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# Goodness of Fit Tests based on Information Criterion for Randomly Censored Data

Fatemeh Omidi, Arezou Habibirad* and Vahid Fakoor

Ferdowsi University of Mashhad

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**Abstract.** We propose two goodness of fit test statistics based on Cumulative Kullback–Leibler (CKL) information and Cumulative residual Kullback–Leibler (CRKL) information for exponential distributions with unknown parameter and randomly censored data. Koziol and Green introduced the Cramér-von Mises statistic with randomly censored data for a simple hypothesis based on the Kaplan–Meier product limit of the distribution function. We use their idea to obtain test statistics based on CKL and CRKL for a randomly censored exponential distribution with estimated parameters. The power of the proposed tests for testing exponentiality is compared with the test statistic based on the empirical distribution function using the opinion of Koziol and Green. A simulation study is performed under a special censorship model introduced by Koziol and Green. Simulation studies show a relatively high power of proposed test statistics in many alternatives.

**Keywords.** Cramér-von Mises statistic; cumulative Kullback–Leibler information; cumulative residual Kullback–Leibler information; exponential distribution; Kaplan–Meier estimator; randomly censored data.

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* Corresponding author

# 1 Introduction

In survival studies, one frequently wants to obtain inferences about an unknown survivor function $\overline{F}(\cdot)$. In the statistical analysis of lifetime data, the goodness-of-fit (gof) test procedures are important to select the proper distribution of data. Classical gof tests are usually defined based on graphical analysis, moments such as skewness or kurtosis, chi-squared type, the empirical distribution function (EDF), or regression and correlations. Many studies are performed such that these analysis are expanded to censored data.
This censorship model was researched by Kaplan and Meier (1958), Efron (1967), Breslow and Crowley (1974). Furthermore, the gof tests for randomly censored data have been studied by many authors, for example, Shapiro and Francia (1972), Koziol and Green (1976), Aalen and Hoem (1978), Hall and Wellner (1980), Koziol (1980), Nair (1981), Chen et al. (1982, 1983), Akritas (1988), Hollander and Pena (1992), Wang and Zheng (1997), Lu and Cheng (2007), Koziol (2009).
Kim (2012) extended Kolmogorov–Smirnov statistic using the idea of Koziol–Green for an exponential distribution with randomly censored data and an unknown scale parameter. Kim (2017, 2019) researched the EDF statistic for randomly censored Weibull and normal distribution, respectively, when some parameters are unknown. In this paper, we first propose two new statistics based on CKL information and CRKL information for a complete sample. Then we apply them to test exponential distributions with estimated parameters when data are randomly censored. The Kaplan–Meier estimator and censoring distribution are applied as in Koziol and Green (1976).
Koziol (1980) tested exponential distribution using Kolmogorov-Smirnov, Kuiper and Cramer-von Mises statistics with censored data. Similar to Koziol (1980), we have tested exponentiality with censored data but we have introduced two new test statistics for test exponentiality. In comparison with Koziol, we have shown that the introduced test statistics perform better in many of the considered alternative examples.
The article is structured as follows: In Section 2, we introduce two new test statistics based on CKL information and CRKL information for a complete sample and generalize them to randomly censored data. In Section 3, we provide a simulation study and make some power comparisons to illustrate the performance of the proposed procedure. An example of randomly censored data from Prostat research (Hollander and Proschan, 1979) is explained in Section 4. The results are summarized in Section 5.

# 2 Proposed Tests

This section is divided into two parts. In the first part, we review divergence measures. In the second part, we introduce new divergence measures type and use them to make test statistics.

## 2.1 Divergence Measures

The concept of the entropy was described by Shannon (1948) to measure the uncertainty of a random variable. It is formally defined as

$$H(f) = -\int_{-\infty}^{+\infty} f(x) \ln f(x) \mathrm{d}x,$$

where $f$ is the probability density function (pdf) of a continuous random variable $X$.
Using the survival function instead of the pdf, Rao et al. (2004) defined the cumulative residual entropy (CRE) by

$$CRE(F) = -\int_{0}^{\infty} \overline{F}(x) \ln \overline{F}(x) \mathrm{d}x.$$

The Kullback–Leibler (KL) (1951) information is a nonsymmetric measure of the discrepancy between two distributions $F$ and $G$ with pdfs $f$ and $g$, respectively, which is defined as

$$KL(F : G) = -\int_{-\infty}^{+\infty} f(x) \ln \frac{f(x)}{g(x)} \mathrm{d}x.$$

Ebrahimi et al. (1992), Choi et al. (2004), and Gurevich and Davidson (2008) have extended tests of fit based on KL information. Baratpour and Habibirad (2012) introduced the CRKL information and applied it for gof tests. This measure calculates the discrepancy between two continuous distributions based on CRE, which is described as

$$CRKL(F : G) = \int_{0}^{\infty} \overline{F}(x) \ln \frac{\overline{F}(x)}{\overline{G}(x)} \mathrm{d}x - [E(X) - E(Y)],$$

where $\overline{F}(\cdot)$ and $\overline{G}(\cdot)$ are the survival functions of $X$ and $Y$, respectively. Another extension based on the cumulative distribution function (cdf) called

CKL information was proposed by Park et al. (2012), which is introduced as

$$CKL(F:G) = \int_0^\infty F(x) \ln \frac{F(x)}{G(x)} \mathrm{d}x - [E(Y) - E(X)].$$

Clearly, using $\ln x \leq x - 1$ for $x > 0$, we obtain that $CKL(F:G) \geq 0$, that $CRKL(F:G) \geq 0$, and that the equalities hold if and only if $F = G$.
These information criteria are used for tests of fit for different types of data and distributions by Chamany and Baratpour (2014), Baratpour and Habibirad (2016), Park et al. (2018), Zohrevand et al. (2020), and others.

## 2.2 Simple GOF Tests

In this section, we first introduce two new divergence criteria that are defined based on CKL information and CRKL information. Then, using these divergence criteria, we develop two test statistics when data are randomly censored.

**Definition 1.** If $X$ and $Y$ are two continuous and nonnegative random variables, respectively, with cdfs $F(\cdot)$ and $G(\cdot)$, then the divergence criterion between these distributions can be describe as follows:

$$T_1(F, G) = \int \overline{F}(x) \ln \frac{\overline{F}(x)}{\overline{G}(x)} \mathrm{d}\overline{G}(x). \tag{1}$$

**Definition 2.** If $X$ and $Y$ are two continuous and nonnegative random variables, respectively, with cdfs $F(\cdot)$ and $G(\cdot)$, then the divergence criterion between these distributions can be introduced as follows:

$$T_2(F, G) = \int F(x) \ln \frac{F(x)}{G(x)} \mathrm{d}G(x). \tag{2}$$

We consider testing

$$H_0 : F = F_0, \quad \text{vs.} \quad H_1 : F \neq F_0, \tag{3}$$

where $F_0$ is a specified continuous distribution. We introduce the test statistics $T_{n1}$ and $T_{n2}$ based on $T_1(F, G)$ and $T_2(F, G)$, respectively, when there is no censoring. These test statistics can be used for different types of data such as complete data, randomly censored data, interval– censored data, and so on.
Let $X_1, \ldots, X_n$ be an independently and identically distributed (iid) ran-

dom sample from cdf $F_0$ with the EDF $F_n$. We propose a test statistic using introduced in (1) as follows

$$T_{n1} = \sqrt{n} \int_0^{\tau} \overline{F}_n(x) \ln \frac{\overline{F}_n(x)}{\overline{F}_0(x)} \mathrm{d}\overline{F}_0(x), \tag{4}$$

where $\overline{F}_0(x) = 1 - F_0(x)$, $\overline{F}_n(x) = 1 - F_n(x)$, and $\tau = \sup\{x; \overline{F}_0(x) > 0\}$.
Let $X_1, \ldots, X_n$ be an iid random sample from cdf $F_0$ with the EDF $F_n$. We defined a test statistic using introduced in (2) as follows

$$T_{n2} = \sqrt{n} \int_0^{\tau} F_n(x) \ln \frac{F_n(x)}{F_0(x)} \mathrm{d}F_0(x), \tag{5}$$

where $\tau = \sup\{x; \overline{F}_0(x) > 0\}$.
Let $X_1, \ldots, X_n$ be lifetimes with distribution function (df) $F$; moreover, let $C_1, \ldots, C_n$ display censoring times with df $G$, and the random vectors $X_1, \ldots, X_n$ and $C_1, \ldots, C_n$ (non-informative censoring assumption) be independent for all $n$. In this paper, we assume that the randomly censored sample is displayed by $n$ iid random pairs $O_i$=($Y_i$, $\delta_i$), $i = 1, \ldots, n$, where

$$Y_i = \min(X_i, C_i) = \begin{cases} X_i, & X_i \leq C_i, \\ C_i, & X_i > C_i, \end{cases} \qquad \delta_i = \begin{cases} 1, & X_i \leq C_i, \\ 0, & X_i > C_i. \end{cases} \tag{6}$$

We consider that $Y_i$'s are the ordered observations without loss of generality for the observed random pairs $(Y_i, \delta_i)$ in (6). We can obtain the probability integral transformation for test statistics in (4) and (5) when $F_0$ is fully specified in the null hypothesis. Since the EDF is not fully clear for censored data, we apply the nonparametric maximum likelihood estimator (NPMLE) for the lifetime distribution $F$, that is, the Kaplan–Meier estimator $\widehat{F}_n(t)$. The Kaplan–Meier estimator was researched by Kaplan and Meier (1958), Efron (1967), Breslow and Crowley (1974), and Meier (1975) can be defined as

$$1 - \widehat{F}_n(t) = \prod_{Y_i \leq t} (\frac{n-i}{n-i+1})^{\delta_i}.$$

The Kaplan–Meier estimator is rewritten as

$$1 - \widehat{p}_j = \prod_{i \leq j} \left( \frac{n-i}{n-i+1} \right)^{\delta_i}. \tag{7}$$

As mentioned, when we encounter with a censored sample, the complete sample $X_1, \ldots, X_n$ and $F_n$ are not accessible. Hence, we replace $F_n$ by the Kaplan–Meier estimator $\widehat{F}_n$ in (4) and (5), and we obtain the test statistics $\widehat{T}_{n1}$ and $\widehat{T}_{n2}$ as

$$\widehat{T}_{n1} = \sqrt{n} \int_0^\tau \overline{\widehat{F}}_n(x) \ln \frac{\overline{\widehat{F}}_n(x)}{\overline{F}_0(x)} \mathrm{d}\overline{F}_0(x), \tag{8}$$

where $\overline{\widehat{F}}_n(x) = 1 - \widehat{F}_n(x)$, and

$$\widehat{T}_{n2} = \sqrt{n} \int_0^\tau \widehat{F}_n(x) \ln \frac{\widehat{F}_n(x)}{F_0(x)} \mathrm{d}F_0(x). \tag{9}$$

In following theorems, we show that $\widehat{T}_{n1}$ in (8) and $\widehat{T}_{n2}$ in (9) asymptotically have the Gaussian distribution.

***Theorem* 1.** *Consider randomly censored data* (6) *with a continuous df* $F$ *such that* $(1 - G) = (1 - F)^\beta$ *with* $\beta$ *a positive constant, where* $\beta$ *is called the censoring parameter. Also, we suppose for* $0 \leq x \leq \tau$ *that*

$$n^{1/2}(\widehat{F}_n(x) - F(x)) = O_p(1),$$

*where* $O_p(1)$ *means bounded in probability. Then, under* $H_0$*, we have*

$$\lim_{n\to\infty} \sup_{0<x<\infty} |P\{\widehat{T}_{n1} \leq x | O^n\} - P\{V \leq x\}| = 0,$$

*in probability, where* $O^n = \{ O_i; i = 1, \ldots, n\}$ *and* $V(x)$ *is a Gaussian process with mean zero and*

$$\sigma^2 = \int_0^\tau \int_0^\tau E\{\Gamma(x)\Gamma(y)\} \mathrm{d}\overline{F}(x) \mathrm{d}\overline{F}(x). \tag{10}$$

**Proof.** Using the Taylor expansion of the logarithm function, we have

$$\begin{aligned}
\widehat{T}_{n1} &= \sqrt{n} \int_0^\tau \overline{\widehat{F}}_n(x) \ln \frac{\overline{\widehat{F}}_n(x)}{\overline{F}_0(x)} \mathrm{d}\overline{F}_0(x) \\
&= \sqrt{n} \left[ \int_0^\tau \overline{\widehat{F}}_n(x) \frac{1}{\overline{\widehat{F}}^*_n(x)} \left( \overline{\widehat{F}}_n(x) - \overline{F}_0(x) \right) \mathrm{d}\overline{F}_0(x) \right],
\end{aligned}$$

where $\min\{\overline{F}_0(x), \overline{\widehat{F}}_n(x)\} \leq \overline{\widehat{F}}^*_n(x) \leq \max\{\overline{F}_0(x), \overline{\widehat{F}}_n(x)\}$ for $0 \leq x \leq \tau$. Using Theorem (2.1) in Koziol and Green (1976), we can conclude that $\sqrt{n}\left(\widehat{F}_n(x) - F_0(x)\right)$ converges weakly to a Gaussian process $\Gamma(x)$ with the mean zero and covariance function given in (2.3) Koziol and Green (1976). Since $\widehat{T}_{n1} = \int_0^\tau \Gamma(x)dx$ and $V = \int_0^\tau \Gamma(x)dx$ is a Gaussian random variable with mean zero and variance

$$\sigma^2 = \int_0^\tau \int_0^\tau E\{\Gamma(x)\Gamma(y)\}\mathrm{d}\overline{F}(x)\mathrm{d}\overline{F}(x). \tag{11}$$

Thus, theorem is proved. □

***Theorem 2.*** *Consider randomly censored data* (6) *with a continuous df* $F$ *such that* $(1-G) = (1-F)^\beta$ *with* $\beta$ *a positive constant, where* $\beta$ *is called the censoring parameter. Also, we suppose for* $0 \leq x \leq \tau$ *that*

$$n^{1/2}(\widehat{F}_n(x) - F(x)) = O_p(1),$$

*Then, under* $H_0$*, we have*

$$\lim_{n\to\infty} \sup_{0<x<\infty} |P\{\widehat{T}_{n2} \leq x|O^n\} - P\{V \leq x\}| = 0,$$

*in probability, where* $V(x)$ *is a Gaussian process with mean zero and*

$$\sigma^2 = \int_0^\tau \int_0^\tau E\{\Gamma(x)\Gamma(y)\}\mathrm{d}\overline{F}(x)\mathrm{d}\overline{F}(x). \tag{12}$$

**Proof.** The proof is similar to the proof of Theorem 1 and then it is omitted. □

Now, let $X_1, \ldots, X_n$ be from a uniform distribution on $(0, 1)$ and $F_0$ is the uniform distribution on $(0, 1)$. Using the opinion of Koziol and Green (1976) for randomly censored data, we can obtain the test statistics in (8) and (9) as follows

$$\widehat{T}_{n1} = \sqrt{n}\int_0^1 \overline{\widehat{F}}_n(t) \ln \frac{\overline{\widehat{F}}_n(t)}{t}\mathrm{d}t, \tag{13}$$

$$\widehat{T}_{n2} = \sqrt{n}\int_0^1 \widehat{F}_n(t)\ln\frac{\widehat{F}_n(t)}{t}\mathrm{d}t. \tag{14}$$

These test statistics measure the difference between $\widehat{F}_n$ and $U(0,1)$. By Koziol and Green (1976), the statistics in (13) and (14) can be defined as

$$\begin{aligned}\widehat{T}_{n1} &= \sqrt{n}\sum_{i=1}^{n+1}(Y_i - Y_{i-1})(1-\widehat{F}_n(Y_i))\ln(1-\widehat{F}_n(Y_i))\\ &\quad - \sqrt{n}\sum_{i=1}^{n+1}(1-\widehat{F}_n(Y_i))(Y_i\ln Y_i - Y_i - Y_{i-1}\ln(Y_{i-1}) + Y_{i-1}),\end{aligned} \tag{15}$$

$$\begin{aligned}\widehat{T}_{n2} &= \sqrt{n}\sum_{i=1}^{n+1}(Y_i - Y_{i-1})(\widehat{F}_n(Y_i))\ln(\widehat{F}_n(Y_i))\\ &\quad - \sqrt{n}\sum_{i=1}^{n+1}\widehat{F}_n(Y_i)(Y_i\ln Y_i - Y_i - Y_{i-1}\ln(Y_{i-1}) + Y_{i-1}),\end{aligned} \tag{16}$$

where $Y_0 = 0$ and $Y_{n+1} = 1$.
In this article, we intend to test if the lifetimes $X_1, \ldots, X_n$ with df $F$ follow an exponential distribution $\varepsilon(\lambda)$ for some $\lambda$. Therefore, $F_0$ in the null hypothesis in (3) is

$$F_0(x) = 1 - \exp(-\lambda x), \qquad x > 0.$$

We suppose that $Y_i = \min(X_i, C_i)$ are the observed random variables. We can consider

$$U_i = F_0(Y_i) = \min(F_0(X_i), F_0(C_i))$$

and use the test statistics (15) and (16) for the simple null hypothesis with the parameter $\lambda$ given. Therefore, we can write

$$H_0 : F(x) = F_0(x,\lambda) = 1 - \exp(-\lambda x), \qquad for\ some\ \ \lambda > 0.$$

Obviously, the null hypothesis is composite and contains some unknown parameter. Thus, we require to estimate the unknown parameter $\lambda$. Kim (2012) obtained MLEs $\lambda$ and $\beta$,

$$\widehat{\lambda} = \frac{\overline{\delta}}{\overline{Y}}, \qquad \widehat{\beta} = \frac{1-\overline{\delta}}{\overline{\delta}}.$$

If we replace $\lambda$ by its MLE, then we can take

$$\widehat{U} = F_0(Y, \widehat{\lambda}) = \widehat{F}_0(Y).$$

If we suppose that $\widehat{F}_n(\cdot)$ is the Kaplan–Meier estimator of $\widehat{U}_i = \widehat{F}_0(Y_i)$, then describe the similar test statistics $\widehat{T}^*_{n1}$ and $\widehat{T}^*_{n2}$ as in (15) and (16)

$$\begin{aligned}\widehat{T}^*_{n1} = &\sqrt{n}\sum_{i=1}^{n+1}(\widehat{U}_i - \widehat{U}_{i-1})(1-\widehat{F}_n(\widehat{U}_i))\ln(1-\widehat{F}_n(\widehat{U}_i))\\ &-\sqrt{n}\sum_{i=1}^{n+1}(1-\widehat{F}_n(\widehat{U}_i))\Big(\widehat{U}_i\ln\widehat{U}_i - \widehat{U}_i - \widehat{U}_{i-1}\ln(\widehat{U}_{i-1}) + \widehat{U}_{i-1}\Big),\end{aligned} \tag{17}$$

$$\begin{aligned}\widehat{T}^*_{n2} = &\sqrt{n}\sum_{i=1}^{n+1}(\widehat{U}_i - \widehat{U}_{i-1})(\widehat{F}_n(\widehat{U}_i))\ln(\widehat{F}_n(\widehat{U}_i))\\ &-\sqrt{n}\sum_{i=1}^{n+1}\widehat{F}_n(\widehat{U}_i)\Big(\widehat{U}_i\ln\widehat{U}_i - \widehat{U}_i - \widehat{U}_{i-1}\ln(\widehat{U}_{i-1}) + \widehat{U}_{i-1}\Big),\end{aligned} \tag{18}$$

where $\widehat{U}_0 = 0$ and $\widehat{U}_{n+1} = 1$.
We may choose the percentiles of $V$ as the critical value when the test statistic $\widehat{T}^*_{n1}$ in (17) is used as the test statistic for testing (3). Specially, at $\alpha 100\%$ significance level we

$$Reject\ \ H_0 \quad if \quad |\widehat{T}^*_{n1}| \geq C_{\alpha/2}, \tag{19}$$

where $P(|V| \geq C_{\alpha/2}) = \alpha$. Therefore, we apply only one sample for the decision (19) and call it NL1-test.
Suppose we choose repeatedly $M$ samples $\widehat{U}^*_{j1}, \ldots, \widehat{U}^*_{jn}$ from $\widehat{F}_n(\widehat{U})$, $j = 1, \ldots, M$, and compute $\widehat{T}^*_{n1j}$ for each of these samples, then frequent event of $|\widehat{T}^*_{n1j}| \geq C_{\alpha/2}$ should lead to the rejection of $H_0$. Using this opinion, we present another test as follows

$$\bar{V} = M^{-1}\sum_{j=1}^{M} I\{|\widehat{T}^*_{n1j}| \geq C_{\alpha/2}\} \quad and \quad p_n = P_n\{|\widehat{T}^*_{n1j}| \geq C_{\alpha/2}\}, \tag{20}$$

where "$P_n$" shows the conditional probability given $\widehat{F}_n$. Obviously, $M\bar{V}$ has a binomial distribution with parameters $p_n$ and $M$, which is asymptotically normal for large $M$. If $z_\alpha$ is the $(1-\alpha)100$th percentile of the standard normal distribution $N(0,1)$ and for some $\beta$ such that $0<\beta<\alpha<1$ we choose

$$M=\max\left\{1,\frac{p_n(1-p_n)(z_{\alpha-\beta}-z_\alpha)^2}{(\alpha-p_n)^2}\right\} \tag{21}$$

then, for testing (3), at $\alpha 100$ % significance level we show NL-GOF as follows

$$Reject\ \ H_0 \quad if \quad \bar{V}\geq\alpha+z_{\alpha-\beta}\sqrt{\frac{\alpha(1-\alpha)}{M}}, \tag{22}$$

that is asymptotically consistent. We prove its consistency in Theorem 3.

***Theorem* 3.** *Under the assumptions of Theorem 1, for NL-GOF test (22) we have*
*(i) under $H_0$,‘*

$$\lim_{n\to\infty}P\left\{\bar{V}\geq\alpha+z_{\alpha-\beta}\sqrt{\frac{\alpha(1-\alpha)}{M}}|H_0\right\}\leq\alpha, \tag{23}$$

*(ii) under $H_1$: $F=F_1\neq F_0$,*

$$\lim_{n\to\infty}P\left\{\bar{V}\geq\alpha+z_{\alpha-\beta}\sqrt{\frac{\alpha(1-\alpha)}{M}}|H_1\right\}=1. \tag{24}$$

**Proof.** The proof is similar to the proof of Theorem 3 in omidi et al. (2021) and then it is omitted. □

We can conveniently show that (19) and Theorem 3 holds for $\widehat{T}^*_{n2}$ in (18) under the assumptions of Theorem 2.

## 3 Monte Carlo Study

A simulation study is performed to determine the null distribution and compares the power of the test statistics $\widehat{T}^*_{n1}$ in (17) and $\widehat{T}^*_{n2}$ in (18) with test statistics based on EDF. In this section, following Koziol and Green (1976), the competing test statistics are considered as follows:

- Kolmogorov–Smirnov (1933)

$$\widehat{D}_n = \sup_{0<u<1} |\widehat{F}_n(\widehat{U}_i) - \widehat{U}_i| = \max\left(D_n^+, D_n^-\right),$$

where

$$D_n^+ = \max_{1<j<n+1, \delta_j=1} \{\widehat{p}_j - \widehat{U}_j\}, \qquad D_n^- = \max_{1<j<n+1, \delta_j=1} \{\widehat{U}_j - \widehat{p}_{j-1}\}$$

with $\widehat{U}_{n+1} = 1$, $\widehat{p}_0 = 0$, $\widehat{p}_{n+1} = 1$, and $\widehat{p}_j$ in (7).

- Cramer-von Mises

$$\widehat{W}_n^2 = \sum_{i=1}^{n} \widehat{F}_n(\widehat{U}_i)(\widehat{U}_{i+1} - \widehat{U}_i)\{\widehat{F}_n(\widehat{U}_i) - (\widehat{U}_{i+1} + \widehat{U}_i)\} - n/3.$$

- Anderson–Darling (1954)

$$\begin{aligned}\widehat{A}_n^2 = n\sum_{i=1}^{n-1}(\widehat{F}_n(\widehat{U}_i))^2 &\Big(-\ln(1-\widehat{U}_{i+1}) + \ln(\widehat{U}_{i+1}) + \ln(1-\widehat{U}_i) - \ln(\widehat{X}_i)\Big)\\ &- 2n\sum_{i=1}^{n-1}(\widehat{F}_n(\widehat{U}_i))^2 \Big(-\ln(1-\widehat{U}_{i+1}) + \ln(1-\widehat{U}_i)\Big)\\ &- n\ln(1-\widehat{U}_n) - n\ln(\widehat{U}_n) - n.\end{aligned}$$

Here we estimate the unknown parameter $\lambda$ using the MLE method and use the MLE $\widehat{\lambda}$ in (17) and (18). The upper percentage points of the statistics are presented in Tables 1–2 for sample sizes $n = 20, 30, 40, 50, 100$, censored proportion s $\gamma = 0.2, 0.4, 0.5, 0.6$, and the significance level s $\alpha$ =0.01, 0.025, 0.05, 0.10, 0.15, 0.25, 0.50. The values are computed based on the Monte Carlo simulation with $10,000$ iterations. We apply the random censorship model suggested in Koziol and Green (1976) to control the censored proportion. It is

$$1 - G = (1 - F)^{\beta}, \quad for\ some\ \ \beta > 0,$$

where $G$ and $\beta$ are the df of the censoring time $C$ and a censoring parameter, respectively. Using this model, we obtain

$$\gamma = P(X > C) = \int_{-\infty}^{\infty} (1 - F)\mathrm{d}G = \int_0^1 \beta(1-x)^{\beta}\mathrm{d}x = \frac{\beta}{\beta+1}.$$

In Tables 1–2, $\gamma$ is the expected proportion of the censored data, and it is equivalent to $\dfrac{\beta}{\beta+1}$. We can see that the critical values decrease when the censored proportion decreases and the significance level increases. Csörgő and Horváth (1981), Chen et al. (1982), and Kim (2011, 2012) investigated the properties of this model.
We study the asymptotic null distributions of $\widehat{T}^*_{n1}$ and $\widehat{T}^*_{n2}$ only when the null hypothesis is fully determined with a known parameter. The introduced test statistics are distribution-free under the simple null hypothesis. Thus, the critical values do not depend on the unknown parameter $\lambda$ when the parameter is estimated by the MLE. We calculate all the statistics when $\lambda = 1$ without loss of generality. Also, we can obtain the $p$-value of the gof tests approximately using the points in Tables 1–2.

In this section, to study the performance of the introduced tests, we consider eight alternative distributions with censoring distribution suggested by Koziol and Green (1976) as follows:

- The exponential distribution with parameter 1.
- The Weibull distribution $W(\theta)$, with scale parameter 1 and shape parameter $\theta$.
- The Gamma distribution $G(\theta)$, with scale parameter 1 and shape parameter $\theta$.
- The log-normal distribution with pdf, $\frac{1}{\theta x\sqrt{2\pi}}\exp\{-(\log x)^2/(2\theta^2)\}$.
- The standard Uniform distribution.
- The half-normal distribution with pdf, $\sqrt{\frac{2}{\pi}}\exp(-x^2/2)$.
- The log-logistic distribution with pdf, $\dfrac{1}{1+x^2}$.
- The half-logistic distribution with pdf, $2\exp(-x)/(1+\exp(-x))^2$.

According to Ahrari et al. (2019), mentioned distributions have densities $f$ with decreasing hazard rates ($DHR$), increasing hazard rates ($IHR$) and models with non-monotone hazard rates ($NFR$). Since the power values of the $\widehat{T}^*_{n1}$ and $\widehat{T}^*_{n2}$ statistics depend on the $\alpha$ value, the maximum power of these tests were assumed in terms of a particular value of $\alpha$. Tables 3–8 display the power of the statistics at the significance level $\alpha = 0.05$ for sample sizes $n = 20, 30, 35, 40, 50, 100$. Moreover, $N = 10,000$ samples are generated for each alternative. The power estimates are obtained by taking the ratio of rejections.

The greatest powers are displayed in bold in each alternative. The power results show the following. First, $\widehat{T}^*_{n2}$ displays the highest power for all alternatives except for Weibull (0.5) and Gamma (0.5) (with $(DHR)$ function). Thus, $\widehat{T}^*_{n2}$ for distributions with $(DHR)$ function has low power. It is obvious that, $\widehat{A}_n$ is the best for Weibull (0.5) and $\widehat{T}^*_{n1}$ is the best for Gamma (0.5). Second, $\widehat{D}_n$ has the lowest power for each alternative considered. Third, for the half-logistic, half-normal, and log-normal alternatives, all of the statistics except $\widehat{T}^*_{n2}$ have relatively low power. Moreover, $\widehat{T}^*_{n2}$ has high power for these alternatives. All of the statistics have relatively low power for the log-logistic alternative.

# 4 Illustrative Example

We analyze the provided data by Koziol and Green (1980). They reported the data related to 211 state IV prostate cancer patients treated with estrogen in a Veterans Administration Cooperative Urological Research Group study, listed in Table 5 obtained from Table 2 of Hollander and Proschan (1979). Among this data, there are 90 died of prostate cancer, 105 died of other diseases, and 16 still alive. Those observations corresponding to deaths due to other causes and those corresponding to the 16 survivors are considered as censored observations (withdrawals). We review the hypothesis of exponentiality for their survival distribution for deaths from cancer of the prostate at a 5% significance level.

First, we need to estimate the unknown parameter. The MLE of the parameter is $\widehat{\lambda} = 0.0134$. Then, we calculate similar to Table 1 for $n = 211$ and $\gamma = 0.6$ the critical values of these data, the test statistics, and the $p$-values of the proposed tests. The results are displayed in Table 6. Based on this table, the data set is from an exponential distribution. Also, $\widehat{T}^*_{n2}$ supports more the hypothesis of exponentiality.

# 5 Conclusions

We proposed two new test statistics based on CKL information and CRKL information for testing exponentiality with an unknown parameter when data

**Table 1.** Upper tail percentage points of the test statistic $\widehat{T}^*_{n1}$ with $\gamma$ the proportion of censored data

| Statistic | $n$ | $\gamma$ | $\alpha$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0.01 | 0.025 | 0.05 | 0.1 | 0.15 | 0.25 | 0.5 |
| $\widehat{T}^*_{n1}$ | 20 | 0.6 | 4.52 | 4.41 | 4.29 | 4.17 | 4.15 | 4.03 | 3.77 |
| | | 0.5 | 4.18 | 4.05 | 3.93 | 3.81 | 3.75 | 3.61 | 3.32 |
| | | 0.4 | 3.80 | 3.65 | 3.52 | 3.37 | 3.28 | 3.11 | 2.77 |
| | | 0.2 | 2.65 | 2.33 | 2.24 | 2.12 | 1.96 | 1.76 | 1.40 |
| | 30 | 0.6 | 4.89 | 4.75 | 4.63 | 4.47 | 4.25 | 4.13 | 3.95 |
| | | 0.5 | 4.75 | 4.72 | 4.61 | 4.45 | 4.24 | 4.11 | 3.89 |
| | | 0.4 | 4.39 | 4.25 | 4.12 | 4.02 | 3.97 | 3.87 | 3.77 |
| | | 0.2 | 3.07 | 2.94 | 2.78 | 2.59 | 2.46 | 2.17 | 1.95 |
| | 40 | 0.6 | 5.12 | 4.97 | 4.77 | 4.50 | 4.38 | 4.17 | 4.03 |
| | | 0.5 | 4.98 | 4.05 | 3.93 | 3.81 | 3.75 | 3.61 | 3.32 |
| | | 0.4 | 4.80 | 3.65 | 3.52 | 3.37 | 3.28 | 3.11 | 2.77 |
| | | 0.2 | 3.45 | 3.33 | 3.24 | 3.12 | 2.96 | 2.56 | 2.10 |
| | 50 | 0.6 | 5.35 | 5.25 | 5.16 | 5.07 | 4.98 | 4.83 | 4.77 |
| | | 0.5 | 5.18 | 5.05 | 4.93 | 4.81 | 4.75 | 4.61 | 4.32 |
| | | 0.4 | 4.82 | 4.56 | 4.48 | 4.29 | 4.01 | 3.75 | 3.17 |
| | | 0.2 | 3.65 | 3.43 | 3.34 | 3.26 | 3.05 | 2.81 | 2.60 |
| | 100 | 0.6 | 6.19 | 6.08 | 5.99 | 5.87 | 5.81 | 5.68 | 5.46 |
| | | 0.5 | 5.61 | 5.47 | 5.34 | 5.20 | 5.12 | 4.97 | 4.76 |
| | | 0.4 | 4.84 | 4.67 | 4.51 | 4.40 | 4.29 | 4.13 | 3.87 |
| | | 0.2 | 3.75 | 3.53 | 3.46 | 3.35 | 3.21 | 3.07 | 2.95 |

**Table 2.** Upper tail percentage points of the test statistic $\widehat{T}^*_{n2}$ with $\gamma$ the proportion of censored data

| Statistic | $n$ | $\gamma$ | $\alpha$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0.01 | 0.025 | 0.05 | 0.1 | 0.15 | 0.25 | 0.5 |
| $\widehat{T}^*_{n2}$ | 20 | 0.6 | 0.082 | 0.005 | -0.005 | -0.025 | -0.071 | -0.115 | -0.180 |
| | | 0.5 | 0.03 | -0.02 | -0.054 | -0.095 | -0.127 | -0.177 | -0.27 |
| | | 0.4 | -0.067 | -0.107 | -0.152 | -0.195 | -0.222 | -0.27 | -0.305 |
| | | 0.2 | -0.238 | -0.274 | -0.303 | -0.336 | -0.354 | -0.382 | -0.42 |
| | 30 | 0.6 | -0.066 | -0.017 | -0.046 | -0.083 | -0.11 | -0.156 | -0.25 |
| | | 0.5 | -0.077 | -0.109 | -0.142 | -0.187 | -0.216 | -0.261 | -0.351 |
| | | 0.4 | -0.160 | -0.19 | -0.234 | -0.275 | -0.3 | -0.34 | -0.412 |
| | | 0.2 | -0.303 | -0.328 | -0.349 | -0.373 | -0.387 | -0.406 | -0.435 |
| | 40 | 0.6 | -0.039 | -0.07 | -0.106 | -0.144 | -0.170 | -0.215 | -0.305 |
| | | 0.5 | -0.135 | -0.17 | -0.205 | -0.246 | -0.274 | 0.317 | -0.394 |
| | | 0.4 | -0.21 | -0.24 | -0.279 | -0.317 | - 0.33 | -0.372 | -0.433 |
| | | 0.2 | -0.329 | -0.35 | -0.36 | -0.38 | -0.39 | -0.411 | -0.435 |
| | 50 | 0.6 | -0.077 | -0.11 | -0.139 | -0.179 | -0.208 | -0.255 | -0.334 |
| | | 0.5 | -0.170 | -0.208 | -0.23 | -0.27 | -0.30 | -0.34 | -0.416 |
| | | 0.4 | -0.249 | -0.280 | -0.30 | -0.33 | -0.36 | -0.389 | -0.442 |
| | | 0.2 | -0.34 | -0.35 | -037 | -0.38 | -0.39 | -0.4 | -0.429 |
| | 100 | 0.6 | -0.176 | -0.205 | -0.23 | -0.269 | -0.295 | -0.336 | -0.422 |
| | | 0.5 | -0.257 | -0.28 | -0.303 | -0.33 | -0.35 | -0.39 | -0.453 |
| | | 0.4 | -0.307 | -0.327 | -0.344 | -0.368 | -0.378 | -0.41 | -0.421 |
| | | 0.2 | -0.349 | -0.354 | -0.359 | -0.367 | -0.371 | -0.378 | -0.392 |

**Table 3.** Power comparison of the test statistics $\widehat{T}^*_{n1}$, $\widehat{T}^*_{n2}$, $\widehat{D}_n$, $\widehat{W}^2_n$ and $\widehat{A}^2_n$ for $\alpha = 0.05$ and $n = 20$

| Distribution | Censoring rate ($\gamma$) | $\widehat{D}_n$ | $\widehat{W}^2_n$ | $\widehat{A}^2_n$ | $\widehat{T}^*_{n1}$ | $\widehat{T}^*_{n2}$ |
|---|---|---|---|---|---|---|
| $exponential(1)$ | 0.6 | 0.049 | 0.050 | 0.051 | 0.049 | 0.051 |
| | 0.5 | 0.050 | 0.051 | 0.050 | 0.051 | 0.049 |
| | 0.4 | 0.049 | 0.049 | 0.051 | 0.051 | 0.050 |
| | 0.2 | 0.051 | 0.050 | 0.050 | 0.050 | 0.051 |
| $Weibull(0.5)$ | 0.6 | 0.06 | 0.07 | **0.14** | 0.07 | 0.10 |
| | 0.5 | 0.09 | 0.11 | **0.20** | 0.08 | 0.13 |
| | 0.4 | 0.12 | 0.15 | **0.23** | 0.11 | 0.15 |
| | 0.2 | 0.14 | 0.16 | **0.35** | 0.13 | 0.20 |
| $Gamma(0.5)$ | 0.6 | 0.08 | 0.21 | 0.19 | **0.22** | 0.09 |
| | 0.5 | 0.09 | 0.25 | 0.20 | **0.28** | 0.13 |
| | 0.4 | 0.10 | 0.30 | 0.26 | **0.32** | 0.14 |
| | 0.2 | 0.12 | 0.37 | 0.35 | **0.39** | 0.18 |
| $Weibull(2)$ | 0.6 | 0.07 | 0.13 | 0.07 | 0.09 | **0.25** |
| | 0.5 | 0.10 | 0.15 | 0.074 | 0.11 | **0.28** |
| | 0.4 | 0.11 | 0.19 | 0.087 | 0.13 | **0.31** |
| | 0.2 | 0.13 | 0.26 | 0.091 | 0.17 | **0.35** |
| $Gamma(2)$ | 0.6 | 0.09 | 0.10 | 0.09 | 0.08 | **0.14** |
| | 0.5 | 0.10 | 0.14 | 0.11 | 0.10 | **0.17** |
| | 0.4 | 0.13 | 0.20 | 0.14 | 0.13 | **0.22** |
| | 0.2 | 0.14 | 0.23 | 0.17 | 0.15 | **0.29** |
| $half - logistic$ | 0.6 | 0.062 | 0.10 | 0.13 | 0.11 | **0.21** |
| | 0.5 | 0.076 | 0.11 | 0.14 | 0.15 | **0.26** |
| | 0.4 | 0.089 | 0.12 | 0.15 | 0.17 | **0.31** |
| | 0.2 | 0.11 | 0.14 | 0.16 | 0.20 | **0.34** |
| $log - norm(1)$ | 0.6 | 0.089 | 0.083 | 0.11 | 0.085 | **0.12** |
| | 0.5 | 0.095 | 0.14 | 0.14 | 0.09 | **0.19** |
| | 0.4 | 0.11 | 0.16 | 0.19 | 0.12 | **0.22** |
| | 0.2 | 0.12 | 0.17 | 0.20 | 0.16 | **0.29** |
| $U(0, 1)$ | 0.6 | 0.062 | 0.106 | 0.17 | 0.19 | **0.22** |
| | 0.5 | 0.089 | 0.25 | 0.22 | 0.23 | **0.31** |
| | 0.4 | 0.102 | 0.34 | 0.32 | 0.29 | **0.37** |
| | 0.2 | 0.187 | 0.39 | 0.36 | 0.35 | **0.42** |
| $half - norm$ | 0.6 | 0.08 | 0.09 | 0.09 | 0.10 | **0.19** |
| | 0.5 | 0.01 | 0.10 | 0.11 | 0.11 | **0.23** |
| | 0.4 | 0.13 | 0.14 | 0.13 | 0.12 | **0.29** |
| | 0.2 | 0.14 | 0.15 | 0.14 | 0.17 | **0.32** |
| $log - logistic$ | 0.6 | 0.04 | 0.06 | 0.08 | 0.07 | 0.06 |
| | 0.5 | 0.06 | 0.07 | 0.09 | 0.07 | 0.07 |
| | 0.4 | 0.07 | 0.08 | 0.11 | 0.08 | 0.08 |
| | 0.2 | 0.08 | 0.09 | 0.12 | 0.09 | 0.09 |

For each alternative, the greatest powers are in bold.

**Table 4.** Power comparison of the test statistics $\widehat{T}^*_{n1}$, $\widehat{T}^*_{n2}$, $\widehat{D}_n$, $\widehat{W}^2_n$ and $\widehat{A}^2_n$ for $\alpha = 0.05$ and $n = 30$

| Distribution | Censoring rate ($\gamma$) | $\widehat{D}_n$ | $\widehat{W}^2_n$ | $\widehat{A}^2_n$ | $\widehat{T}^*_{n1}$ | $\widehat{T}^*_{n2}$ |
|---|---|---|---|---|---|---|
| $exponential(1)$ | 0.6 | 0.050 | 0.051 | 0.050 | 0.051 | 0.051 |
| | 0.5 | 0.049 | 0.049 | 0.050 | 0.048 | 0.050 |
| | 0.4 | 0.049 | 0.05 | 0.051 | 0.049 | 0.049 |
| | 0.2 | 0.051 | 0.048 | 0.049 | 0.050 | 0.051 |
| $Weibull(0.5)$ | 0.6 | 0.09 | 0.11 | **0.18** | 0.08 | 0.12 |
| | 0.5 | 0.12 | 0.15 | **0.21** | 0.10 | 0.15 |
| | 0.4 | 0.15 | 0.18 | **0.26** | 0.14 | 0.16 |
| | 0.2 | 0.17 | 0.19 | **0.38** | 0.19 | 0.22 |
| $Gamma(0.5)$ | 0.6 | 0.09 | 0.28 | 0.21 | **0.29** | 0.10 |
| | 0.5 | 0.11 | 0.42 | 0.28 | **0.45** | 0.15 |
| | 0.4 | 0.13 | 0.53 | 0.40 | **0.55** | 0.16 |
| | 0.2 | 0.14 | 0.57 | 0.51 | **0.58** | 0.19 |
| $Weibull(2)$ | 0.6 | 0.10 | 0.15 | 0.10 | 0.13 | **0.28** |
| | 0.5 | 0.11 | 0.17 | 0.11 | 0.15 | **0.35** |
| | 0.4 | 0.12 | 0.24 | 0.12 | 0.19 | **0.42** |
| | 0.2 | 0.16 | 0.36 | 0.13 | 0.28 | **0.55** |
| $Gamma(2)$ | 0.6 | 0.10 | 0.11 | 0.10 | 0.11 | **0.16** |
| | 0.5 | 0.12 | 0.16 | 0.13 | 0.12 | **0.19** |
| | 0.4 | 0.15 | 0.21 | 0.15 | 0.14 | **0.25** |
| | 0.2 | 0.16 | 0.24 | 0.19 | 0.16 | **0.35** |
| $half-logistic$ | 0.6 | 0.092 | 0.11 | 0.14 | 0.12 | **0.25** |
| | 0.5 | 0.10 | 0.13 | 0.16 | 0.16 | **0.31** |
| | 0.4 | 0.11 | 0.15 | 0.18 | 0.18 | **0.39** |
| | 0.2 | 0.15 | 0.17 | 0.20 | 0.21 | **0.44** |
| $log-norm(1)$ | 0.6 | 0.10 | 0.11 | 0.14 | 0.12 | **0.15** |
| | 0.5 | 0.11 | 0.16 | 0.20 | 0.14 | **0.22** |
| | 0.4 | 0.12 | 0.17 | 0.21 | 0.15 | **0.25** |
| | 0.2 | 0.13 | 0.19 | 0.23 | 0.19 | **0.31** |
| $U(0,1)$ | 0.6 | 0.12 | 0.25 | 0.22 | 0.26 | **0.27** |
| | 0.5 | 0.14 | 0.34 | 0.28 | 0.38 | **0.39** |
| | 0.4 | 0.18 | 0.45 | 0.42 | 0.46 | **0.47** |
| | 0.2 | 0.25 | 0.51 | 0.53 | 0.52 | **0.55** |
| $half-norm$ | 0.6 | 0.09 | 0.11 | 0.13 | 0.11 | **0.30** |
| | 0.5 | 0.10 | 0.12 | 0.14 | 0.12 | **0.38** |
| | 0.4 | 0.11 | 0.15 | 0.15 | 0.14 | **0.41** |
| | 0.2 | 0.13 | 0.17 | 0.16 | 0.19 | **0.53** |
| $log-logistic$ | 0.6 | 0.09 | 0.09 | 0.10 | 0.09 | 0.10 |
| | 0.5 | 0.10 | 0.10 | 0.11 | 0.10 | 0.11 |
| | 0.4 | 0.11 | 0.13 | 0.12 | 0.11 | 0.13 |
| | 0.2 | 0.12 | 0.15 | 0.13 | 0.12 | 0.14 |

For each alternative, the greatest powers are in bold.

**Table 5.** Power comparison of the test statistics $\widehat{T}^*_{n1}$, $\widehat{T}^*_{n2}$, $\widehat{D}_n$, $\widehat{W}^2_n$ and $\widehat{A}^2_n$ for $\alpha = 0.05$ and $n = 35$

| Distribution | Censoring rate ($\gamma$) | $\widehat{D}_n$ | $\widehat{W}^2_n$ | $\widehat{A}^2_n$ | $\widehat{T}^*_{n1}$ | $\widehat{T}^*_{n2}$ |
|---|---|---|---|---|---|---|
| $exponential(1)$ | 0.6 | 0.050 | 0.051 | 0.050 | 0.051 | 0.051 |
| | 0.5 | 0.049 | 0.049 | 0.050 | 0.051 | 0.050 |
| | 0.4 | 0.05 | 0.05 | 0.051 | 0.049 | 0.049 |
| | 0.2 | 0.051 | 0.050 | 0.049 | 0.051 | 0.050 |
| $Weibull(0.5)$ | 0.6 | 0.11 | 0.12 | **0.20** | 0.10 | 0.13 |
| | 0.5 | 0.13 | 0.16 | **0.28** | 0.11 | 0.16 |
| | 0.4 | 0.16 | 0.19 | **0.36** | 0.16 | 0.17 |
| | 0.2 | 0.18 | 0.20 | **0.40** | 0.20 | 0.24 |
| $Gamma(0.5)$ | 0.6 | 0.11 | 0.30 | 0.22 | **0.31** | 0.15 |
| | 0.5 | 0.12 | 0.45 | 0.37 | **0.46** | 0.17 |
| | 0.4 | 0.13 | 0.57 | 0.42 | **0.58** | 0.21 |
| | 0.2 | 0.17 | 0.60 | 0.59 | **0.62** | 0.29 |
| $Weibull(2)$ | 0.6 | 0.11 | 0.18 | 0.15 | 0.14 | **0.29** |
| | 0.5 | 0.12 | 0.20 | 0.16 | 0.17 | **0.38** |
| | 0.4 | 0.15 | 0.25 | 0.17 | 0.20 | **0.55** |
| | 0.2 | 0.18 | 0.38 | 0.18 | 0.29 | **0.61** |
| $Gamma(2)$ | 0.6 | 0.11 | 0.12 | 0.11 | 0.12 | **0.17** |
| | 0.5 | 0.13 | 0.17 | 0.14 | 0.14 | **0.20** |
| | 0.4 | 0.16 | 0.23 | 0.18 | 0.16 | **0.27** |
| | 0.2 | 0.17 | 0.26 | 0.21 | 0.18 | **0.38** |
| $half-logistic$ | 0.6 | 0.10 | 0.12 | 0.16 | 0.17 | **0.39** |
| | 0.5 | 0.11 | 0.14 | 0.17 | 0.18 | **0.48** |
| | 0.4 | 0.14 | 0.16 | 0.19 | 0.19 | **0.57** |
| | 0.2 | 0.17 | 0.18 | 0.22 | 0.24 | **0.65** |
| $log-norm(1)$ | 0.6 | 0.11 | 0.12 | 0.15 | 0.13 | **0.16** |
| | 0.5 | 0.13 | 0.18 | 0.22 | 0.15 | **0.25** |
| | 0.4 | 0.14 | 0.20 | 0.25 | 0.17 | **0.29** |
| | 0.2 | 0.15 | 0.23 | 0.27 | 0.20 | **0.35** |
| $U(0,1)$ | 0.6 | 0.20 | 0.39 | 0.25 | 0.31 | **0.43** |
| | 0.5 | 0.22 | 0.51 | 0.31 | 0.52 | **0.51** |
| | 0.4 | 0.25 | 0.65 | 0.48 | 0.61 | **0.61** |
| | 0.2 | 0.28 | 0.76 | 0.59 | 0.74 | **0.78** |
| $half-norm$ | 0.6 | 0.10 | 0.12 | 0.14 | 0.12 | **0.35** |
| | 0.5 | 0.11 | 0.15 | 0.15 | 0.13 | **0.42** |
| | 0.4 | 0.14 | 0.16 | 0.16 | 0.15 | **0.50** |
| | 0.2 | 0.15 | 0.17 | 0.17 | 0.18 | **0.65** |
| $log-logistic$ | 0.6 | 0.10 | 0.11 | 0.12 | 0.10 | 0.11 |
| | 0.5 | 0.12 | 0.12 | 0.13 | 0.11 | 0.12 |
| | 0.4 | 0.14 | 0.13 | 0.14 | 0.12 | 0.14 |
| | 0.2 | 0.19 | 0.18 | 0.17 | 0.14 | 0.17 |

For each alternative, the greatest powers are in bold.

**Table 6.** Power comparison of the test statistics $\widehat{T}^*_{n1}$, $\widehat{T}^*_{n2}$, $\widehat{D}_n$, $\widehat{W}^2_n$ and $\widehat{A}^2_n$ for $\alpha = 0.05$ and $n = 40$

| Distribution | Censoring rate ($\gamma$) | $\widehat{D}_n$ | $\widehat{W}^2_n$ | $\widehat{A}^2_n$ | $\widehat{T}^*_{n1}$ | $\widehat{T}^*_{n2}$ |
|---|---|---|---|---|---|---|
| $exponential(1)$ | 0.6 | 0.050 | 0.051 | 0.050 | 0.051 | 0.051 |
| | 0.5 | 0.049 | 0.049 | 0.050 | 0.051 | 0.050 |
| | 0.4 | 0.05 | 0.05 | 0.051 | 0.049 | 0.049 |
| | 0.2 | 0.051 | 0.050 | 0.049 | 0.051 | 0.050 |
| $Weibull(0.5)$ | 0.6 | 0.13 | 0.14 | **0.23** | 0.11 | 0.14 |
| | 0.5 | 0.15 | 0.17 | **0.30** | 0.12 | 0.17 |
| | 0.4 | 0.18 | 0.20 | **0.39** | 0.17 | 0.18 |
| | 0.2 | 0.20 | 0.22 | **0.51** | 0.21 | 0.26 |
| $Gamma(0.5)$ | 0.6 | 0.12 | 0.32 | 0.25 | **0.33** | 0.16 |
| | 0.5 | 0.13 | 0.49 | 0.39 | **0.48** | 0.18 |
| | 0.4 | 0.15 | 0.58 | 0.44 | **0.61** | 0.23 |
| | 0.2 | 0.18 | 0.65 | 0.63 | **0.67** | 0.34 |
| $Weibull(2)$ | 0.6 | 0.12 | 0.19 | 0.17 | 0.16 | **0.32** |
| | 0.5 | 0.14 | 0.22 | 0.19 | 0.19 | **0.41** |
| | 0.4 | 0.17 | 0.27 | 0.21 | 0.22 | **0.58** |
| | 0.2 | 0.20 | 0.41 | 0.23 | 0.32 | **0.65** |
| $Gamma(2)$ | 0.6 | 0.12 | 0.13 | 0.13 | 0.09 | **0.21** |
| | 0.5 | 0.14 | 0.19 | 0.16 | 0.09 | **0.28** |
| | 0.4 | 0.17 | 0.25 | 0.18 | 0.17 | **0.32** |
| | 0.2 | 0.19 | 0.28 | 0.19 | 0.20 | **0.41** |
| $half-logistic$ | 0.6 | 0.11 | 0.13 | 0.17 | 0.18 | **0.41** |
| | 0.5 | 0.12 | 0.15 | 0.18 | 0.19 | **0.51** |
| | 0.4 | 0.16 | 0.17 | 0.20 | 0.20 | **0.69** |
| | 0.2 | 0.18 | 0.21 | 0.25 | 0.26 | **0.80** |
| $log-norm(1)$ | 0.6 | 0.12 | 0.13 | 0.17 | 0.14 | **0.19** |
| | 0.5 | 0.13 | 0.14 | 0.24 | 0.16 | **0.26** |
| | 0.4 | 0.14 | 0.16 | 0.26 | 0.18 | **0.31** |
| | 0.2 | 0.15 | 0.18 | 0.29 | 0.22 | **0.38** |
| $U(0,1)$ | 0.6 | 0.21 | 0.41 | 0.27 | 0.32 | **0.46** |
| | 0.5 | 0.23 | 0.56 | 0.35 | 0.55 | **0.59** |
| | 0.4 | 0.26 | 0.69 | 0.50 | 0.63 | **0.65** |
| | 0.2 | 0.29 | 0.78 | 0.61 | 0.76 | **0.80** |
| $half-norm$ | 0.6 | 0.12 | 0.13 | 0.15 | 0.14 | **0.38** |
| | 0.5 | 0.13 | 0.16 | 0.16 | 0.15 | **0.45** |
| | 0.4 | 0.16 | 0.17 | 0.18 | 0.16 | **0.52** |
| | 0.2 | 0.18 | 0.19 | 0.19 | 0.20 | **0.68** |
| $log-logistic$ | 0.6 | 0.12 | 0.13 | 0.14 | 0.12 | 0.12 |
| | 0.5 | 0.14 | 0.15 | 0.15 | 0.13 | 0.13 |
| | 0.4 | 0.16 | 0.17 | 0.16 | 0.14 | 0.15 |
| | 0.2 | 0.20 | 0.21 | 0.18 | 0.16 | 0.18 |

For each alternative, the greatest powers are in bold.

**Table 7.** Power comparison of the test statistics $\widehat{T}^*_{n1}$, $\widehat{T}^*_{n2}$, $\widehat{D}_n$, $\widehat{W}^2_n$ and $\widehat{A}^2_n$ for $\alpha = 0.05$ and $n = 50$

| Distribution | Censoring rate ($\gamma$) | $\widehat{D}_n$ | $\widehat{W}^2_n$ | $\widehat{A}^2_n$ | $\widehat{T}^*_{n1}$ | $\widehat{T}^*_{n2}$ |
|---|---|---|---|---|---|---|
| $exponential(1)$ | 0.6 | 0.050 | 0.051 | 0.050 | 0.051 | 0.051 |
| | 0.5 | 0.049 | 0.049 | 0.050 | 0.051 | 0.050 |
| | 0.4 | 0.05 | 0.05 | 0.051 | 0.049 | 0.049 |
| | 0.2 | 0.051 | 0.050 | 0.049 | 0.051 | 0.050 |
| $Weibull(0.5)$ | 0.6 | 0.14 | 0.16 | **0.30** | 0.12 | 0.15 |
| | 0.5 | 0.16 | 0.18 | **0.35** | 0.14 | 0.19 |
| | 0.4 | 0.19 | 0.21 | **0.42** | 0.19 | 0.20 |
| | 0.2 | 0.21 | 0.23 | **0.52** | 0.22 | 0.27 |
| $Gamma(0.5)$ | 0.6 | 0.14 | 0.36 | 0.28 | **0.35** | 0.18 |
| | 0.5 | 0.15 | 0.52 | 0.41 | **0.52** | 0.20 |
| | 0.4 | 0.17 | 0.61 | 0.48 | **0.68** | 0.26 |
| | 0.2 | 0.19 | 0.70 | 0.65 | **0.72** | 0.39 |
| $Weibull(2)$ | 0.6 | 0.12 | 0.20 | 0.18 | 0.17 | **0.34** |
| | 0.5 | 0.14 | 0.24 | 0.20 | 0.21 | **0.45** |
| | 0.4 | 0.19 | 0.32 | 0.22 | 0.24 | **0.61** |
| | 0.2 | 0.22 | 0.45 | 0.25 | 0.35 | **0.75** |
| $Gamma(2)$ | 0.6 | 0.14 | 0.15 | 0.14 | 0.11 | **0.25** |
| | 0.5 | 0.16 | 0.21 | 0.18 | 0.14 | **0.30** |
| | 0.4 | 0.18 | 0.27 | 0.19 | 0.18 | **0.38** |
| | 0.2 | 0.21 | 0.31 | 0.20 | 0.22 | **0.45** |
| $half-logistic$ | 0.6 | 0.12 | 0.14 | 0.18 | 0.19 | **0.45** |
| | 0.5 | 0.14 | 0.16 | 0.19 | 0.21 | **0.61** |
| | 0.4 | 0.17 | 0.19 | 0.22 | 0.24 | **0.73** |
| | 0.2 | 0.20 | 0.23 | 0.27 | 0.28 | **0.85** |
| $log-norm(1)$ | 0.6 | 0.13 | 0.14 | 0.18 | 0.15 | **0.23** |
| | 0.5 | 0.14 | 0.16 | 0.25 | 0.18 | **0.30** |
| | 0.4 | 0.18 | 0.19 | 0.29 | 0.20 | **0.41** |
| | 0.2 | 0.19 | 0.20 | 0.33 | 0.24 | **0.49** |
| $U(0,1)$ | 0.6 | 0.23 | 0.42 | 0.29 | 0.35 | **0.49** |
| | 0.5 | 0.24 | 0.45 | 0.38 | 0.48 | **0.61** |
| | 0.4 | 0.28 | 0.61 | 0.55 | 0.65 | **0.71** |
| | 0.2 | 0.30 | 0.79 | 0.71 | 0.82 | **0.86** |
| $half-norm$ | 0.6 | 0.14 | 0.15 | 0.16 | 0.17 | **0.41** |
| | 0.5 | 0.15 | 0.17 | 0.17 | 0.18 | **0.46** |
| | 0.4 | 0.17 | 0.18 | 0.19 | 0.19 | **0.58** |
| | 0.2 | 0.19 | 0.21 | 0.21 | 0.22 | **0.73** |
| $log-logistic$ | 0.6 | 0.13 | 0.16 | 0.17 | 0.13 | 0.14 |
| | 0.5 | 0.16 | 0.17 | 0.18 | 0.14 | 0.17 |
| | 0.4 | 0.18 | 0.18 | 0.19 | 0.16 | 0.18 |
| | 0.2 | 0.22 | 0.23 | 0.20 | 0.18 | 0.20 |

For each alternative, the greatest powers are in bold.

**Table 8.** Power comparison of the test statistics $\widehat{T}^*_{n1}$, $\widehat{T}^*_{n2}$, $\widehat{D}_n$, $\widehat{W}^2_n$ and $\widehat{A}^2_n$ for $\alpha = 0.05$ and $n = 100$

| Distribution | Censoring rate ($\gamma$) | $\widehat{D}_n$ | $\widehat{W}^2_n$ | $\widehat{A}^2_n$ | $\widehat{T}^*_{n1}$ | $\widehat{T}^*_{n2}$ |
|---|---|---|---|---|---|---|
| $exponential(1)$ | 0.6 | 0.050 | 0.051 | 0.050 | 0.051 | 0.051 |
| | 0.5 | 0.049 | 0.049 | 0.050 | 0.051 | 0.050 |
| | 0.4 | 0.051 | 0.051 | 0.051 | 0.049 | 0.049 |
| | 0.2 | 0.051 | 0.050 | 0.049 | 0.051 | 0.050 |
| $Weibull(0.5)$ | 0.6 | 0.18 | 0.21 | **0.39** | 0.19 | 0.18 |
| | 0.5 | 0.28 | 0.45 | **0.55** | 0.28 | 0.22 |
| | 0.4 | 0.42 | 0.53 | **0.67** | 0.35 | 0.35 |
| | 0.2 | 0.56 | 0.65 | **0.72** | 0.41 | 0.42 |
| $Gamma(0.5)$ | 0.6 | 0.18 | 0.52 | 0.47 | **0.55** | 0.19 |
| | 0.5 | 0.21 | 0.65 | 0.51 | **0.67** | 0.22 |
| | 0.4 | 0.25 | 0.81 | 0.58 | **0.83** | 0.28 |
| | 0.2 | 0.29 | 0.85 | 0.70 | **0.87** | 0.42 |
| $Weibull(2)$ | 0.6 | 0.14 | 0.28 | 0.21 | 0.19 | **0.52** |
| | 0.5 | 0.16 | 0.31 | 0.23 | 0.25 | **0.75** |
| | 0.4 | 0.21 | 0.45 | 0.25 | 0.31 | **0.90** |
| | 0.2 | 0.25 | 0.60 | 0.29 | 0.45 | **0.95** |
| $Gamma(2)$ | 0.6 | 0.18 | 0.22 | 0.17 | 0.16 | **0.47** |
| | 0.5 | 0.22 | 0.28 | 0.19 | 0.18 | **0.59** |
| | 0.4 | 0.28 | 0.31 | 0.21 | 0.25 | **0.77** |
| | 0.2 | 0.32 | 0.38 | 0.28 | 0.36 | **0.85** |
| $half-logistic$ | 0.6 | 0.13 | 0.18 | 0.19 | 0.22 | **0.58** |
| | 0.5 | 0.18 | 0.23 | 0.21 | 0.24 | **0.67** |
| | 0.4 | 0.21 | 0.32 | 0.24 | 0.29 | **0.77** |
| | 0.2 | 0.26 | 0.37 | 0.29 | 0.32 | **0.90** |
| $log-normal(1)$ | 0.6 | 0.16 | 0.15 | 0.20 | 0.17 | **0.28** |
| | 0.5 | 0.18 | 0.19 | 0.27 | 0.19 | **0.37** |
| | 0.4 | 0.20 | 0.21 | 0.32 | 0.22 | **0.45** |
| | 0.2 | 0.27 | 0.23 | 0.37 | 0.29 | **0.57** |
| $U(0,1)$ | 0.6 | 0.25 | 0.72 | 0.42 | 0.61 | **0.81** |
| | 0.5 | 0.31 | 0.78 | 0.55 | 0.71 | **0.90** |
| | 0.4 | 0.45 | 0.85 | 0.62 | 0.82 | **0.97** |
| | 0.2 | 0.55 | 0.94 | 0.7 | 0.88 | **1.00** |
| $half-normal$ | 0.6 | 0.16 | 0.17 | 0.19 | 0.18 | **0.51** |
| | 0.5 | 0.17 | 0.18 | 0.21 | 0.22 | **0.74** |
| | 0.4 | 0.18 | 0.19 | 0.22 | 0.25 | **0.95** |
| | 0.2 | 0.21 | 0.22 | 0.24 | 0.27 | **1.00** |
| $log-logistic$ | 0.6 | 0.14 | 0.18 | 0.19 | 0.16 | 0.17 |
| | 0.5 | 0.18 | 0.18 | 0.19 | 0.15 | 0.18 |
| | 0.4 | 0.20 | 0.19 | 0.21 | 0.17 | 0.19 |
| | 0.2 | 0.23 | 0.24 | 0.25 | 0.20 | 0.22 |

For each alternative, the greatest powers are in bold.

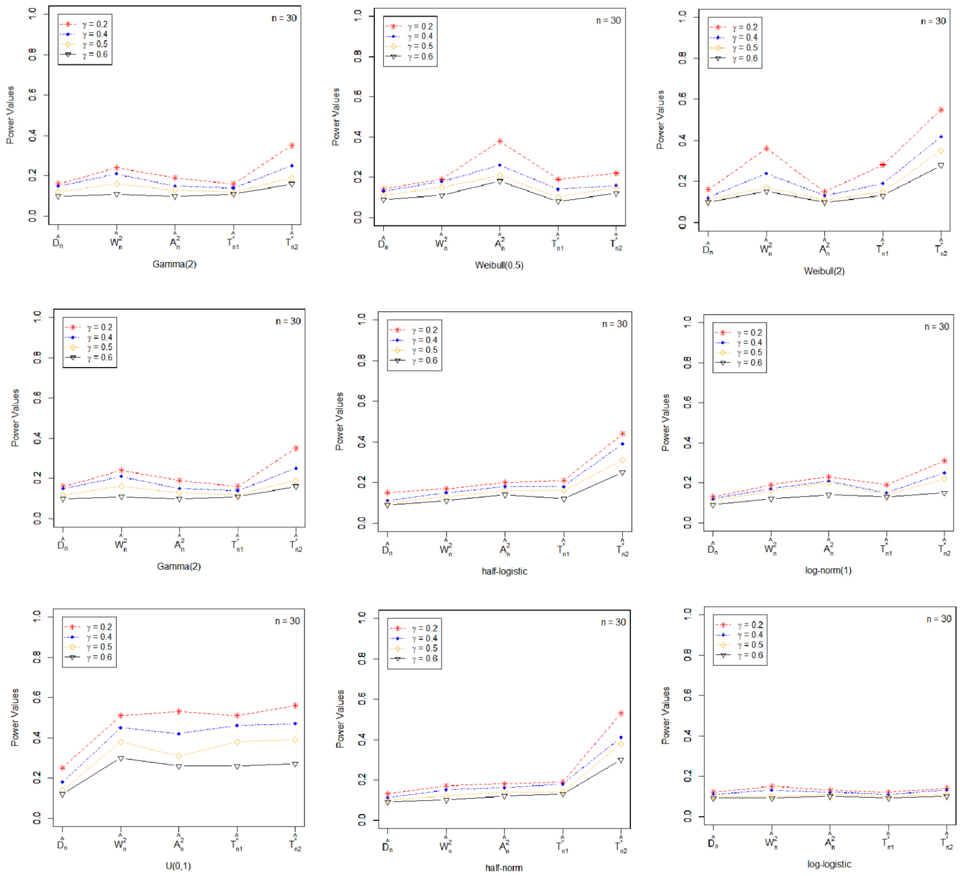


**Figure 1**. Power values of the proposed tests with competing tests for alternative distributions, and censoring rates, 0.2, 0.4, 0.5, 0.6, and $n = 30$

**Table 9.** Survival times and Withdrawal times in months for 211 patients (with number of ties given in parentheses)

| | |
|---|---|
| *Survivaltimes* | 0(3), 2, 3, 4, 6, 7(2), 8, 9(2), 11(3), 12(3), 15(2), 16(3), 17(2), 18, 19(2), 20, 21, 22(2), 23, 24, 25(2), 26(3), 27(2), 28(2), 29(2), 30, 31, 32(3), 33(2), 34, 35, 36, 37(2), 38, 40, 41(2), 42(2), 43, 45(3), 46, 47(2), 48(2), 51, 53(2), 54(2), 57, 60, 61, 62(2), 67, 69, 87, 97(2), 100, 145, 158, |
| Withdrawal times | 0(6), 1(5), 2(4), 3(3), 4, 6(5), 7(5), 8, 9(2), 10, 11, 12(3), 13(3), 14(2), 15(2), 16, 17(2), 18(2), 19(3), 21, 23, 25, 27, 28, 31, 32, 34, 35, 37, 38(4), 39(2), 44(3), 46, 47, 48, 49, 50, 53(2), 55, 56, 59, 61, 62, 65, 66(2), 72(2), 74, 78, 79, 81, 89, 93, 99, 102, 104(2), 106, 109, 119(2), 125, 127, 129, 131, 133(2), 135, 136(2), 138, 141, 142, 143, 144, 148, 160, 164(3). |

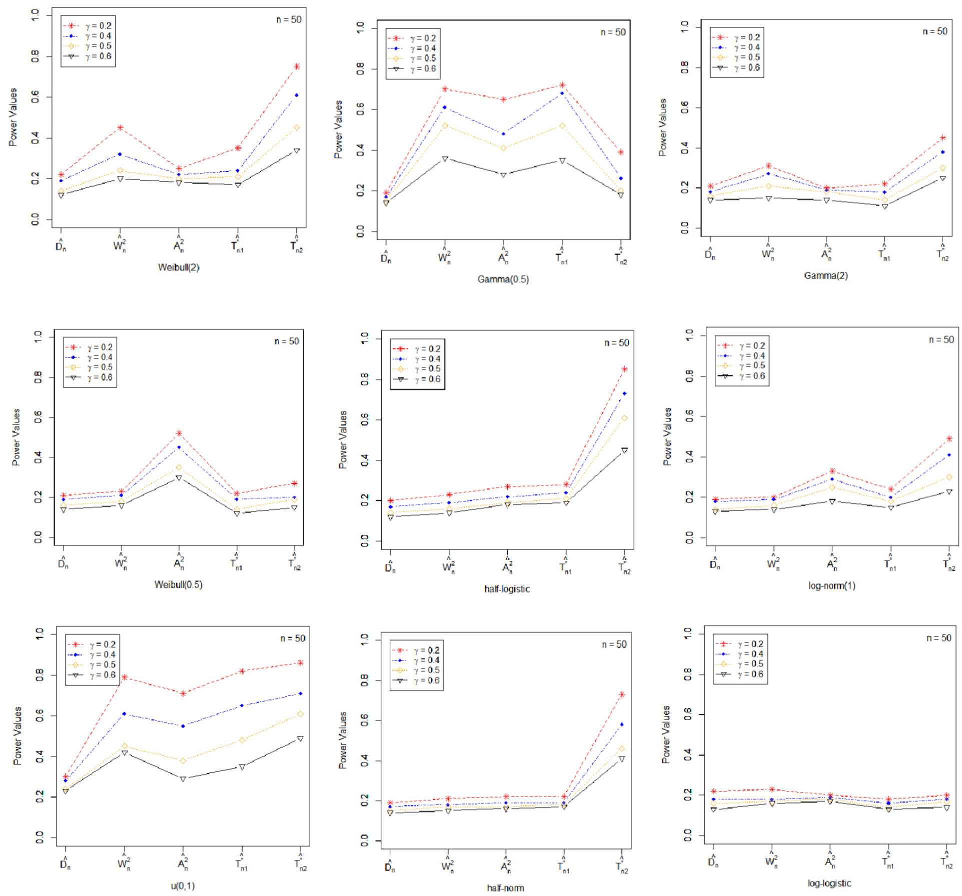

**Figure 2**. Power values of the proposed tests with competing tests for alternative distributions, and censoring rates, 0.2, 0.4, 0.5, 0.6, and $n = 50$

**Table 10.** Critical values, Test statistics and p-values of the proposed tests

| Tests | $\widehat{T}^*_{n1}$ | $\widehat{T}^*_{n2}$ |
|---|---|---|
| Critical value | 9.47 | -0.242 |
| Test statistic | 7.41 | -0.516 |
| p-value | 0.345 | 0.751 |

are randomly censored. We displayed that these tests are asymptotically consistent. Using Monte Carlo simulations, we compare the power values of our tests with the power values of the test statistics based on EDF. The simulations study detected that the $\widehat{T}^*_{n2}$ test performs well in competition with the competitors for many alternatives and has higher power than the other suggested tests.

Though this paper focused on the exponential distribution, we will extend these tests to pdfs such as Weibull distribution, Normal distribution, and so on.

# References

Aalen, O.O., and Hoem, J.M. (1978). Random Time Changes for Multivariate Counting Processes. *Scandinavian Actuarial Journal*, **1978**, 81-101.

Ahrari, V., Habibirad, A., and Baratpour, S. (2019). Exponentiality Test based on Alpha-Divergence and Gamma-Divergence. *Communications in Statistics-Simulation and Computation*, **48**, 1138-1152.

Anderson, T.W., and Darling, D.A. (1954). A Test of Goodness of Fit. *Journal of American Statistical Association*, **49**, 765-769.

Akritas, M.G. (1988). Pearson-Type Goodness-of-Fit Tests: the Univariate Case. *Journal of the American Statistical Association*, **83**, 222-230.

Baratpour, S., and Habibirad, A. (2012). Testing Goodness-of-Fit for Exponential Distribution based on Cumulative Residual Entropy. *Communications in Statistics-Theory and Methods*, **41**, 1387-1396.

Baratpour, S., and Habibirad, A. (2016). Exponentiality Test based on the Progressive Type II Censoring via Cumulative Entropy. *Communications in Statistics-Simulation and Computation*, **45**, 2625-2637.

Breslow, N., and Crowley, J. (1974). A Large Sample Study of the Life Table and Product Limit Estimates under Random Censorship. *The Annals of Statistics*, 437-453.

Chamany, A., and Baratpour, S. (2014). A Dynamic Discrimination Information based on Cumulative Residual Entropy and its Properties. *Communications in Statistics-Theory and Methods*, **43**, 1041–1049.

Chen, Y.Y., Hollander, M., and Langberg, N.A. (1982). Small-Sample Results for the Kaplan–Meier Estimator. *Journal of the American Statistical Association*, **77**, 141-144.

Chen, Y.Y., Hollander, M., and Langberg, N.A. (1983). Testing Whether New is Better than Used with Randomly Censored Data. *The Annals of Statistics*, **11**, 267-274.

Choi, B., Kim, K., and Song, S. (2004). Goodness-of-Fit Test for Exponentiality based on Kullback–Leibler Information. *Communications in Statistics*, **33**, 525-536.

Csörgő, O., and Horváth, L. (1981). On the Koziol–Green Model for Random Censorship. *Biometrika*, **68**, 391-401.

Ebrahimi, n., Soofi, E.S., and Habibullah, M. (1992). Testing Exponentiality based on Kullback–Leibler Information. *Journal of the Royal Statistical Society*, **54**, 739-748.

Efron, B. (1967). The Two Sample Problems with Censored Data. *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, **4**, 831-853.

Gurevich, G., and Davidson, A. (2008). Standardized Forms of Kullback–Leibler Information based Statistics for Normality and Exponentiality. *Computer Modelling and New Technologies*, **12**, 14-25.

Hall, W.J., and Wellner, J.A. (1980). Confidence Bands for a Survival Curve from Censored Data. *Biometrika*, **67**, 133-43.

Hollander, M., and Pena, A. (1992). A Chi-Squared Goodness-of-Fit Test for Randomly Censored Data. *Journal of the American Statistical Association*, **87**, 458-463.

Hollander, M., and Proschan, F. (1979). Testing to Determine the Underlying Distribution Using Randomly Censored Data. *Biometrics*, 393-401.

Kaplan, E.l., and Meier, P. (1958). Nonparametric Estimation from Incomplete Observations. *Journal of the American Statistical Association*, **53**, 457-481.

Kim, N. (2011). Testing Log Normality for Randomly Censored Data. *The Korean Journal of Applied Statistics*, **24**, 883–891.

Kim, N. (2012). Testing Exponentiality based on EDF Statistics for Randomly Censored Data when the Scale Parameter is Unknown. *The Korean Journal of Applied Statistics*, **25**, 311-319.

Kim, N. (2017). Goodness-of-Fit tests for Randomly Censored Weibull Distributions with Estimated Parameters. *Communications for Statistical Applications and Methods*, **24**, 519-531.

Kim, N. (2019). Tests based on EDF Statistics for Randomly Censored Normal Distributions when Parameters are Unknown. *Communications for Statistical Applications and Methods*, **26**, 431-443.

Kolmogorov, A.N. (1933). Sulla Determinazione Empirica di une Legge di Distribuzione. *Giornale del l'Intituto Italiano degli Attuari*, **4**, 83-91.

Koziol, J.A. (1980). Goodness-of-Fit Tests for Randomly Censored Data. *Biometrika*, **67**, 693-696.

Koziol, J.A. (2009). The Concordance Index C and the Mann–Whitney Parameter $\Pr(X > Y)$ with Randomly Censored Data. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, **51**, 467-474.

Koziol, J.A., and Green, S.B. (1976). A Cramer-von Mises Statistic for Randomly Censored Data. *Biometrika*, **63**, 465-474.

Kullback, S., and Leibler, R.A. (1951). On Information and Sufficiency. *The Annals of Mathematical Statistics*, **22**, 79-86.

Lu, X., and Cheng, T. (2007). Randomly Censored Partially Linear Single-Index Models. *Journal of Multivariate Analysis*, **98**, 1895-1922.

Meier, P. (1975). Estimation of a Distribution Function from Incomplete Observations. *Journal of Applied Probability*, **12**, 67-87.

Nair, v. (1981). Plots and Tests for Goodness of Fit with Randomly Censored Data. *Biometrika*, **68**, 99-103.

Omidi, F., Fakoor, V., and Habibirad, A. (2021). Goodness of Fit Test based on Information Criterion for Interval Censored Data. *Communications in Statistics-Theory and Methods*, 1-21.

Park, S., Rao, M., and Wan Shin, D. (2012). On Cumulative Residual Kullback–Leibler Information. *Statistics and Probability Letters*, **82**, 2025-2032.

Park, S., Noughabi, H., Alizadeh, A., and Kim, I. (2018). General Cumulative Kullback–Leibler Information. *Communications in Statistics-Theory and Methods*, **47**, 1551-1560.

Rao, M., Chen, Y., Vemuri, B.C., and Wang, F. (2004). Cumulative Residual Entropy: A New Measure of Information. *IEEE Transactions on Information Theory*, **50**, 1220-1228.

Shannon, C.E. (1948). A Mathematical Theory of Communication. *The Bell System Technical Journal*, **27**, 379-432.

Shapiro, S.S., and Francia, R.S. (1972). An Approximate Analysis of Variance Test for Normality. *Journal of the American Statistical Association*, **67**, 215-226.

Wang, Q., and Zheng, Z. (1997). Asymptotic Properties for the Semiparametric Regression Model with Randomly Censored Data. *Science in China Series A: Mathematics*, **40**, 945-957.

Zohrevand, Y., Hashemi, R., and Asadi, M. (2020). An Adjusted Cumulative Kullback–Leibler Information with Application to Test of Exponentiality. *Communications in Statistics-Theory and Methods*, **49**, 44-60.

**Fatemeh Omidi**
Department of Statistics,
Ferdowsi University of Mashhad,
Mashhad, Iran.
email: *fthomidi@gmail.com*

**Arezou Habibirad**
Department of Statistics,
Ferdowsi University of Mashhad,
Mashhad, Iran.
email: *ahabibi@um.ac.ir*

**Vahid Fakoor**
Department of Statistics,
Ferdowsi University of Mashhad,
Mashhad, Iran.
email: *fakoor@um.ac.ir*