Search Probability for Non-zero Effects Detection under Skew-Normal/Independent Search Model

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Abstract. Shirakura et al. (1996) has been introduced and calculated the search probability (SP) for normal search model. However, in practical situations the normality assumption may fail. In this study, we consider a more realistic underlying skew-normal/independent (SNI) model and obtain the SP. This is a general case, in a sense that the result in Shirakura et al. (1996) is its special case. The proposed SP carries some reliable properties and can be used as a design comparison criterion to compare and rank the search designs (SD).

Keywords. Design comparison criterion; search design; search linear model; search probability; skew-normal distribution.

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1 Introduction

Expensive treatments in many factorial designs rally the cost of experiments. The challenge for saving money is running the experiments with the lower number of runs. So, experimenters often apply a fraction of factorial experiment, assuming the higher order interactions are all negligible; e.g. in

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main effect plans (MEP) it is assumed that all interaction effects are zero. Meanwhile, the existence of non-zero interactions may lead to the biased estimation of main effects, though the sparsity principle indicates that the number of such the interactions is small.

A search design resolves the search and estimation non-zero interaction effects along with estimating the overall mean and main effects. For k non-zero interactions, such a SD is called a main effect plus k plan ($MEP.k$). Construction of $MEP.k$ has been investigated by several authors and were reviewed in Ghosh et al. (2007).

Following Srivastava et al. (1975), for an $N \times 1$ vector of observations $\mathbf{y}$, the search linear model is defined as

$$\mathbf{y} = \mathbf{A}_1\boldsymbol{\xi}_1 + \mathbf{A}_2\boldsymbol{\xi}_2 + \mathbf{e}, \qquad Var(\mathbf{e}) = \sigma^2\boldsymbol{\Sigma}_{\mathbf{e}}, \tag{1}$$

where $\mathbf{A}_i$'s ($i = 1, 2$) are known design ($N \times \nu_i$) matrices and $\boldsymbol{\xi}_i$'s ($i = 1, 2$) are ($\nu_i \times 1$) vectors of factorial effects for $i = 1, 2$, $\mathbf{e}$ is an ($N \times 1$) error vector, σ^2 is unknown coefficient of covariance matrix $\boldsymbol{\Sigma}_{\mathbf{e}}$. It is known that k elements of $\boldsymbol{\xi}_2$ are non-zero but do not know which ones they are. Meanwhile, $\boldsymbol{\xi}_1$ is the set of effects which the interest is to be estimated all along with identifying and estimating the non-zero effects in $\boldsymbol{\xi}_2$. A design which resolve such the problem is called a search design. For each vector of possible k non-zero effects $\boldsymbol{\zeta}(k \times 1)$ in $\boldsymbol{\xi}_2$ let $\mathbf{A}_{21}(\boldsymbol{\zeta})$ be the $N \times k$ submatrix of $\mathbf{A_2}$ corresponding to $\boldsymbol{\zeta}$. The emanated reduced models are written as below,

$$\mathbf{y} = \mathbf{A}_1\boldsymbol{\xi}_1 + \mathbf{A}_{21}(\boldsymbol{\zeta})\boldsymbol{\zeta} + \mathbf{e}. \tag{2}$$

There are $l = \binom{\nu_2}{k}$ rival models in (2), only a model is true, i.e.:

$$\mathbf{y} = \mathbf{A}_1\boldsymbol{\xi}_1 + \mathbf{A}_{21}(\boldsymbol{\zeta}_0)\boldsymbol{\zeta}_0 + \mathbf{e}, \tag{3}$$

where $\boldsymbol{\zeta}_0$ is $k \times 1$ correct vector of non-zero effects in $\boldsymbol{\xi}_2$. The search problem is to identify and estimate $\boldsymbol{\zeta}_0$ as well as estimate $\boldsymbol{\xi}_1$. To identify $\boldsymbol{\zeta}_0$, Srivastava et al. (1975) suggested to calculate sum of square error (SSE) for all possible l models (2) and appointed the model with minimum SSE as the true model. Shirakura et al. (1996) argued about the stochastic property of SSE and proposed to initiate $P(SSE_0 \leqslant SSE)$ on SSE_0 of model (3) and SSE of any other competing model and came up with the search probability in an explicit form for $k = 1$ and underlying normal search linear model. That is, the search probability is the minimum value of the probability of

paired comparing of the SSEs for all possible rival models. SP is design dependent and thus, introduced as a measure for comparing SDs with respect to their search performances. That is, the SD with a higher SP is superior in identifying the true model. Several SP-based criteria have been proposed by researchers for comparing search ability of SDs. As examples, Ghosh and Teschmacher (2002) and Talebi and Esmailzadeh (2011b) developed the above SP-criterion, all are restricted to $k = 1$. Talebi and Esmailzadeh (2011a) gave a so-called KL-criterion which is applicable for $k \geqslant 1$.

All of the above criteria including Shirakura et al. (1996) have been obtained for normal errors; however, in practical situations such an assumption may not be well adapted to real data. There are numerous number of non-normal real data set, e.g. with skewed or heavy tailed distributions. For example, hidden truncation problem which has been described by Arnold et al. (2002), was led to data with skew-normal (SN) model; Data set in Bahrami and Qasemi (2015) were analyzed by a flexible skew-generalized normal model; AIS data in Marchenko (2010) and wind speed data in Arellano-Valle and Genton (2010) were considered and analyzed as SN and skew-t models; In analyzing fiber-glass data, Lachos and Labra (2014) applied distributions from SNI class. They developed SNI family distributions including models with skew-normal, skew-t, skew-slash and skew-contaminated normal distributions. Additionally, this family comprise the normal/independent distributions as a special case.

For the non-normal real data, the developed SP-based criteria are ineffective. So, in order to cover the real problem situations and due to practical flexibility and utility of the SNI family, SP for SNI search models was studied in this research. This generalized the results of Shirakura et al. (1996), i.e. the SP for normal model would be a special case of this study. So, the criterion obtained under the SNI model is more general which allows for more flexibility in design comparisons.

Some preliminaries on distributional issues are required that are presented in Section 2. In Section 3, the formulization of SP is given. Under SN and SNI error, the SP-criterion will be obtained and presented in Section 4. The performance of the proposed criterion in comparing the search ability and ranking of some SDs are given in Section 5.

2   Distributional Perspective

Because in this research a general form of SP for linear search models with SNI error will be investigated, more understandings of SNI and SN distributions and some of features are helpful.

A random variable Z is distributed as a SN distribution with skewness parameter λ, denoted by $SN(\lambda)$, if it has the probability density function (*pdf*) below:

$$f(z) = 2\phi(z)\Phi(\lambda z), \qquad z \in R, \lambda \in R,$$

where $\phi(z)$ and $\Phi(z)$ are the *pdf* and the cumulative distribution function (*cdf*) of the standard normal distribution, respectively (Azzalini, 1985). When $\lambda = 0$, Z follows $N(0,1)$. Now, let's extend this to a general N-dimensional random vector; that is, an $N \times 1$ random vector $\mathbf{Y}$ has a multivariate SN distribution with location vector $\boldsymbol{\mu} \in R^N$, variance matrix $\boldsymbol{\Sigma}$, and skewness vector $\boldsymbol{\lambda} \in R^N$, denoted by $\mathbf{Y} \sim SN_N(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$, if its *pdf* is

$$f(\mathbf{y}) = 2\phi_N(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma})\Phi(\boldsymbol{\lambda}'\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{y} - \boldsymbol{\mu})), \qquad \mathbf{y} \in R^N, \tag{4}$$

where $\phi_N(\cdot)$ is the *pdf* of the $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, (Arellano-Valle et al., 2005). Note that for $\boldsymbol{\lambda} = \mathbf{0}$ the random vector $\mathbf{Y}$ follows $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

To define the SNI distribution, consider an $N \times 1$ random vector $\mathbf{Y}$ with pdf

$$f(\mathbf{y}) = 2 \int_0^\infty \phi_N(\mathbf{y}|\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma})\Phi_1(u^{\frac{1}{2}}\boldsymbol{\lambda}'\boldsymbol{\Sigma}^{-\frac{1}{2}}(\mathbf{y} - \boldsymbol{\mu}))dH(u), \tag{5}$$

where U is a positive random variable with cdf $H(u; \boldsymbol{\nu})$ indexed by the parameter vector $\boldsymbol{\nu}$, $\boldsymbol{\mu} \in R^N$ and $\boldsymbol{\lambda} \in R^N$ are location and skewness parameters, respectively, and $\boldsymbol{\Sigma}$ is a positive definite (*pd*) scale matrix. Then, $\mathbf{Y}$ has an SNI distribution, denoted by $\mathbf{Y} \sim SNI_N(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$ (Lachos and Labra, 2014). As an example, for the skew-t distribution with ν degrees of freedom, $ST_N(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$, $U \sim Gamma(\nu/2, \nu/2)$, where $Gamma(\alpha, \beta)$ stands for gamma distribution with mean α/β. When $\boldsymbol{\lambda} = \mathbf{0}$, $\mathbf{Y}$ follows an NI distribution. For $\mathbf{Y}$ with SNI distribution, it is also true that for any $\mathbf{a} \in R^N$ and $b \in R$

$$\mathbf{a}'\mathbf{Y} + b \sim SNI_1(\mathbf{a}'\boldsymbol{\mu} + b, \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}, \lambda^*; H), \tag{6}$$

where $\lambda^* = \alpha/(1 - \alpha^2)^{1/2}$ and $\alpha = \{\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}(1 + \boldsymbol{\lambda}'\boldsymbol{\lambda})\}^{-1/2}\mathbf{a}'\boldsymbol{\Sigma}^{1/2}\boldsymbol{\lambda}$.

We utilize random variables X_1 and X_2 with SNI distribution functions F_1 and F_2 respectively, and use the following Farlie-Gumbel-Morgenstern

(FGM) copula function (Balakrishnan and Lai, 2009) with $\alpha \in [-1, 1]$, due to the small correlations between X_1 and X_2, to come up with joint distribution function $F(x_1, x_2)$ as below:

$$F(x_1, x_2) = F_1(x_1)F_2(x_2)\{1 + \alpha[1 - F_1(x_1)][1 - F_2(x_2)]\}. \tag{7}$$

3   *SP* Formulization

It is supposed to investigate the general form of SP for model (2). Following Shirakura et al. (1996) in using their notations for preserving the similarity, yet in a more general context, some formulization may simplify the derivation of SP in the next section.

Let $\mathbf{A}(\boldsymbol{\zeta}) = [\mathbf{A}_1 : \mathbf{A}_{21}(\boldsymbol{\zeta})]$ and $\boldsymbol{M}(\boldsymbol{\zeta}) = \mathbf{A}(\boldsymbol{\zeta})'\boldsymbol{\Sigma}_e^{-1}\mathbf{A}(\boldsymbol{\zeta})$, SSE of model (2) can be written as $s^2(\boldsymbol{\zeta}) = \mathbf{y}'\mathbf{Q}(\boldsymbol{\zeta})\mathbf{y}$, where $\mathbf{Q}(\boldsymbol{\zeta}) = \boldsymbol{\Sigma}_e^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P}(\boldsymbol{\zeta}))\boldsymbol{\Sigma}_e^{-\frac{1}{2}}$, in which $\boldsymbol{P}(\boldsymbol{\zeta}) = \boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{A}(\boldsymbol{\zeta})M(\boldsymbol{\zeta})^{-1}\mathbf{A}(\boldsymbol{\zeta})'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}$. By taking $\mathbf{M} = \mathbf{A}_1'\boldsymbol{\Sigma}_e^{-1}\mathbf{A}_1$, $\mathbf{P} = \boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{A}_1\mathbf{M}^{-1}\mathbf{A}_1'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}$, $\mathbf{Z} = \boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{Y}$ and $h(\boldsymbol{\zeta}, \mathbf{y}) = \mathbf{Z}'(\mathbf{P}(\boldsymbol{\zeta}) - \mathbf{P})\mathbf{Z}$, $s^2(\boldsymbol{\zeta})$ can be written as:

$$s^2(\boldsymbol{\zeta}) = \mathbf{Z}'(\mathbf{I} - \mathbf{P})\mathbf{Z} - h(\boldsymbol{\zeta}, \mathbf{y}). \tag{8}$$

Accordingly, the minimization of $s^2(\boldsymbol{\zeta})$ for identifying the true model is equivalent to the maximization of $h(\boldsymbol{\zeta}, \mathbf{y})$ on $\boldsymbol{\zeta}$. For the noiseless case, $\sigma^2 = 0$, $P(s^2(\boldsymbol{\zeta}_0) < s^2(\boldsymbol{\zeta})) = P(h(\boldsymbol{\zeta}_0, \mathbf{y})) > P(h(\boldsymbol{\zeta}, \mathbf{y})) = 1$; however, for $\sigma^2 > 0$ it is less than 1. Shirakura et al. (1996) defined $SP = \min_{\boldsymbol{\zeta}_0 \in \xi_2} \min_{\boldsymbol{\zeta} \neq \boldsymbol{\zeta}_0} P(h(\boldsymbol{\zeta}_0, \mathbf{y}) > h(\boldsymbol{\zeta}, \mathbf{y}))$, and argued that a larger value of SP is the higher capability of search design in identifying the true model. For $\mathbf{B} = \mathbf{A}_{21}(\boldsymbol{\zeta})'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P})\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{A}_{21}(\boldsymbol{\zeta})$, $h(\boldsymbol{\zeta}, \mathbf{y})$ is:

$$h(\boldsymbol{\zeta}, \mathbf{y}) = \mathbf{Z}'(\mathbf{I} - \mathbf{P})\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{A}_{21}(\boldsymbol{\zeta})\mathbf{B}^{-1}\mathbf{A}_{21}(\boldsymbol{\zeta})'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P})\mathbf{Z}. \tag{9}$$

Consider the case $k = 1$ and suppose $\mathbf{a}(\zeta)$ is the column in $\mathbf{A}_2$ corresponding to ζ. By taking $r(\zeta) = \mathbf{a}(\zeta)'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P})\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{a}(\zeta)$, the $h(\boldsymbol{\zeta}, \mathbf{y})$ in (9) is expressed as

$$h(\boldsymbol{\zeta}, \mathbf{y}) = \left(\frac{\mathbf{a}(\zeta)'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P})\mathbf{Z}}{\sqrt{r(\zeta)}} \right)^2. \tag{10}$$

By inserting the true model (3) in (10) and from $\boldsymbol{\Sigma}_e^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P})\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{A}_1 = \mathbf{0}$,

we have

$$h(\boldsymbol{\zeta}, \mathbf{y}) = \left(\frac{\mathbf{a}(\zeta)'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P})\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{a}(\zeta_0)}{\sqrt{r(\zeta)}}\zeta_0 + \frac{\mathbf{a}(\zeta)'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}(\mathbf{I} - \mathbf{P})\boldsymbol{\Sigma}_e^{-\frac{1}{2}}}{\sqrt{r(\zeta)}}\mathbf{e} \right)^2$$

$$= \left(\mathbf{b}(\zeta)'\mathbf{b}(\zeta_0)\sqrt{r(\zeta_0)}\zeta_0 + \mathbf{b}(\zeta)'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{e} \right)^2,$$

in which $\mathbf{b}(\zeta) = (\mathbf{I} - \mathbf{P})\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{a}(\zeta)/\sqrt{r(\zeta)}$ and $\mathbf{b}(\zeta)'\mathbf{b}(\zeta) = 1$.

Now, by $Z = \mathbf{b}(\zeta)'\mathbf{b}(\zeta_0)\delta_0 + \mathbf{b}(\zeta)'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{e}$, where $\delta_0 = \sqrt{r(\zeta_0)}\zeta_0$, calculation of Z for $\zeta = \zeta_0$ gives $Z_0 = \delta_0 + \mathbf{b}(\zeta_0)'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{e}$. For $\zeta \neq \zeta_0$,

$$\begin{aligned} P(h(\zeta_0, \mathbf{y}) > h(\zeta, \mathbf{y})) &= P(Z_0^2 > Z^2) \\ &= P((Z_0 + Z)(Z_0 - Z) > 0) \\ &= P(X_1 > 0, X_2 > 0) + P(X_1 < 0, X_2 < 0), \quad (11) \end{aligned}$$

where X_1 and X_2 are as below:

$$X_1 = Z_0 - Z = (\mathbf{b}(\zeta_0) - \mathbf{b}(\zeta))'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{e} + (1 - \mathbf{b}(\zeta)'\mathbf{b}(\zeta_0))\delta_0, \quad (12)$$

$$X_2 = Z_0 + Z = (\mathbf{b}(\zeta_0) + \mathbf{b}(\zeta))'\boldsymbol{\Sigma}_e^{-\frac{1}{2}}\mathbf{e} + (1 + \mathbf{b}(\zeta)'\mathbf{b}(\zeta_0))\delta_0. \quad (13)$$

Distributions of X_1 and X_2 under the error distributions of model (2) will be given in the next section, while $(\mathbf{b}(\zeta_0) - \mathbf{b}(\zeta))'(\mathbf{b}(\zeta_0) + \mathbf{b}(\zeta)) = 0$ imply that X_1 and X_2 are uncorrelated.

4   *SP* for *SN/SNI* Search Model

Apriority, in model (2) let's assume that $\mathbf{e} \sim SN_N(\mathbf{0}, \sigma^2\boldsymbol{\Sigma}, \boldsymbol{\lambda})$, and let $\boldsymbol{\delta} = \boldsymbol{\lambda}/\sqrt{1 + \boldsymbol{\lambda}'\boldsymbol{\lambda}}$. Consider the variables X_1 and X_2 in (12) and (13), respectively, from (6) distributions of them are easily provided, i.e.,

$$X_1 \sim SN\left((1 - x(\zeta, \zeta_0))\delta_0, \sigma^2 v(\zeta, \zeta_0), \lambda_1^*\right)$$

and

$$X_2 \sim SN\left((1 + x(\zeta, \zeta_0))\delta_0, \sigma^2 w(\zeta, \zeta_0), \lambda_2^*\right),$$

where, $x(\zeta, \zeta_0) = \mathbf{b}(\zeta)'\mathbf{b}(\zeta_0)$, $x(\zeta, \zeta_0) \in [-1, 1]$, $v(\zeta, \zeta_0) = \mathbf{a}(\zeta, \zeta_0)'\mathbf{\Sigma}\mathbf{a}(\zeta, \zeta_0)$, $w(\zeta, \zeta_0) = \mathbf{b}(\zeta, \zeta_0)'\mathbf{\Sigma}\mathbf{b}(\zeta, \zeta_0)$, $\mathbf{a}(\zeta, \zeta_0) = \mathbf{\Sigma}_e^{-\frac{1}{2}}(\mathbf{b}(\zeta_0) - \mathbf{b}(\zeta))$, $\mathbf{b}(\zeta, \zeta_0) = \mathbf{\Sigma}_e^{-\frac{1}{2}}(\mathbf{b}(\zeta_0) + \mathbf{b}(\zeta))$, $\lambda_1^* = \frac{\delta_1^*}{(1-\delta_1^{*2})^{\frac{1}{2}}}$, $\delta_1^* = v(\zeta, \zeta_0)^{\frac{-1}{2}}\mathbf{a}(\zeta, \zeta_0)'\mathbf{\Sigma}^{\frac{1}{2}}\boldsymbol{\delta}$, $\lambda_2^* = \frac{\delta_2^*}{(1-\delta_2^{*2})^{\frac{1}{2}}}$, $\delta_2^* = w(\zeta, \zeta_0)^{\frac{-1}{2}}\mathbf{b}(\zeta, \zeta_0)'\mathbf{\Sigma}^{\frac{1}{2}}\boldsymbol{\delta}$, and $\mathbf{\Sigma}_e = \mathbf{\Sigma} - \frac{2}{\pi}\mathbf{\Sigma}^{\frac{1}{2}}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{\Sigma}^{\frac{1}{2}}$.

Remark 1. For the case in which $\mathbf{\Sigma} = \mathbf{I}$ and $\boldsymbol{\lambda} = \lambda\mathbf{j}$, we have $\mathbf{\Sigma_e} = \mathbf{I} - 2\left(\lambda^2/\pi(1 + N\lambda^2)\right)\mathbf{jj}'$ and $\mathbf{\Sigma_e^{-1}} = \mathbf{I} + h\mathbf{jj}'$, where $\mathbf{j}$ is the $N \times 1$ vector of ones and $h = 2\lambda^2/\{\pi(1 + N\lambda^2) - 2N\lambda^2\}$.

Now, we provide the *SP* for the distributional condition of Remark 1 in the following theorem.

Theorem 1. *Under* $\mathbf{e} \sim SN_N(\mathbf{0}, \sigma^2\mathbf{\Sigma}, \boldsymbol{\lambda})$, *the SP in (11) is provided and expressed as:*

$$G(x(\zeta, \zeta_0), v(\zeta, \zeta_0), w(\zeta, \zeta_0), \rho, \alpha) =$$
$$1 - F_{Z_1}(-\frac{1 - x(\zeta, \zeta_0)}{\sqrt{v(\zeta, \zeta_0)}}\sqrt{r(\zeta_0)}\rho) - F_{Z_2}(-\frac{1 + x(\zeta, \zeta_0)}{\sqrt{w(\zeta, \zeta_0)}}\sqrt{r(\zeta_0)}\rho)$$
$$+ 2F_{Z_1, Z_2}(-\frac{1 - x(\zeta, \zeta_0)}{\sqrt{v(\zeta, \zeta_0)}}\sqrt{r(\zeta_0)}\rho, -\frac{1 + x(\zeta, \zeta_0)}{\sqrt{w(\zeta, \zeta_0)}}\sqrt{r(\zeta_0)}\rho), \qquad (14)$$

where $\rho = \zeta_0/\sigma$, F_{Z_1} *and* F_{Z_2}, *are the cdfs of* $SN(\lambda_1^*)$ *and* $SN(\lambda_2^*)$, *respectively, and* F_{Z_1, Z_2} *is the FGM copula given in (7) in which* $x_1 = -\frac{1-x(\zeta,\zeta_0)}{\sqrt{v(\zeta,\zeta_0)}}\sqrt{r(\zeta_0)}\rho$ *and* $x_2 = -\frac{1+x(\zeta,\zeta_0)}{\sqrt{w(\zeta,\zeta_0)}}\sqrt{r(\zeta_0)}\rho$.

Proof. Let take Z_1 and Z_2 as: $Z_1 = \frac{X_1 - (1-x(\zeta,\zeta_0))\delta_0}{\sigma\sqrt{v(\zeta,\zeta_0)}}$ and $Z_2 = \frac{X_2 - (1+x(\zeta,\zeta_0))\delta_0}{\sigma\sqrt{w(\zeta,\zeta_0)}}$. Clearly, Z_1 and Z_2 are distributed as $SN(\lambda_1^*)$ and $SN(\lambda_2^*)$, respectively. Some calculations give us the following for *SP* in (11),

$$1 - P(Z_1 < -\frac{1 - x(\zeta, \zeta_0)}{\sqrt{v(\zeta, \zeta_0)}}\sqrt{r(\zeta_0)}\rho) - P(Z_2 < -\frac{1 + x(\zeta, \zeta_0)}{\sqrt{w(\zeta, \zeta_0)}}\sqrt{r(\zeta_0)}\rho)$$
$$+ 2P(Z_1 < -\frac{1 - x(\zeta, \zeta_0)}{\sqrt{v(\zeta, \zeta_0)}}\sqrt{r(\zeta_0)}\rho, Z_2 < -\frac{1 + x(\zeta, \zeta_0)}{\sqrt{w(\zeta, \zeta_0)}}\sqrt{r(\zeta_0)}\rho),$$

which can be easily written as (14), and the proof is complete . $\square$

It should be noted that $G(\cdot)$ function in (14) depends on the parameters ρ, $\boldsymbol{\lambda}$ and α and is also design-dependent. For normal Z_1 and Z_2, i.e. $\lambda_1^* = \lambda_2^* = 0$, the SP in (14) is reduced to a more simple form given in the following corollary.

Corollary 1. *If $\delta_1^* = \delta_2^* = 0$, then Z_1 and Z_2 follow $N(0,1)$. Therefore, the SP given in Theorem 1 is simplified as:*

$$
\begin{aligned}
G(x(\zeta,\zeta_0), v(\zeta,\zeta_0), w(\zeta,\zeta_0), \rho) = {} & 1 - \Phi\big(\frac{1 - x(\zeta,\zeta_0)}{\sqrt{v(\zeta,\zeta_0)}}\sqrt{r(\zeta_0)}\rho\big) \\
& - \Phi\big(\frac{1 + x(\zeta,\zeta_0)}{\sqrt{w(\zeta,\zeta_0)}}\sqrt{r(\zeta_0)}\rho\big) \\
& + 2\Phi\big(\frac{1 - x(\zeta,\zeta_0)}{\sqrt{v(\zeta,\zeta_0)}}\sqrt{r(\zeta_0)}\rho\big) \\
& \times \Phi\big(\frac{1 + x(\zeta,\zeta_0)}{\sqrt{w(\zeta,\zeta_0)}}\sqrt{r(\zeta_0)}\rho\big). \qquad (15)
\end{aligned}
$$

Remark 2. The $G(\cdot)$ function in (15) is symmetric about ρ for $x(\zeta,\zeta_0) \in [-1,1]$, $v(\zeta,\zeta_0) > 0$, $w(\zeta,\zeta_0) > 0$, and $G(x(\zeta,\zeta_0), v(\zeta,\zeta_0), w(\zeta,\zeta_0), 0) = 0.5$. Furthermore, $G(\cdot)$ is strictly increasing on $\rho > 0$ for fixed $x(\zeta,\zeta_0) \in (-1,1)$, $v(\zeta,\zeta_0) > 0$ and $w(\zeta,\zeta_0) > 0$.

Remark 3. Although, the expression (15) and the properties given in Remark 2 is very similar to what was obtained by Shirakura et al. (1996), it is different in quiddity due to be obtained under a more general underlying model with *SN* error.

Remark 4. Three designs D_1, D_2 and D_3, given in the next section, comprise the condition in Corollary 1. This provides application of (15) on such the designs.

Following Shirakura et al. (1996) in characterizing and deriving the properties of SP, the results in the following corollaries will be shown that in addition to some new findings all of the former properties in the above study are preserved, yet under more general present conditions.

Corollary 2. *Under the conditions of Remark 1 and Corollary 1, the followings are true.*

   i) The function $G(x(\zeta,\zeta_0), v(\zeta,\zeta_0), w(\zeta,\zeta_0), \rho)$ in (15) is symmetric about λ.

ii) If $\mathbf{b}(\zeta)'\mathbf{j} = 0$ *and* $\mathbf{b}(\zeta_0)'\mathbf{j} = 0$, *then*

1. $v(\zeta, \zeta_0) = 2(1 - x(\zeta, \zeta_0))$ *and* $w(\zeta, \zeta_0) = 2(1 + x(\zeta, \zeta_0))$, *and*
2. *The function* $G(x(\zeta, \zeta_0), v(\zeta, \zeta_0), w(\zeta, \zeta_0), \rho)$ *in (15) is reduced to*

$$G(x(\zeta, \zeta_0), d) = 1 - \Phi(\sqrt{1 - x(\zeta, \zeta_0)}d) - \Phi(\sqrt{1 + x(\zeta, \zeta_0)}d)$$
$$+ 2\Phi(\sqrt{1 - x(\zeta, \zeta_0)}d)\Phi(\sqrt{1 + x(\zeta, \zeta_0)}d), \quad (16)$$

where $d = \sqrt{\frac{r(\zeta_0)}{2}}\rho$.

Corollary 3. *The function* $G(x(\zeta, \zeta_0), d)$ *in (16) has the following properties:*

i) $G(x(\zeta, \zeta_0), d) = G(-x(\zeta, \zeta_0), d)$ *for* $x(\zeta, \zeta_0) \in [-1, 1]$.

ii) For a fixed $d \neq 0$, $G(x(\zeta, \zeta_0), d)$ *is strictly decreasing on* $x(\zeta, \zeta_0) \in (-1, 1)$.

Now, let's take $\mathbf{\Sigma} = \mathbf{I}$, $\boldsymbol{\lambda} = \lambda\mathbf{j}$ and assume that the conditions of Corollaries 1 and 2(ii) are satisfied. Furthermore, for a fixed ζ_0, let $x(\zeta_0) = max_{\zeta \neq \zeta_0}|\mathbf{b}(\zeta)'\mathbf{b}(\zeta_0)|$. Then, from Corollaries 2 and 3, the *SP* can be expressed as

$$P(\rho, \lambda) = min_{\zeta_0} G\left(x(\zeta_0), \sqrt{\frac{r(\zeta_0)}{2}}\rho\right). \quad (17)$$

Remark 2 give $P(\rho, \lambda) = P(-\rho, \lambda)$, thus it can be focused on $P(\rho, \lambda)$ for only $\rho > 0$. Furthermore, $P(\rho, \lambda)$ is strictly increasing for $\rho > 0$ and $P(\rho, \lambda) > 0.5$. The following definition is utilized for ranking *SD*s based on the design-dependent *SP*-criterion.

Definition 1. Suppose $P_{P_1}(\rho, \lambda)$ and $P_{P_2}(\rho, \lambda)$ are the *SP* for search designs P_1 and P_2, with N treatments, respectively. Then P_1 is said to be better than P_2 for detecting non-zero effect, if $P_{P_1}(\rho, \lambda) > P_{P_2}(\rho, \lambda)$ for any $\rho > 0$. That is, P_1 identifies the true model (3) with a higher probability than P_2.

Corollary 4. *If* $r(\zeta)$ *is constant for any* $\zeta \in \boldsymbol{\xi}_2$, *then from Corollary 3(ii)* $P(\rho, \lambda)$ *in (17), as a function of* ρ *and* λ, *can be written as*

$$P_D(\rho, \lambda) = G\left(x^*, \sqrt{\frac{r(\zeta)}{2}}\rho\right),$$

where $x^* = max_{\zeta_0} x(\zeta_0)$.

Corollary 4 will be used in the application Section 5.

Now, suppose that in model (2) $\mathbf{e} \sim SNI_N(\mathbf{0}, \sigma^2\mathbf{\Sigma}, \mathbf{\lambda}; H)$, then from (6) we have:

$$X_1 \sim SNI_1\left((1 - x(\zeta, \zeta_0))\delta_0, \sigma^2 v(\zeta, \zeta_0), \lambda_1^*; H\right)$$

and

$$X_2 \sim SNI_1((1 + x(\zeta, \zeta_0))\delta_0, \sigma^2 w(\zeta, \zeta_0), \lambda_2^*; H).$$

Note that for finite $E[U^{-1/2}]$ and $E[U^{-1}]$, we have $\mathbf{\Sigma_e} = E[U^{-1}]\mathbf{\Sigma} - \frac{2}{\pi}E^2[U^{-1/2}]\mathbf{\Sigma}^{1/2}\mathbf{\delta\delta}'\mathbf{\Sigma}^{1/2}$. Under the above scenario, the *SP* in (11) is the same as what was obtained in (14), in which F_{Z_1} and F_{Z_2} are the *cdf*s of $SNI_1(0, 1, \lambda_1^*; H)$ and $SNI_1(0, 1, \lambda_2^*; H)$, respectively, and hence F_{Z_1, Z_2} is the *FGM* copula in (7) based on these *SNI* marginal distributions.

5　An Application

The search model in this section, is assumed to have *SN* and skew-t error distributions. The adequacy of the proposed *SP*-criterion in comparing search performance of designs is assessed by applying it on three 12-run *SD*s D_1, D_2 and D_3, given in Talebi and Esmailzadeh (2015) for a 2^4 factorial experiment. In model (1), let $\mathbf{\xi}_1$ be the vector of the overall mean and main effects, and $\mathbf{\xi}_2$ be the vector of 2- and 3-factor interactions and assume $k = 1$. Furthermore, assume that 4-factor interaction is zero. In the following examples the capability of the proposed criterion in ranking the MEP.1's with respect to their search performances is presented. The calculations were done by Matlab and using "*sn*" package in R software.

Let $\mathbf{e} \sim SN_{12}(\mathbf{0}, \sigma^2\mathbf{I}, \mathbf{\lambda})$. For $\alpha = 1/6$, ρ in [-1,1] with increments of 0.1 and $\mathbf{\delta} = (0.04, 0.17, 0.28, 0.30, 0.35, 0.39, 0.15, 0.22, 0.43, 0.11, 0.40, 0.29)'$, (14) was used to calculate *SP*, for D_1, D_2 and D_3. Note that, based on our findings, results are robust to the different values of $\alpha \in [-1, 1]$. The results in Tables 1 and 2, show that for all values of ρ, D_2 is less efficient than both D_1 and D_3, while D_1 is better than D_3 for ρ in [-1,-0.5] and [0.4, 1]. For ρ in [-0.4,0.2], which is in the vicinity of zero, D_3 is better than D_1. This means D_3 is superior for small effect sizes.

For the special case $\mathbf{\lambda} = \mathbf{j}$, we have $\mathbf{\lambda}_1^* = \mathbf{\lambda}_2^* = 0$ and $\mathbf{b}(\zeta)'\mathbf{j} = 0$ for any ζ and for all three above designs. Note that, $r(\zeta)$ is not constant for any ζ. We used (17) to calculate the *SP*s for different values of ρ. The results are

Table 1. SP under SN model for $\rho < 0$.

ρ	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
D_1	0.9361	0.9106	0.8770	0.8341	0.7815	0.7211	0.6573	0.5974	0.5496	0.5207
D_2	0.9196	0.8897	0.8519	0.8055	0.7514	0.6902	0.6278	0.5736	0.5341	0.5116
D_3	0.9338	0.9080	0.8745	0.8319	0.7802	0.7208	0.6579	0.5984	0.5508	0.5232

Table 2. SP under SN model for $\rho \geqslant 0$.

ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
D_1	0.5176	0.5201	0.5480	0.5982	0.6622	0.7308	0.7886	0.8381	0.8787	0.9108	0.9356
D_2	0.5096	0.5169	0.5321	0.5695	0.6229	0.6844	0.7465	0.8035	0.8523	0.8918	0.9224
D_3	0.5181	0.5233	0.5515	0.5982	0.6567	0.7204	0.7808	0.8323	0.8746	0.9080	0.9337

given in Table 3, for D_1, D_2 and D_3. It can be seen that for all values of ρ, D_1 performs better than D_2 and for $\rho < 0.8$, D_1 is better than D_3. For $\rho < 0.6$ D_2 performs better than D_3. Then, as ρ gets values greater than 0.5, D_3 overcomes D_2. As ρ tends to 1, the SP for D_1 is similar to D_3 and they are better than D_2. Such the orders preserve for larger values of ρ, although all get closer.

Hereafter, let $\boldsymbol{\xi}_2$ is restricted to the set of 2-factor interactions and 3- and 4-factor interactions are all assumed to be zero. Suppose $\boldsymbol{\lambda} = \lambda\mathbf{j}$ and let take $\lambda = 1$, then for every 2- factor interaction, say ζ, $\boldsymbol{\lambda}_1^* = \boldsymbol{\lambda}_2^* = 0$ and $\mathbf{b}(\zeta)'\mathbf{j} = 0$, for D_1, D_2 and D_3. Moreover, for every ζ, $r(\zeta) = 10.67$ for both D_1 and D_3 and is 9.333 for D_2. Therefore, from Corollary 4 we have $P_{D_1}(\rho) = P_{D_2}(\rho) = G(0.5, 5.33\rho)$ and $P_{D_2}(\rho) = G(0.4286, 4.66\rho)$. The above are valid for other values of λ. Maple software plot command has been run to plot SP against ρ for designs D_1, D_2 and D_3, and is shown in figure 1. This plot indicates that D_1 and D_3 have the same search ability and they are better than D_2. Now, for skew-t model let assume $\mathbf{e} \sim ST_{12}(\mathbf{0}, \sigma^2\mathbf{I}, \boldsymbol{\lambda}, 4)$.

Table 3. SP under SN model with $\boldsymbol{\lambda} = \mathbf{j}$, for some values of ρ.

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
D_1	0.5144	0.5548	0.6135	0.6805	0.7468	0.8062	0.8557	0.8947	0.9245	0.9467
D_2	0.5132	0.5505	0.6055	0.6698	0.7353	0.7956	0.8471	0.8886	0.9205	0.9444
D_3	0.5126	0.5483	0.6019	0.6661	0.7330	0.7963	0.8515	0.8947	0.9245	0.9467
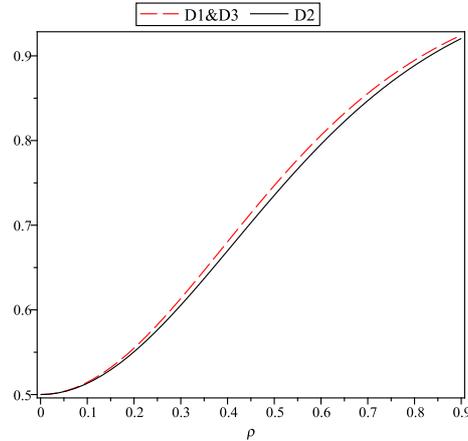
Figure 1. SP versus ρ for D_1, D_2 and D_3 under SN model with $\boldsymbol{\lambda} = \mathbf{j}$.

Let take $\boldsymbol{\delta} = (0.04, 0.17, 0.28, 0.30, 0.35, 0.39, 0.15, 0.22, 0.43, 0.11, 0.40, 0.29)'$ and $\alpha = 1/6$. Based on (14) with skew-t marginal distributions, SPs are calculated for different values of ρ. The findings for D_1, D_2 and D_3 are given in Tables 4 and 5. The results show that the performance behavior depends on ρ values. D_3 is the superior design for ρ in [-0.6,0] except for $\rho = -0.1$ and D_2 is less efficient than both D_1 and D_3 for values of ρ in [-1,0.8]. It should be noted that for all of the above comparisons the values of criteria are very similar and ranking order of designs is minor, because these 3 designs are highly efficient designs.

6   Conclusion

SNI model is more reliable in modeling the real data, due to possible skewness, heavy-tails and correlated error components; moreover, SNI model includes the normal model as a special case. Considering such the more general model, we came up with an intricate and unexplicit distributional format of SP-criterion; however, for SN model with $\boldsymbol{\lambda} = \lambda\mathbf{j}$ and subject to some conditions, we derived an explicit SP-criterion which is different from SP in Shirakura et al. (1996) in quiddity, but carries all its properties. In application section it was shown that the performance of the new proposed SP-criterion depends on the values of components in $\boldsymbol{\lambda}$. This revealed that how ignoring the skewness may cause to misleading values of SP and

Table 4. SP under ST model for $\rho < 0$.

ρ	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
D_1	0.8759	0.8492	0.8170	0.7787	0.7343	0.6850	0.6335	0.5846	0.5454	0.5228
D_2	0.8713	0.8432	0.8092	0.7693	0.7234	0.6732	0.6205	0.5716	0.5359	0.5201
D_3	0.8752	0.8486	0.8166	0.7786	0.7347	0.6859	0.6346	0.5857	0.5458	0.5224

Table 5. SP under ST model for $\rho \geqslant 0$.

ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
D_1	0.5177	0.5247	0.5528	0.5956	0.6458	0.6970	0.7450	0.7876	0.8243	0.8550	0.8805
D_2	0.5103	0.5177	0.5395	0.5772	0.6271	0.6815	0.7343	0.7785	0.8168	0.8492	0.8761
D_3	0.5195	0.5224	0.5464	0.5867	0.6357	0.6870	0.7357	0.7795	0.8173	0.8491	0.8756

possibly failing in identifying the true model as well as the superior design. Although this votes in favor of the underlying model in this research against the normal model, the SP values could be independent of $\boldsymbol{\lambda}$ for special case $\boldsymbol{\lambda} = \lambda\mathbf{j}$, namely the values of SP are the same as SP for normal model $\boldsymbol{\lambda} = \mathbf{0}$. The condition $\mathbf{b}(\zeta)'\mathbf{j} = 0$ and $\mathbf{b}(\zeta_0)'\mathbf{j} = 0$ in Corollary 2(ii) depends on the structure of design, which may lead to some simplification in SP. Designs D_1, D_2 and D_3 which have been taken in Section 4 carry such a structure. That is, the mentioned condition, by which we came up with some simple and explicit form of criterion, is meaningful.

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