r_2 = 32$).

Case 4: (Burr XII and Burr III) and ($r_1 = 5$, $r_2 = 25$).

Thus, for Cases 1, 2, 3 and 4, the tracking intervals are (-0.136504, -0.003690), (-0.215939, -0.086732), (-0.054068, 0.036941) and (-0.237397, 0.064241) respectively. Similar to the dataset I, it is clear that, for Cases 1 and 2, the Burr XII is better than the inverse Weibull to estimate the true model and for Cases 3 and 4, the two models are equivalent to consider as an estimate for the Weibull distribution (true model).
Another important problem in engineering experiments namely the prediction intervals of the future observations, based on the current available observation. So, we obtain the Prediction interval of the future observation (y_{r_2+1}) and the ratio of two future consecutive data (y_{j+1}/y_j; $j = r_2, \ldots, n-1$), see Wu (2008).
Suppose that, $y_{r_1}, \ldots, y_{r_2}$ present the doubly censored sample from a Weibull

distribution with parameters α and β. Then the $100(1-\gamma)\%$ prediction interval of y_{r_2+1} and y_{j+1}/y_j; $j = r_2, \ldots, n-1$ are given by

$$\left(\kappa_1 F_{1-\frac{\gamma}{2}}(2, 2(r_2 - r_1)) + y_{r_2}\ ,\ \kappa_1 F_{\frac{\gamma}{2}}(2, 2(r_2 - r_1)) + y_{r_2}\right)$$

and

$$\left(\kappa_2 F_{1-\frac{\gamma}{2}}(2, 2(r_2 - r_1)) + 1\ ,\ \kappa_2 F_{\frac{\gamma}{2}}(2, 2(r_2 - r_1)) + 1\right)$$

respectively, where,

$$\kappa_1 = \frac{(n - r_2 + 1)y_{r_2} + \sum_{i=r_1+1}^{r_2-1} y_i - (n - r_1)y_{r_1}}{(n - r_2)(r_2 - r_1)}$$

and

$$\kappa_2 = (n - r_2 + 1)y_{r_2} + \sum_{i=r_1+1}^{r_2-1} \frac{y_i - (n - r_1)y_{r_1}}{y_j(n - j)(r_2 - r_1)}.$$

See Appendix C for more details.
Now, we consider the dataset II and we wish to find the prediction interval of the y_{31} and y_{j+1}/y_j; $j = 30, \ldots, 33$, based on the observed sample when $r_1 = 5$ and $r_2 = 30$. The prediction interval of y_{31} is (5.12654, 8.45481), therefore the real value of y_{31}(=5.17215) falls into this interval. The prediction intervals of y_{j+1}/y_j; $j = 30, \ldots, 33$ and their real values are presented in Table 5. It is observed that all intervals include the corresponding real values. Furthermore, we can see that the length of the prediction interval increases when i increase, ($j = 30, \ldots, 33$).

Table 5. The 0.95% prediction intervals for the ratio of two future consecutive data and their real values.

y_{j+1}/y_j; $j = 30, \ldots, 33$	Rediction Intervals	Real Values
(y_{31}/y_{30})	(1.00055, 1.64253)	1.00792
(y_{32}/y_{31})	(1.00073, 1.84783)	1.01047
(y_{33}/y_{32})	(1.00108, 2.24593)	1.02073
(y_{34}/y_{33})	(1.00199, 3.33760)	1.06599

Data set 2: in this example, we provide another data analysis for more illustrative purposes. The data have been taken from Rupert and Miller (1997) and it represents the 23 patients who were diagnosed with acute myelogeneous cancer. Moradi et al. (2014) observed that the Burr III (BIII) and exponentiated Burr III (EBIII) distributions work quite well for this data and the AIC's of them are very similar. Now, we want to compare the BIII and EBIII distributions using the tracking interval. So, we consider BIII (say f) and EBIII (say g) as rival models and create three artificially doubly censored data sets from the uncensored data set, using the following cases of censoring schemes:

Case 1:= $r_1 = 2, r_2 = 21$.

Case 1:= $r_1 = 4, r_2 = 18$.

Case 1:= $r_1 = 8, r_2 = 16$.

In all the cases we have estimated the unknown parameters using the MLEs and then constructed the tracking intervals. For Cases 1, 2 and 3, the tracking intervals are $(-2.4844 \times 10^{-5}, 6.6271 \times 10^{-5})$, $(-6.6627 \times 10^{-4}, 2.2624 \times 10^{-4})$ and $(-8.1747 \times 10^{-4}, 1.1074 \times 10^{-4})$. It is observed that the tracking intervals contain zero (as we expected). So there is no confidence that we incur a lower risk using BIII rather than EBIII distribution. Also in all the cases, the tracking interval lengths are negligible. This indicates that the Burr XII and Burr III are very similar in information criteria sense.

5 Conclusion

In the present work, we examined how the two rival models behave under doubly censoring scheme. We considered an interval, say, tracking interval for differences of the expected KL of two rival models. Our approach enlightens the variability of any criterion based on log-likelihood function, like AIC and their variants. To introduce the tracking interval, we proposed a statistic which tracks the difference of the expected KL risks between maximum likelihood estimators in two non-nested rival models. We also obtained the asymptotic distribution of maximum likelihood estimator under doubly censoring scheme. It is observed that the asymptotic distribution of maximum likelihood estimator is asymptotically normal. The results of our simulation study were in agreement with the asymptotic results. For an application, we considered several statistical distribution functions to analyze the micro plasma spread factor datasets. Using several statistical criteria, like minimum Kolmogorov distance and minimum AIC value, the Weibull distribution with estimated parameters $\alpha \cong 9.5$ and $\beta \cong 0.6$ appears to be more appropriate statistical distribution for datasets I (0°) and II (30°). These results can be to extend for other angels of plasma spraying. We also have obtained the tracking intervals for comparing two rival models based on different censoring schemes and found that the Burr XII is closer than inverse Weibull and Burr III to Weibull distribution.

References

Akaike, H. (1973). Information Theory and an Extension of Maximum Likelihood Principle. In: Second International Symposium on Information Theory, Akademia Kiado, 267–281.

Asadi, S., Passandideh-Fard, M. and Moghiman, M. (2008). Numerical and Analytical Model of the Inclined Impact of a Droplet on a Solid Surface in a Thermal Spray Coating Process, *Iranian Journal of Surface and Engineering*, **4**, 1-14.

Asadi, S. (2012). Simulation of Nanodroplet Impact on a Solid Surface, *International Journal of Nano Dimension*, **3**, 19-26.

Burr, I.W. (1942). Cumulative Frequency Functions, *Annals Mathematics Statistics*, **13**, 215–232.

Cai, T. and Cheng, S. (2004). Semiparametric Regression Analysis for Doubly Censored Data, *Biometrika*, **91**, 277-290.

Commenges, D., Sayyareh, A., Letenneur, L., Guedj J. and Bar- Hen, A. (2008). Estimating a Difference of Kullback–Leibler Risks Using a Normalized Difference of AIC, *The Annals of Applied Statistics*, **2**, 1123–1142.

Cox, D.R. (1961). Test of Separate Families of Hypothesis, In: Proceeding of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, **1**, 105–123.

Cox, D.R. (1962). Further Result on Tests of Separate Families of Hypothesis, *Journal of the Royal Statistical Society B*, **24**, 406–424.

Cramér, H. (1946). *Mathematical Methods of Statistics*, Princeton University Press, Princeton.

Jones, G. and Rocke, D.M. (2002). Multivariate Survival Analysis with Doubly-Censored Data: Application to The Assessment of Accutane Treatment for Fibrodysplasia Ossificans Progressiva, *Stat. Med.*, **21**, 2547-2562.

Kang, C.W. and Ng, H.W. (2006). Splat Morphology and Spreading Behavior due to Oblique Impact of Droplets onto Substrates in Plasma Spray Coating Process, *Surface and Coatings Technology*, **200**, 5462-5477.

Khan, H.M.R., Albatineh, A., Alshahrani, S., Jenkins, N. and Ahmed, N.U. (2011). Sensitivity Analysis of Predictive Modeling for Responses from the Three-Parameter Weibull Model with a Follow-up Doubly Censored, Sample of Cancer Patients, *Computational Statistics and Data Analysis*, **55**, 3093–3103.

Kullback, S. and Leibler, R. (1951). On Information and Sufficiency, *Annals of Mathematical Statistics*, **22**, 79–86.

Linhart, H. and Zucchini, W. (1986). *Model Selection.* Wiley, New York. MR0866144.

Louis, T.A. (1982). Finding the Observed Information Matrix when Using the EM Algorithm, *Journal of Royal Statistics Society Series B*, **44**, 226–233.

Moradi, N., Sayyareh, A. and Panahi, H. (2014). Estimation of the Parameters of a Exponentiated Burr Type III Distribution under Type II Censoring, *Journal of Statistical Scienses*, **8**, 93-109.

Panahi, H. and Sayyareh, A. (2013). Parameter Estimation and Prediction of Order Statistics for the Burr Type XII Distribution with Type-II Censoring, *Journal of Applied Statistics*, **41**, 215-232.

Panahi, H. and Sayyareh, A. (2014). Tracking Interval for Type II Hybrid Censoring Scheme, *JIRSS*, **13**, 187-208.

Panahi, H. and Sayyareh, A. (2015). Estimation and Prediction for a Unified Hybrid-Censored Burr Type XII Distribution, *Journal of Statistical Computation and Simulation*, 1-19.

Raqab, M.Z. and Kundu, D. (2005). Comparison of Different Estimators of P(Y < X) for a Scaled Burr Type X Distribution, *Communications in Statistics Simulation and Computation*, **34**, 465-483.

Rastogi, M.k. and Tripathi, Y.M. (2012). Estimating the Parameters of a Burr Distribution under Progressive Type II Censoring, *Statistical Methodology*, **9**, 381–391.

Ren, J. and Peer, P.G. (2000). A Study on Effectiveness of Screening Mammograms, *International journal of Epidemiology*, **29**, 803-806.

Rupert, G. and Miller, J.R. (1997). *Survival analysis*, John Wiley, 0-471-25218-2.

Sayyareh, A., Obeidi, R. and Bar-Hen, A. (2011). Empirical Comparison of Some Model Selection Criteria, *Communication in Statistics-Simulation and Computation*, **40**, 72–86.

Sayyareh, A. (2012a). Inference after Separated Hypotheses Testing: an Investigation for Linear Models, *Journal of Statistical Computation and Simulation*, **82**, 1275-1286.

Sayyareh, A. (2012b). Tracking Interval for Selecting between Non-nested Models: An Investigation for Type II Right Censored Data, *Journal of Statistical Planning and Inference*, **142**, 3201–3208.

Thomas, D.R. and Wilson, W.M. (1972). Linear Order Statistics Estimation for the Two-Parameter Weibull and Extreme-Value Distribution form Type II Progressive Censored Samples, *Technometrics*, **14**, 679–691.

Vuong, Q.H. (1989). Likelihood Ratio Tests for Model Selection and Non-nested Hypothesis, *Econometrica*, **57**, 307–333.

Wu, S. (2008). Interval Estimation for a Pareto Distribution based on a Doubly Type II Censored Sample, *Computational Statistics and Data Analysis*, **52**, 3779–3788. .

Appendix A

To prove Theorem 1, we use the idea of missing information principle (Louis, 1982) to obtain the asymptotic normality of the MLE under doubly censoring scheme. For this purpose, we consider the Taylor expansion of $n^{-1}\frac{\partial L_n^f(\alpha)}{\partial \alpha}$ around $\alpha = \alpha_0$ as:

$$\begin{aligned} n^{-1}\frac{\partial L_n^f(\alpha)}{\partial \alpha} &= n^{-1}\frac{\partial L_n^f(\alpha)}{\partial \alpha}\,|_{\alpha=\alpha_0} + n^{-1}(\alpha-\alpha_0)\frac{\partial^2 L_n^f(\alpha)}{\partial \alpha\,\partial \alpha'}\,|_{\alpha=\alpha_0} + o_p(1) \\ &= A_1 + A_2(\alpha-\alpha_0) + o_p(1) \qquad (6) \end{aligned}$$

where,

$$\begin{aligned} A_1 = \frac{1}{n}\Bigg[\Bigg\{ \sum_{i=r_1}^{r_2} \frac{\partial}{\partial \alpha}\log f^{\alpha}(y_i) \Bigg\} + (r_1-1)\frac{\partial}{\partial \alpha}\log\{F(y_{r_1})\} \\ + (n-r_2)\frac{\partial}{\partial \alpha}\log\{\overline{F}(y_{r_2})\} \Bigg]_{\alpha=\alpha_0} \end{aligned}$$

and

$$\begin{aligned} A_2 = \frac{1}{n}\Bigg[\Bigg\{ \sum_{i=r_1}^{r_2} \frac{\partial^2}{\partial \alpha\,\partial \alpha'}\log f^{\alpha}(y_i) \Bigg\} + (r_1-1)\frac{\partial^2}{\partial \alpha\,\partial \alpha'}\log\{F(y_{r_1})\} \\ + (n-r_2)\frac{\partial^2}{\partial \alpha\,\partial \alpha'}\log\{\overline{F}(y_{r_2})\} \Bigg]_{\alpha=\alpha_0} \end{aligned}$$

We will show that, $A_1 \xrightarrow{\text{P}} 0$ and $A_2 \xrightarrow{\text{P}} -J_{f_d}$, where J_{f_d} is constant. Using the missing information principle, the observed information under doubly censoring scheme is

$$\sum_{i=r_1}^{r_2} \log f^{\alpha}(y_i) = \sum_{i=1}^{n} \log f^{\alpha}(w_i) - \sum_{i=1}^{r_1-1} \log f^{\alpha}(z_i \mid Y) - \sum_{i=1}^{n-r_2} \log f^{\alpha}(u_i \mid Y), \tag{7}$$

where, $w = (w_1, \ldots, w_n) =$ the complete data, $z = (z_1, \ldots, z_{r_1-1})$ and $u = (u_1, \ldots, u_{n-r_2}) =$ the complete data of size $r_1 - 1$ and $n - r_2$, from the right and left truncated population with density functions:
$h_1^* = \frac{f^{\alpha}(z)}{F^{\alpha}(y_{r_1})}$; $z < y_{r_1}$ and $h_2^* = \frac{f^{\alpha}(u)}{\overline{F}^{\alpha}(y_{r_2})}$; $u > y_{r_2}$ respectively. Note that, the sequences of random variables W's, Z's and U's are independent. For simplicity, we use $f^{\alpha}(z_i)$ and $f^{\alpha}(u_i)$ instead of $f^{\alpha}(z_i \mid Y)$ and $f^{\alpha}(u_i \mid Y)$ in what follows. Thus, A_1 can be rewritten as

$$\begin{aligned} A_1 = \frac{1}{n}\Bigg[& \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_0} \log f^{\alpha}(w_i) \right\} - \sum_{i=1}^{r_1-1} \frac{\partial}{\partial \alpha_0} \log f^{\alpha}(z_i) - \sum_{i=1}^{n-r_2} \frac{\partial}{\partial \alpha_0} \log f^{\alpha}(u_i) \\ & + (n - r_2) \frac{\partial}{\partial \alpha_0} \log\{\overline{F}^{\alpha}(y_{r_2})\} + (r_1 - 1) \frac{\partial}{\partial \alpha_0} \log\{F^{\alpha}(y_{r_1})\} \Bigg] \\ \equiv & \frac{1}{n}(A_1^* - A_1^{**}). \end{aligned} \tag{8}$$

So, from Cramer (1946), $\frac{1}{n} A_1^* = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_0} \log f^{\alpha}(w_i) \xrightarrow{P} 0$ and we will prove that

$$\begin{aligned} \frac{1}{n} A_1^{**} = \frac{1}{n} & \sum_{i=1}^{r_1-1} \frac{\partial}{\partial \alpha_0} \log f^{\alpha}(z_i) + \sum_{i=1}^{n-r_2} \frac{\partial}{\partial \alpha_0} \log f^{\alpha}(u_i) \\ & - (n - r_2) \frac{\partial}{\partial \alpha_0} \log\{\overline{F}^{\alpha}(y_{r_2})\} - (r_1 - 1) \frac{\partial}{\partial \alpha_0} \log\{F^{\alpha}(y_{r_1})\} \xrightarrow{P} 0. \end{aligned}$$

We can rewritten A_1^{**} as

$$\begin{aligned} A_1^{**} &= \sum_{i=1}^{r_1-1} \frac{\partial}{\partial\alpha_0} \log f^{\alpha}(z_i) - \sum_{i=1}^{r_1-1} E\left\{\frac{\partial}{\partial\alpha_0} \log f^{\alpha}(Z_i)\right\} \\ &+ \sum_{i=1}^{r_1-1} E\left\{\frac{\partial}{\partial\alpha_0} \log f^{\alpha}(Z_i)\right\} + \sum_{i=1}^{n-r_2} \frac{\partial}{\partial\alpha_0} \log f^{\alpha}(u_i) \\ &- \sum_{i=1}^{n-r_2} E\left\{\frac{\partial}{\partial\alpha_0} \log f^{\alpha}(U_i)\right\} + \sum_{i=1}^{n-r_2} E\left\{\frac{\partial}{\partial\alpha_0} \log f^{\alpha}(U_i)\right\} \\ &- (n-r_2)\frac{\partial}{\partial\alpha_0} \log\{\overline{F}^{\alpha}(y_{r_2})\} - (r_1-1)\frac{\partial}{\partial\alpha_0} \log\{F^{\alpha}(y_{r_1})\}. \end{aligned}$$

Under $\Re_3$, we have

$$\begin{aligned} E\left\{\frac{\partial}{\partial\alpha_0} \log f^{\alpha}(Z)\right\} &= \int_{-\infty}^{y_{r_1}} \frac{\partial}{\partial\alpha_0} \log f^{\alpha}(z) \frac{f^{\alpha}(z)}{F^{\alpha}(y_{r_1})}\, d\mu(z) \\ &= \frac{\frac{\partial}{\partial\alpha_0} F^{\alpha}(y_{r_1})}{F^{\alpha}(y_{r_1})} = \frac{\partial}{\partial\alpha_0} \log\{F^{\alpha}(y_{r_1})\} \end{aligned} \tag{9}$$

and similarly,

$$E\left\{\frac{\partial}{\partial\alpha_0} \log f^{\alpha}(U)\right\} = \frac{\partial}{\partial\alpha_0} \log\{\overline{F}^{\alpha}(y_{r_2})\} \tag{10}$$

So, $\frac{A_1^{**}}{n} \xrightarrow{P} 0$. Now by using Slutsky's Theorem, the result follows ($A_1 \xrightarrow{P} 0$). Similarly, we consider, $A_2 = \frac{1}{n}(A_2^* - A_2^{**})$, where,

$$A_2^* = \sum_{i=1}^{n} \frac{\partial^2}{\partial\alpha_0 \partial\alpha'_0} \log f^{\alpha}(w_i)$$

and

$$\begin{aligned} A_2^{**} &= \sum_{i=1}^{r_1-1} \frac{\partial^2}{\partial\alpha_0 \partial\alpha'_0} \log f^{\alpha}(z_i) + \sum_{i=1}^{n-r_2} \frac{\partial^2}{\partial\alpha_0 \partial\alpha'_0} \log f^{\alpha}(u_i) \\ &- (n-r_2)\frac{\partial^2}{\partial\alpha_0 \partial\alpha'_0} \log\{\overline{F}^{\alpha}(y_{r_2})\} - (r_1-1)\frac{\partial^2}{\partial\alpha_0 \partial\alpha'_0} \log\{F^{\alpha}(y_{r_1})\} \end{aligned}$$

We know that, $\frac{A_2^*}{n} \xrightarrow{P} -\wp$, and

$$\begin{aligned}
\frac{A_2^{**}}{n} = \frac{n-r_2}{n}\Bigg[\frac{1}{n-r_2}\Bigg\{\sum_{i=1}^{n-r_2}\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\log f^{\alpha}(u_i) \\
-\sum_{i=1}^{n-r_2}E\left(\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\log f^{\alpha}(U_i)\right)\Bigg\}\Bigg] \\
+\frac{r_1-1}{n}\Bigg[\frac{1}{r_1-1}\Bigg\{\sum_{i=1}^{r_1-1}\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\log f^{\alpha}(z_i) \\
-\sum_{i=1}^{r_1-1}E\left(\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\log f^{\alpha}(Z_i)\right)\Bigg\}\Bigg] \\
-\frac{1}{n}\Bigg[(n-r_2)\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\log\{\overline{F}^{\alpha}(y_{r_2})\} \\
-\sum_{i=1}^{n-r_2}E\left\{\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\log(f^{\alpha}(U_i)\right\}\Bigg] \\
-\frac{1}{n}\Bigg[(r_1-1)\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\log\{F^{\alpha}(y_{r_1})\} \\
-\sum_{i=1}^{r_1-1}E\left\{\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\log(f^{\alpha}(Z_i)\right\}\Bigg] \qquad (11)
\end{aligned}$$

The first and second terms in (11) converges in probability to zero. So, based on (9), (10) and after some simplification, we obtain

$$\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\log\overline{F}^{\alpha}(y_{r_2}) = \frac{\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\overline{F}^{\alpha}(y_{r_2})}{\overline{F}^{\alpha}(y_{r_2})} - \left[E\left\{\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(U)\right\}\right]^2 \qquad (12)$$

and

$$\begin{aligned}
E\left\{\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\log f^{\alpha}(U)\right\} = \frac{\frac{\partial^2}{\partial\alpha_0\partial{\alpha'}_0}\overline{F}^{\alpha}(y_{r_2})}{\overline{F}^{\alpha}(y_{r_2})} \\
-\int_{y_{r_2}}^{\infty}\left\{\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(u)\right\}^2\frac{f^{\alpha}(u)}{\overline{F}^{\alpha}(y_{r_2})}d\mu(u). \qquad (13)
\end{aligned}$$

Thus, from (11), (12) and (13), we have

$$\frac{1}{r_1-1}\sum_{i=1}^{r_1-1}\left[\frac{\partial^2}{\partial\alpha_0\partial\alpha'_0}\log\{F^{\alpha}(y_{r_1})\}-E\left\{\frac{\partial^2}{\partial\alpha_0\partial\alpha'_0}\log f^{\alpha}(Z_i)\right\}\right]$$

$$=\frac{1}{r_1-1}\sum_{i=1}^{r_1-1}Var\left\{\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(Z_i)\right\}=B^{*}$$

Similarly,

$$\frac{1}{n-r_2}\sum_{i=1}^{n-r_2}\left[\frac{\partial^2}{\partial\alpha_0\partial\alpha'_0}\log\{\overline{F}^{\alpha}(y_{r_2})\}-E\left\{\frac{\partial^2}{\partial\alpha_0\partial\alpha'_0}\log f^{\alpha}(U_i)\right\}\right]$$

$$=\frac{1}{n-r_2}\sum_{i=1}^{n-r_2}Var\left\{\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(U_i)\right\}=B^{**}$$

where, B^{*} and B^{**} converges to bounded values, say ϑ_1 and ϑ_2 respectively. Thus, $-\frac{A_2^{**}}{n}\xrightarrow{P}p_1\vartheta_1+(1-p_2)\vartheta_2$, and combining of this results gives, $A_2=\frac{1}{n}(A_2^{*}-A_2^{**})\xrightarrow{P}-J_{f_d}$, where, $J_{f_d}\equiv\wp+p_1\vartheta_1+(1-p_2)\,\vartheta_2$. Now, from (6) and (8), we have

$$\begin{aligned}\sqrt{nJ_{f_d}}(\hat{\alpha}_n-\alpha_0)&=\frac{\frac{\sqrt{n}A_1}{\sqrt{J_{f_d}}}}{-\frac{A_2}{J_{f_d}}}\\&=(nJ_{f_d})^{-\frac{1}{2}}\left(-\frac{A_2}{J_{f_d}}\right)^{-1}\left\{\sum_{i=1}^{n}\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(w_i)\right\}\\&\quad-\sum_{i=1}^{r_1-1}\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(z_i)\\&\quad+(nJ_{f_d})^{-\frac{1}{2}}\left(-\frac{A_2}{J_{f_d}}\right)^{-1}\left[-\sum_{i=1}^{n-r_2}\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(u_i)\right.\\&\quad\left.+(n-r_2)\frac{\partial}{\partial\alpha_0}\log\{\overline{F}^{\alpha}(y_{r_2})\}\right]\\&\quad+(nJ_{f_d})^{-\frac{1}{2}}\left\{-\frac{A_2}{J_{f_d}}\right\}^{-1}\left[(r_1-1)\frac{\partial}{\partial\alpha_0}\log\{F^{\alpha}(y_{r_1})\}\right]\end{aligned}\tag{14}$$

where, $-A_2/J_{f_d} \xrightarrow{P} 1$. So, it suffices to show that the numerator is asymptotically $N(0,1)$. Using (9), (10) and Slutsky Theorem, we have

$$\frac{\sqrt{r_1-1}}{\sqrt{n}}\left[\frac{1}{\sqrt{r_1-1}}\left\{\sum_{i=1}^{r_1-1}\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(z_i)-\sum_{i=1}^{r_1-1}E\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(Z_i)\right)\right\}\right] \xrightarrow{P} N\left(0,p_1\vartheta_1\right) \tag{15}$$

$$\frac{\sqrt{n-r_2}}{\sqrt{n}}\left[\frac{1}{\sqrt{n-r_2}}\left\{\sum_{i=1}^{n-r_2}\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(z_i)-\sum_{i=1}^{n-r_2}E\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(U_i)\right)\right\}\right] \xrightarrow{P} N\left(0,(1-p_2)\vartheta_2\right) \tag{16}$$

Now, using Slutsky Theorem, we obtain,

$$\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(w_i),\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n-r_2}\left\{\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(u_i)\right)-E\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(U_i)\right)\right\}\right.$$
$$\left.-\frac{1}{\sqrt{n}}\sum_{i=1}^{r_1-1}\left\{\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(z_i)\right)-E\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(Z_i)\right)\right\}\right] \xrightarrow{D} (V,U),$$

where, $V=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(w_i)\sim N\left(0,\wp\right)$ and $U\sim N\left(0,p_1\vartheta_1+(1-p_2)\vartheta_2\right)$ and V and U are independent. Now, using continuous mapping Theorem and (14)-(16), we conclude that

$$(nJ_{f_d})^{-\frac{1}{2}}\left[\sum_{i=r_1}^{r_2}\frac{\partial}{\partial\alpha}\log f^{\alpha}(y_i)+(n-r_2)\frac{\partial}{\partial\alpha}\log\{\overline{F}^{\alpha}y_{r_2})\}\right.$$
$$\left.+(r_1-1)\frac{\partial}{\partial\alpha}\log\{F^{\alpha}(y_{r_1})\}\right]_{\alpha=\alpha_0}$$

convergence in distribution to $N(0,1)$, and the proof is complete.

Appendix B

Proof: Based on Theorem 3.3 of Voung (1989) and Theorem 1, we have

$$\sqrt{n}\left\{\frac{1}{n}L_n^{\frac{f}{g}}(\hat{\alpha}_n,\hat{\beta}_n)-(p_2-p_1)\,E_{h*}(\tau_1)+p_1\tau_2+(1-p_2)\,\tau_3\right\}$$
$$=\sqrt{n}\left\{\frac{1}{n}L_n^{\frac{f}{g}}(\alpha_*,\beta_*)-(p_2-p_1)\,E_{h*}(\tau_1)+p_1\tau_2+(1-p_2)\,\tau_3\right\}+o_p(1)$$

where, $\tau_1=\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}$, $\tau_2=\log\frac{F^{\alpha_*}(\zeta_1)}{G^{\beta_*}(\zeta_1)}$ and $\tau_3=\log\frac{\overline{F}^{\alpha_*}(\zeta_2)}{\overline{G}^{\beta_*}(\zeta_2)}$. But from the multivariate central Theorem, the first term in the right hand side converges in distribution to $N(0,\omega^2_{*d})$. It now suffices to show that

$$\omega^2_{*d}=Var_h\left\{\log\frac{f^{\alpha_*}(W)}{g^{\beta_*}(W)}\right\}+p_1Var_{h_1^*}\left\{\log\frac{f^{\alpha_*}(Z)}{g^{\beta_*}(Z)}\right\}$$
$$+(1-p_2)Var_{h_2^*}\left\{\log\frac{f^{\alpha_*}(U)}{g^{\beta_*}(U)}\right\}.$$

Using the missing information principle (7), we can write $Var(\frac{1}{n}L_n^{f/g}(\hat{\alpha}_n,\hat{\beta}_n))$ as

$$Var\left\{\frac{1}{n}L_n^{\frac{f}{g}}\left(\hat{\alpha}_n,\hat{\beta}_n\right)\right\}=Var\left\{\frac{1}{n}\eta_1+\frac{(r_1-1)}{n}\eta_2+\frac{(n-r_2)}{n}\eta_3\right\}$$
$$=Var\;\left[n^{-1}\left\{\eta_6-\eta_7-\;\eta_8+(r_1-1)\eta_2+(n-r_2)\eta_3\right\}\right].$$

where

$$\eta_1=\sum_{i=r_1}^{r_2}\log\frac{f^{\hat{\alpha}_n}(Y_i)}{g^{\hat{\beta}_n}(Y_i)},\quad \eta_2=\log\frac{F^{\hat{\alpha}_n}(y_{r_1})}{G^{\hat{\beta}_n}(y_{r_1})},\quad \eta_3=\log\frac{\overline{F}^{\hat{\alpha}_n}(y_{r_2})}{\overline{G}^{\hat{\beta}_n}(y_{r_2})},$$

$$\eta_6=\sum_{i=1}^{n}\log\frac{f^{\hat{\alpha}_n}(W_i)}{g^{\hat{\beta}_n}(W_i)},\quad \eta_7=\sum_{i=1}^{r_1-1}\log\frac{f^{\hat{\alpha}_n}(Z_i)}{g^{\hat{\beta}_n}(Z_i)},\quad \eta_8=\sum_{i=1}^{n-r_2}\log\frac{f^{\hat{\alpha}_n}(U_i)}{g^{\hat{\beta}_n}(U_i)}.$$

If, $\frac{r_1-1}{n}\to p_1$ and $\frac{n-r_2}{n}\to 1-p_2$ as $n\to\infty$ such that $Y_{r_1}=\xi_{1n}\to\xi_1$ and $Y_{r_2}=\xi_{2n}\to\xi_2$ in probability, then

$$\omega^2_{*d}=Var_h\left\{\log\frac{f^{\alpha_*}(W)}{g^{\beta_*}(W)}\right\}+p_1Var_{h_1^*}\left\{\log\frac{f^{\alpha_*}(Z)}{g^{\beta_*}(Z)}\right\}$$
$$+(1-p_2)Var_{h_2^*}\left\{\log\frac{f^{\alpha_*}(U)}{g^{\beta_*}(U)}\right\}.$$

Appendix C

Let $y_{r_1},\ldots,y_{r_2}$ be a doubly censored sample from a Weibull distribution. Thus $v_{r_1}=\beta y_{r_1}^{\alpha},\ldots,v_{r_2}=\beta y_{r_2}^{\alpha}$ is a doubly censored sample from an exponential distribution with parameter one and based on Thomas and Wilson (1972), we know that,

$$s_{r_1+1}=(n-r_1)(v_{r_1+1}-v_{r_1}),\ s_{r_1+2}=(n-r_1-1)(v_{r_1+2}-v_{r_1+1}),\ldots,$$

$$s_{r_2}=(n-r_2+1)(v_{r_2}-v_{r_2-1}),\ s_{r_2+1}=(n-r_2)(v_{r_2+1}-v_{r_2})$$

are *iid* random variables from a exponential distribution with parameter one. So, $2s_{r_2+1}\sim\chi^2_2$ and $2\sum_{i=r_1+1}^{r_2}s_i\sim\chi^2_{2(r_2-r_1)}$ and they are independent. Now, we consider the statistic λ_1 as

$$\lambda_1=\frac{\frac{(n-r_2)(2\beta)(y_{r_2}+1-y_{r_2})}{2}}{\frac{2\beta\{(n-r_2+1)y_{r_2}+\sum_{i=r_1+1}^{r_2-1}y_i-(n-r_1)y_{r_1}\}}{(y_{r_2}-y_{r_1})}}.$$

Thus,

$$P\left(\ F_{\frac{\gamma}{2}}(2\ ,\ 2r_2-2r_1)<\lambda_1<F_{1-\frac{\gamma}{2}}(2\ ,\ 2r_2-2r_1)\ \right)=1-\gamma.$$

After some simplification, we have:

$$\left(\kappa_1F_{1-\frac{\gamma}{2}}(2,2(r_2-r_1))+y_{r_2}<y_{r_2+1}<\kappa_1F_{\frac{\gamma}{2}}(2,2(r_2-r_1))+y_{r_2}\right),$$

where, $\kappa_1=((n-r_2+1)y_{r_2}+\sum_{i=r_1+1}^{r_2-1}y_i-(n-r_1)y_{r_1})/((n-r_2)(r_2-r_1)\ )$. Considering the statistic λ_2 as

$$\lambda_2=\frac{(n-j)(r_2-r_1)(y_{j+1}-y_j)}{(n-r_2+1)y_{r_2}+\sum_{r_1+1}^{r_2-1}y_i-(n-r_1)y_{r_1}};\qquad j=r_2,\ldots,n-1,$$

the prediction interval of $y_{j+1}/y_j;\ j=r_2,\ldots,n-1$ can be obtained, similarly.

H. Panahi
Department of Mathematics and Statistics,
Islamic Azad University, Lahijan Branch,
Lahijan, Iran.
email: *panahi@liau.ac.ir*

A. Sayyareh
Department of Statistics,
Razi University,
Kermanshah, Iran.
email: *asayyareh@razi.ac.ir*