© Statistical Research and Training Center
مجله‌ی پژوهش‌های آماری ایران
پژوهشکده‌ی آمار

J. Statist. Res. Iran **17** (2020): 45–62
دوره‌ی ۱۷، شماره‌ی ۱، بهار و تابستان ۱۳۹۹، صص ۴۵–۶۲
DOI: 10.52547/jsri.17.1.45
DOR: 20.1001.1.25385771.2020.17.1.3.4

Two-stage Procedure in P-Order Autoregressive Process

Eisa Mahmoudi*, Soudabe Sajjadipanah and Mohammadsadegh Zamani

Yazd University

Received: 2021/30/04 Approved: 2022/01/05

Abstract. In this paper, the two-stage procedure is considered for autoregressive parameters estimation in the p-order autoregressive model (AR(p)). The point estimation and fixed-size confidence ellipsoids construction are investigated which are based on least-squares estimators. Performance criteria are shown including asymptotically risk efficient, asymptotically efficient, and asymptotically consistent. Monte Carlo simulation studies are conducted to investigate the performance of the two-stage procedure. Finally, real-time-series data is provided to investigate to the applicability of the two-stage procedure.

Keywords. Asymptotically consistent, asymptotically efficient, asymptotically risk efficient, fixed size confidence ellipsoids, two-stage procedure.

MSC 2010: 62L12, 62M10, 62L10, 62L15.

1 Introduction

In some situations, for example, in biology, economics, electronics, finance, and management, researchers wish to determine the smallest possible sample size. Sequential procedures help as an alternative approach instead of fixed

* Corresponding author

sample inference when limited resources, gain in sample size, cost effectiveness, etc. Sequential procedures determine sample size by different stopping rules which each procedure is used according to the conditions. Time series models, in particular useful to model data, has been widely considered in many statistical problems and are widely applicable in many contexts. Time series analysis has been used effectively to model, estimate, forecast, and predict real practical problems, but in some cases, analysis can not be examined due to the sample size unknown. In order to overcome such a problem, some researchers have applied sequential procedures. Many investigators have interested to study these procedures in various fields and has done research in this fields, several of which will be mentioned. Stein (1945, 1949) first introduced a two-stage procedure for the problem of constructing a fixed-width confidence interval and hypotheses testing by considering the mean in a normal population.

Mukhopadhyay and Duggan (1997) introduced a modified two-stage procedure to estimate the mean in a normal population for a confidence interval because two-stage procedure oversamples in a confidence interval estimation. Mukhopadhyay (1980) investigated a two-stage procedure to construct a confidence interval of the mean in a normal population under the condition by assuming the variance is unknown Sriram (1987, 1988) studied a purely sequential sampling scheme to estimate the autoregressive parameters in a first-order autoregressive model. Basawa, McCormick and Sriram (1990) examined a sequential sampling procedure for dependent observations and applied it to estimate the autoregressive parameters in a first-order autoregressive process with Weibull errors. Fakhre-Zakeri and Lee (1992) have used a purely sequential procedure to study the estimation of the mean vector parameter in a multivariate linear process. Mukhopadhyay and Sriram (1992) was provided an estimation of the means of p-independent first-order autoregressive models via a purely sequential procedure.

Lee (1994) applied sequential point and confidence interval estimation of parameters in a p-th order autoregressive model. Basu and Das (1997) proposed sequential estimation of autoregressive parameters in a multiple pth-order autoregressive model. Lee and Sriram (1999) by applying a purely sequential procedure estimate parameter in a threshold AR(1) model. Sriram (2001) investigated fixed-size confidence region in single and multiple first-order threshold autoregressive models through a purely sequential procedure. Gombay (2010) used a sequential procedure to derive a confidence interval of time series observations. Also, Mukhopadhyay and Zacks (2018)

© 2020, SRTC Iran

assumed a modified Linex loss function in a normal distribution and investigated a two-stage and purely sequential procedure to estimate the unknown variance. Karmakar and Mukhopadhyay (2018, 2019), proposed a sequential procedure to estimate parameters in a single and multivariate random coefficient autoregressive pth-order model. Mahmoudi, Khalifeh and Nekoukhou (2019) considered studying a two-stage procedure to estimate a parameter in a stress-strength model. Sriram and Samadi (2019) give an overview of a purely sequential procedure to estimate a parameter in an AR(1) model previously studied by Sriram (1988). Khalifeh, Mahmoudi and Chaturvedi (2020) analyzed to investigate the performance of a two-stage procedure to construct a confidence interval for a parameter in an exponential distribution. Hu and Zhuang (2020) designed an innovative and general class of modified two-stage sampling schemes under the squared error loss.

As it mentioned, according to different stopping rules, sequential procedures are defined that the most usable of which are naming purely sequential procedure, two-stage procedure and modified two-stage procedure as the most widely used. As mentioned earlier, the purely sequential procedure has been studied that despite the small sample size, the cost under this procedure is high. The two-stage procedure is simpler in execution than the purely sequential procedure. Also, the two-stage procedure satisfies the operational savings and this is one of the advantages of this procedure that encouraged us to study the procedure. In this paper, the performance of the two-stage procedure is investigated in a p-order autoregressive model that results are presented in form of theorems. These theorems provide asymptotic properties including asymptotically risk efficient, asymptotically efficient, and asymptotically consistent which demonstrate the performance of the two-stage procedure in comparison with the best-fixed sample size procedure. We emphasize that the theorems are proved under the assumption of Lee (1994) to estimate the autoregressive parameters in the pth-order autoregressive model via the purely sequential procedure. In the following, Monte Carlo simulation studies are conducted to investigate our main results.

We have organized the rest of this paper in the following way. In Section 2.1, we stated and proved the asymptotic properties for the point and the fixed-width confidence region of autoregressive parameters. In Section 3, the main recent results are reviewed by comprehensive simulation studies. Finally, in Section 4, numerical studies with an application to a real time series data are considered to illustrate the applicability of the two-stage procedure.

2 Two-stage Estimation

2.1 Point Estimation

An autoregressive model of order p (AR(p)) with $|\beta_i| < 1$ is denoted by,

$$X_i = \beta_1 X_{i-1} + \cdots + \beta_p X_{i-p} + \varepsilon_i, \quad i = 1, 2, ...,$$

where $\{\varepsilon_i, i \geq 1\}$ is a sequence of independent and identically distributed random variables with an unknown distribution F which is assumed $\mathbf{E}[\varepsilon_i] = 0$ and $\mathbf{E}[\varepsilon_i^2] = \sigma^2 \in (0, \infty)$. Also, the initial state $\mathbf{X}_0 = (X_0, \cdots, X_{-p+1})^{'}$ is $\mathcal{F}_0$-measurable random vector with $\mathbf{E}(\mathbf{X}_0) = \mathbf{0}$ and $\mathbf{E}({X_i}^2) < \infty, i = -p+1, ..., 0$ where $\mathcal{F}_0$ is independent of $\{\varepsilon_i, i \geq 1\}$. By supposing $\boldsymbol{X_i} = (X_i, \cdots, X_{i-p+1})^{'}$, the least-squares estimator of $\boldsymbol{\beta}$ is given by

$$\hat{\boldsymbol{\beta}}_n = (\hat{\boldsymbol{\beta}}_{n1}, \ldots, \hat{\boldsymbol{\beta}}_{np})' = (\sum\nolimits_{i=1}^{n} \mathbf{X}_{i-1}\mathbf{X}_{i-1}')^{-1}(\sum\nolimits_{i=1}^{n} \mathbf{X}_{i-1}'\mathbf{X}_i).$$

Let Σ be the $p \times p$ positive definite matrix whose (i, j)-th entry is equal to $\gamma(i-j)$ where $\gamma(.)$ is the autocovariance function of the process $\{X_t\}$. It is well known from green Brockwell and Davis (1987) that as $n \to \infty$,

$$n^{-1}(\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}_{i-1}') \xrightarrow{\text{a.s}} \Sigma. \tag{1}$$

The asymptotic distribution of $\hat{\boldsymbol{\beta}}_n$ from Lai and Wei (1982) is given by

$$(\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}_{i-1}')^{1/2}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \xrightarrow{d} N(\mathbf{0}, \sigma^2 \mathbf{I}_p), \tag{2}$$

where the $p \times p$ identity matrix is denoted by $\boldsymbol{I}_p$. Also, we suppose that $\sum_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}_{i-1}^{'}$ and $\sum_{t=p}^{n} \boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t^{'}$ are non singular for all sufficiently large n where it should be noted that $\boldsymbol{\varepsilon}_t = (\varepsilon_i, \ldots, \varepsilon_{i-p+1})'$. The loss function for estimating the unknown $\boldsymbol{\beta}$ is given by

$$\begin{aligned} L_n(\hat{\boldsymbol{\beta}}_n, \boldsymbol{\beta}) &= A\left[n^{-1}\left(\left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\left(\sum\nolimits_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}_{i-1}'\right)\left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)\right)\right] + n \\ &= An^{-1}Q_n + n, \end{aligned}$$

where $A(>0)$ indicates the cost of one unit per observation and assume Q_n

© 2020, SRTC Iran

as follows

$$\begin{aligned} Q_n &= \left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)' \left(\sum_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right) \left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right) \\ &= (\sum_{i=1}^{n} \boldsymbol{X}_{i-1}\varepsilon_i)'(\sum_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1})^{-1}(\sum_{i=1}^{n} \boldsymbol{X}_{i-1}\varepsilon_i). \end{aligned}$$

Our aim is to minimize the risk function by choosing the appropriate sample size. When σ is known, the risk function is calculated Lee (1994) due to the uniform integrability property by Theorem 1 of Lee (1994) along with the asymptotic normality result for $\boldsymbol{\beta}$,

$$\begin{aligned} R_n = \mathbf{E}[L_n(\hat{\boldsymbol{\beta}}_n, \boldsymbol{\beta})] &= n^{-1}A(\sigma^2 p + o(1)) + n \\ &= n^{-1}(Ap)\sigma^2 + n + o(n^{-1}). \end{aligned}$$

In order to minimize the risk R_n with respect to n, the best fixed-sample size is approximately obtained as $n_A \simeq (Ap)^{1/2}\sigma$ that the $o(n^{-1})$ term is ignored. Then, the corresponding minimum risk function is given by

$$R_{n_A} \simeq (Ap)\sigma^2((Ap)^{1/2}\sigma)^{-1} + \sigma(Ap)^{1/2} = 2(Ap)^{1/2}\sigma.$$

When σ is unknown, which is the usual case, it is difficult to find the best-fixed sample size in practice and we cannot calculate the minimum risk function. We propose utilizing the two-stage sampling scheme to solve this problem that to this end, first an initial sample of size m is selected to estimate σ by $\hat{\sigma}_m$ then a two-stage stopping rule analogy with n_A is defined as follows

$$N_m = \max\{m, \lfloor (Ap)^{1/2}\hat{\sigma}_m \rfloor + 1\}, \tag{3}$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than x (for more details refer to Ghosh, Mukhopadhyay and Sen (1997)) and the least squares estimator $\hat{\sigma}_n$ is defined $\hat{\sigma}_n^2 = n^{-1}\sum_{i=1}^{n}\left(X_i - \hat{\beta}_1 X_{i-1} - \cdots - \hat{\beta}_p X_{i-p}\right)^2 = n^{-1}\sum_{i=1}^{n}\varepsilon_i^2 - n^{-1}Q_n$ for any n.

In the following, we present the main result of this subsection, Theorem 1, that thewhich indicates the two-stage procedure similar to the best-fixed sample size procedure is efficient in terms of asymptotically efficient and asymptotically risk efficient properties. Before presenting the theorem, we need to consider a lemma that it is necessary to establish asymptotically risk efficient properties.

***Lemma* 1.** *Assume that* $\mathbf{E}|\varepsilon_1|^{4s} < \infty$ *and* $\max_{-p+1\leq j\leq 0} \mathbf{E}|X_j|^{4s} < \infty$ *for* $s \geq 1$*. In addition if* $A^{1/2(1+\eta)} \leqslant m = o(A^{1/2})$ *for some* $\eta > 0$*. Then for any* $0 < \theta < 1$*,*

$$P\left(N_m < (1-\theta)(Ap)^{1/2}\sigma\right) = O\left(A^{-s/2(1+\eta)}\right), \tag{4}$$

$$P\left(N_m > \left[(1+\theta)(Ap)^{1/2}\sigma\right] + 1\right) = O\left(A^{-s/2(1+\eta)}\right). \tag{5}$$

Proof. In order to prove equations (4) and (5), we may assume without loss of generality that $\sigma = 1$. Let $\delta = 1-(1-\theta)^2$ and $\delta' = (1+\theta)^2 - 1$. We have

$$\begin{aligned}
&P(N_m \leq (1-\theta)(Ap)^{1/2}) \\
&= P\left(m^{-1}\sum_{i=1}^{m} \varepsilon_i^2 - m^{-1}Q_m < (1-\theta)^2\right) \\
&= P\left(\delta < m^{-1}\sum_{i=1}^{m} (1-\varepsilon_i^2) + m^{-1}Q_m\right) \\
&\leq P\left(m^{-1}\sum_{i=1}^{m} (1-\varepsilon_i^2) > \delta/2\right) \\
&+ P\left(m^{-1}Q_m > \delta/2\right),
\end{aligned}$$

From Rosentall inequality Merlevède and Peligrad (2013), we have

$$\mathbf{E}\left(|m^{-1}\sum_{i=1}^{m} (1-\varepsilon_i^2)|^{2s}\right) = O\left(A^{-s/2(1+\eta)}\right),$$

and hence by using the Markov inequality, we conclude that

$$\begin{aligned}
P\left(m^{-1}\sum_{i=1}^{m} (1-\varepsilon_i^2) > \delta/2\right) &\leq (\frac{2}{\delta})^{2s}\mathbf{E}\left(|m^{-1}\sum_{i=1}^{m} (1-\varepsilon_i^2)|^{2s}\right) \\
&= O\left(A^{-s/2(1+\eta)}\right),
\end{aligned}$$

Moreover, from the Theorem 1 of Lee (1994) and Markov inequality the

© 2020, SRTC Iran

result is yielded

$$\begin{aligned} P\left(m^{-1}Q_m > \delta/2\right) &\leq (\frac{2}{\delta})^{2s}\mathbf{E}\left(|m^{-1}Q_m|^s\right) \\ &= O\left(A^{-s/2(1+\eta)}\right). \end{aligned}$$

Thus from above estimates we have

$$\begin{aligned} &\leq P\left(m^{-1}\sum_{i=1}^{m}(1-\varepsilon_i^2) > \delta/2\right) + P\left(m^{-1}Q_m > \delta/2\right) \\ &= O\left(A^{-s/2(1+\eta)}\right). \end{aligned}$$

and (4) is obtained. In order to prove (5), Note that we have

$$\begin{aligned} &P(N_m > [(1+\theta)(Ap)^{1/2}] + 1) \\ &= P\left((Ap)^{1/2}\hat{\sigma}_m > (1+\theta)(Ap)^{1/2}\right) \\ &\leq P\left(m^{-1}\sum_{i=1}^{m}\varepsilon_i^2 > (1+\theta)^2\right) \\ &= P\left(m^{-1}\sum_{i=1}^{m}(\varepsilon_i^2 - 1) > \delta'\right) \\ &= O\left(A^{-s/2(1+\eta)}\right). \end{aligned}$$

Likewise, the proof follows from Markov inequality and Rosentall inequality. □

As mentioned before, the main theorem of this part is presented.

***Theorem* 1.** *Suppose for* $s > 1$ *that* $\mathbf{E}|\varepsilon_1|^{4s} < \infty$, $\max_{-p+1\leq j\leq 0}\mathbf{E}|X_j|^{4s} < \infty$ *and* $A^{1/2(1+\eta)} \leqslant m = o(A^{1/2})$ *for some* $\eta \in (0, (s+1)/2 - 1)$. *Then as* $A \to \infty$,

$$\frac{N_m}{n_A} \xrightarrow{\text{a.s}} 1, \tag{6}$$

$$\mathbf{E}\left[\frac{N_m}{n_A}\right] \to 1, \quad \text{(asymptotic efficiency)} \tag{7}$$

$$\frac{R_{N_m}}{R_{n_A}} \to 1, \quad \text{(asymptotic risk efficiency)}, \tag{8}$$

where $R_{N_m} = \mathbf{E}[L_{N_m}(\hat{\boldsymbol{\beta}}_{N_m}, \boldsymbol{\beta})]$.

Proof. From (3) note that

$$\hat{\sigma}_m (Ap)^{1/2} \leq N_m \leq \hat{\sigma}_m (Ap)^{1/2} + m.$$

The assertion (6) and (7) follow from above inequality by dividing this inequality by n_A and taking limit and expectation as $A \to \infty$ respectively. redAfterwards, for assertion (8), we can write

$$\begin{aligned} R_{N_m}/R_{n_A} &= \mathbf{E}[L_{N_m}(\hat{\boldsymbol{\beta}}_{N_m}, \boldsymbol{\beta})]/R_{n_A} \\ &= A\mathbf{E}\left[{N_m}^{-1} Q_{N_m}\right]/R_{n_A} + \mathbf{E}[N_m]/R_{n_A}. \end{aligned}$$

In view of (7) and $R_{n_A} \simeq 2(Ap)^{1/2}\sigma$, it is sufficient to prove that

$$\mathbf{E}\left[\frac{A^{1/2}}{p^{1/2}\sigma} N_m^{-1} Q_{N_m}\right] \to 1.$$

as $A \to \infty$. Let $n' = [(1-\theta)(Ap)^{1/2}\sigma]$, $n'' = [(1+\theta)(Ap)^{1/2}\sigma] + 1$ and $B = \{n' \leq N_m \leq n''\}$ for some $\theta \in (0,1)$. It is suffices to show that

$$\mathbf{E}\left[\frac{A^{1/2}}{p^{1/2}\sigma} N_m^{-1} Q_{N_m} I_{B^c}\right] \to 0. \tag{9}$$

and

$$\mathbf{E}\left[\frac{A^{1/2}}{p^{1/2}\sigma} N_m^{-1} Q_{N_m} I_B\right] \to 1. \tag{10}$$

where B^c the complement of set B. Proof of (9) follows from the Cauchy-Schwartz inequality, Theorem 1 of Lee (1994), Lemma 1 and the similar arguments in (3.21) of Lee (1994). In the following, we review the proof for

© 2020, SRTC Iran

$B^c = \{m \le N_m < n'\}$

$$
\begin{aligned}
& \mathbf{E}\left[\frac{A^{1/2}}{\sigma} N_m^{-1}\left(\sum\nolimits_{i=1}^{N_m} X_{i-1}^2\right)\left(\hat{\beta}_{N_m}-\beta\right)^2 I_{B^c}\right] \\
\le\ & \mathbf{E}\left[\frac{A^{1/2}}{\sigma} N_m^{-1}\left(\sum\nolimits_{i=1}^{N_m} X_{i-1}^2\right)\left(\hat{\beta}_{N_m}-\beta\right)^2 I_{B^c} I_{B^c}\right] \\
\le\ & \frac{A^{1/2}}{\sigma} \| \, N_m^{-1}\left(\sum\nolimits_{i=1}^{N_m} X_{i-1}^2\right)\left(\hat{\beta}_{N_m}-\beta\right)^2 I_{B^c} \|_{\,2} P^{1/2}\left(m \le N_m < n'\right) \\
\le\ & \frac{A^{1/2}}{\sigma} \sup_n \| \, \left(\sum\nolimits_{i=1}^{n} X_{i-1}^2\right)\left(\hat{\beta}_{n}-\beta\right)^2 \|_{\,2} \left(\sum\nolimits_{n=m}^{\infty} n^{-2}\right)^{1/2} P^{1/2}\left(m \le N_m < n'\right) \\
=\ & A^{1/2} O\left(m^{-1/2}\right) O\left(A^{-s/2(1+\eta)}\right) \to 0.
\end{aligned}
$$

Note that $I_B \xrightarrow{a.s} 1$. Also, from (2), (6), Lemma 5 of Lee (1994), the Slutsky theorem and the Anscombe's theorem of Woodroofe (1982), assertion (10) follows that

$$\frac{A^{1/2}}{p^{1/2}\sigma} N_m^{-1} Q_{N_m} I_B \xrightarrow{d} \chi_p^2.$$

It should be noted that Anscombe's theorem is concluded from Lemma 5 of Lee (1994). Hence, to complete the proof of the Theorem, we need to prove the uniform integrability property. This property is proved using the Theorem 1 of Lee (1994) for $r > 1$, as follows:

$$
\begin{aligned}
\mathbf{E}\left[\frac{A^{1/2}}{p^{1/2}\sigma} N_m^{-1} Q_{N_m} I_B\right]^r &\le \frac{A^{r/2}}{(p^{1/2}\sigma)^r} \mathbf{E}\left[\max_B (N_m^{-1} Q_{N_m})^r\right] \\
&\le \frac{A^{r/2}}{(n' p^{1/2}\sigma)^r} \mathbf{E}\left[\max_{n' \le n \le n''} (Q_{N_m})^r\right] = O(1).
\end{aligned}
$$

□

2.2 Fixed-width Confidence Region

In this subsection, our purpose is to construct a confidence set for $\boldsymbol{\beta}$ in p-dimensional Euclidean space $\mathbb{R}^p$ with the maximum diameter $2d$ ($d > 0$). The confidence region, based on the random sequence $\{X_t\}$, has the coverage probability approximately equal to $1 - \alpha$ ($0 < \alpha < 1$) as d tends to 0 which is assumed an ellipsoidal confidence region for the unknown $\boldsymbol{\beta}$ at the sample size n:

$$S_n = \{\boldsymbol{z} : \left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{z}\right)' \left(\sum_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right) \left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{z}\right) \\ \leq d^2 \lambda_{min} \left(\sum_{i=1}^{n} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right) \},$$

where $\lambda_{min}(C)$ indicates the smallest eigenvalue of a matrix C. An ellipsoid with maximum axis $2d$ is defined by S_n with the fixed size in this sense that (1) combined with (2) entails for d is sufficiently small,

$$P\left(\boldsymbol{\beta} \in S_n\right) \simeq 1 - \alpha.$$

The best fixed sample size is approximately $k_d \simeq [d^{-2}\lambda_{min}^{-1}(\Sigma)\sigma^2\chi_p^2(1-\alpha)]$ that $P\left(\chi_p^2 \leq \chi_p^2(1-\alpha)\right) = 1 - \alpha$. The best fixed-sample size procedure cannot be calculated when some or all σ and β_i are unknown. As before, we overcome this problem by defining the two-stage stopping rule,

$$N_m^d = \max\{m, \lfloor d^{-2}\hat{\sigma}_m^2\lambda_{min}^{-1}\left(m^{-1}\sum_{i=1}^{m} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right)\chi_p^2(1-\alpha)\rfloor + 1\}. \quad (11)$$

We state the main result of this subsection, Theorem 2, which indicates asymptotically consistent and asymptotically efficient properties as $d \to 0$.

Theorem 2. *Assume for $s > 1$, $\mathbf{E}|\varepsilon_1|^{4s} < \infty$ and $\max_{-p+1 \leq j \leq 0} \mathbf{E}|X_j|^{4s} < \infty$. Then as $d \to 0$,*

$$\frac{N_m^d}{k_d} \xrightarrow{a.s} 1, \quad (12)$$

$$\mathbf{E}\left[\frac{N_m^d}{k_d}\right] \to 1, \quad \text{(asymptotic efficiency)} \quad (13)$$

$$P\left(\boldsymbol{\beta} \in S_{N_m^d}\right) \to 1 - \alpha, \quad \text{(asymptotic consistency)} \quad (14)$$

Proof. From (11) note that

© 2020, SRTC Iran

$$\hat{\sigma}^2_m d^{-2}\chi^2_p(1-\alpha)\lambda^{-1}_{min}\left(m^{-1}\sum\nolimits_{i=1}^{m}\boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right) \leq N^d_m \\ \leq \hat{\sigma}^2_m d^{-2}\chi^2_p(1-\alpha)\lambda^{-1}_{min}\left(m^{-1}\sum\nolimits_{i=1}^{m}\boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right) + m$$

Dividing all side of above equation by k_d and taking the limit and the expection as $d \to 0$ yields (12) and (13) respectively. For (14), we can write

$$\begin{aligned} & P\left(\boldsymbol{\beta} \in S_{N^d_m}\right) \\ = \ & P\Big(\left(\hat{\boldsymbol{\beta}}_{N^d_m} - \boldsymbol{\beta}\right)'\left(\sum\nolimits_{i=1}^{N^d_m}\boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right)\left(\hat{\boldsymbol{\beta}}_{N^d_m} - \boldsymbol{\beta}\right) \\ \leq \ & d^2\lambda_{min}\left(\sum\nolimits_{i=1}^{N^d_m}\boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right)\Big) \end{aligned}$$

We know

$$\lambda_{min}\left(n^{-1}\sum\nolimits_{i=1}^{n}\boldsymbol{X}_{i-1}\boldsymbol{X'}_{i-1}\right) \xrightarrow{a.s} \lambda_{min}\left(\Sigma\right)$$

Then

$$\lambda_{min}\left({N^d_m}^{-1}\sum\nolimits_{i=1}^{N^d_m}\boldsymbol{X}_{i-1}\boldsymbol{X'}_{i-1}\right) \xrightarrow{a.s} \lambda_{min}\left(\Sigma\right)$$

Finally, (14) can be obtained from (2), (12) and the Anscombe's theorem. Note that the uniformly continuous in property condition of sequence $\left\{\left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\left(\sum_{i=1}^{n}\boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1}\right)\left(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)\right\}$ needed in Anscombe's theorem is satisfied from Lemma 5 of Lee (1994). Thus, we have,

$$
\begin{aligned}
& P\Bigg(\frac{\left(\hat{\boldsymbol{\beta}}_{N_m^d} - \boldsymbol{\beta}\right) \left(\sum_{i=1}^{N_m^d} \boldsymbol{X}_{i-1}\boldsymbol{X'}_{i-1}\right) \left(\hat{\boldsymbol{\beta}}_{N_m^d} - \boldsymbol{\beta}\right)'}{\sigma^2} \\
\leq \; & \frac{d^2 N_m^d \lambda_{min} \left({N_m^d}^{-1} \sum_{i=1}^{N_m^d} \boldsymbol{X}_{i-1}\boldsymbol{X'}_{i-1}\right) \chi_p^2(1-\alpha)}{\sigma^2 \lambda_{min}^{-1}(\Sigma)\, \lambda_{min}(\Sigma)\, \chi_p^2(1-\alpha)} \Bigg) \\
= \; & P\Bigg(\frac{\left(\hat{\boldsymbol{\beta}}_{N_m^d} - \boldsymbol{\beta}\right) \left(\sum_{i=1}^{N_m^d} \boldsymbol{X}_{i-1}\boldsymbol{X'}_{i-1}\right) \left(\hat{\boldsymbol{\beta}}_{N_m^d} - \boldsymbol{\beta}\right)'}{\sigma^2} \\
\leq \; & \frac{N_m^d \lambda_{min} \left({N_m^d}^{-1} \sum_{i=1}^{N_m^d} \boldsymbol{X}_{i-1}\boldsymbol{X'}_{i-1}\right) \chi_p^2(1-\alpha)}{k_d \lambda_{min}(\Sigma)} \Bigg) \to 1-\alpha.
\end{aligned}
$$

and the claim follows. □

3 Simulation Study

In this section, we conduct Monte Carlo simulation studies for AR(2) to evaluate the performance of point estimation, confidence region, and confidence interval of the linear combination which $\varepsilon_i \sim N(0,1)$. We investigate the results of theorems for point estimation in terms of stopping variable, the ratio of stopping variable to the best fixed-sample size, estimators, mean of square errors (MSE), and the ratio of two-stage risk functions to the fixed-sample size function. Moreover, the confidence region examines in terms of stopping variable, the ratio of stopping variable to the best fixed-sample size, and coverage probability. Results of point estimation and confidence region are reported in Tables 1 and 2, respectively. The values of (β_1, β_2) are selected based on the stationary conditions of AR(2) model which is given by the following triangular region

$$
\begin{cases}
\beta_1 + \beta_2 < 1 \\
\beta_2 - \beta_1 < 1 \\
|\beta_2| < 1,
\end{cases}
$$

(for more details refer to Brockwell and Davis (1987)). In addition, point estimation and confidence region for $(m, A) = (10, 100), (80, 6400)$ and $(m, d) =$

© 2020, SRTC Iran

Table 1. Estimators of two-stage procedure according to N_m

$(m, n_A, A, \beta_1, \beta_2)$	$\widehat{EN}$	$\widehat{E\left[\frac{N}{n_A}\right]}$	$\widehat{\beta_1}$	$\widehat{\beta_2}$	MSE $(\widehat{\beta}_1)$	MSE $(\widehat{\beta}_2)$	$\hat{R}_{EN}/R_{n_A}$
$(10, 14.14214, 100, 0.1, 0.1)$	12.5886	0.8901	0.0744	0.1731	0.0942	0.1036	0.9519
$(10, 14.14214, 100, 0.2, 0.1)$	12.6921	0.8974	0.1552	0.1885	0.0919	0.1054	0.9562
$(10, 14.14214, 100, -0.1, -0.3)$	12.2221	0.8642	−0.0591	−0.1561	0.0680	0.1328	0.8795
$(10, 14.14214, 100, 0.1, -0.5)$	12.2184	0.8639	0.0581	−0.3212	0.0562	0.1452	0.8706
$(10, 14.14214, 100, 0.2, -0.3)$	12.2955	0.8694	0.1251	−0.1411	0.0704	0.1354	0.9038
$(10, 14.14214, 100, -0.1, -0.7)$	12.3342	0.8721	−0.0546	−0.4957	0.0457	0.1504	0.8707
$(10, 14.14214, 100, 0.2, -0.5)$	12.2245	0.8644	0.1138	−0.3069	0.0621	0.1493	0.9011
$(10, 14.14214, 100, 0.3, -0.3)$	12.4271	0.8787	0.1859	−0.1199	0.0756	0.1427	0.9530
$(80, 113.1371, 6400, 0.1, 0.1)$	110.6679	0.9781	0.1070	0.1170	0.0109	0.0096	1.0658
$(80, 113.1371, 6400, 0.2, 0.1)$	110.9529	0.9806	0.2158	0.1474	0.0109	0.0120	1.1731
$(80, 113.1371, 6400, -0.1, -0.3)$	110.4881	0.9765	−0.0769	−0.2778	0.0055	0.0092	0.9287
$(80, 113.1371, 6400, 0.1, -0.5)$	110.5518	0.9771	0.0651	−0.4772	0.0044	0.0081	0.9331
$(80, 113.1371, 6400, 0.2, -0.3)$	110.7040	0.9784	0.1503	−0.2573	0.0075	0.0108	1.0289
$(80, 113.1371, 6400, -0.1, -0.7)$	110.3865	0.9756	−0.0581	−0.6762	0.0035	0.0059	0.9359
$(80, 113.1371, 6400, 0.2, -0.5)$	110.7043	0.9798	0.1316	−0.4571	0.0078	0.0095	1.0834
$(80, 113.1371, 6400, 0.3, -0.3)$	110.9818	0.9809	0.2268	−0.2195	0.0101	0.0158	1.1792

$(10, 1.2), (10, 0.6)$ are reported using R software by 10,000 replications. Also, results of confidence region are calculated with 95% confidence coefficient for different d and m.

From Table 1, as can be seen, the stopping variables increase with increasing A. By increasing A, the ratios of the stopping variable to the best fixed-sample size are close to 1. Moreover, estimators of β_1 and β_2 approach to values of β_1 and β_2 as a result of which the MSEs decreases. The ratios of the two-stage risk function to the fixed-sample size function are approximately around 1 with increasing A, as we expected.

From Table 2, the stopping variables increase as d decreases. Furthermore, the ratios of the stopping variable to the best fixed-sample size are close to 1 when d decreases. The coverage probability is close to 0.95 for different (β_1, β_2) with decreasing d, as we expected.

Based on the simulation results, the results of Theorem 1 and Theorem 2 are confirmed, which confirms the good performance of the procedure.

4 Data Analysis

In this section, we use the annual record of the numbers of the Canadian lynx trapped in the Mackenzie River district of northwest canada. This data set is modeled by the linear $AR(2)$ model to the logarithm of the lynx data

Table 2. Region estimation of two-stage procedure according to N_m^d

$(m, d, k_d, \beta_1, \beta_2)$	$\widehat{EN_m^d}$	$\widehat{E\left[\frac{N_m^d}{k_d}\right]}$	CP
$(10, 1.2, 5.6357, 0.1, 0.1)$	10.2935	1.8263	0.9621
$(10, 1.2, 5.0209, -0.1, -0.3)$	10.0252	1.9966	0.9773
$(10, 1.2, 4.0987, 0.1, -0.5)$	10.0121	2.4426	0.9801
$(10, 1.2, 5.3796, 0.2, -0.3)$	10.0711	1.8720	0.9741
$(10, 1.2, 2.7666, -0.1, -0.7)$	10.0021	3.6144	0.9742
$(10, 1.2, 2.7666, 0.2, -0.5)$	10.0593	2.3097	0.9693
$(10, 1.2, 2.7666, 0.3, -0.3)$	10.238	1.7841	0.9623
$(10, 0.6, 22.5431, 0.1, 0.1)$	24.7912	1.0990	0.9450
$(10, 0.6, 20.0839, -0.1, -0.3)$	20.0740	0.9999	0.9540
$(10, 0.6, 16.3950, 0.1, -0.5)$	17.3022	1.0553	0.9572
$(10, 0.6, 21.5184, 0.2, -0.3)$	20.8126	0.9771	0.9483
$(10, 0.6, 11.0666, -0.1, -0.7)$	14.0827	1.2702	0.9545
$(10, 0.6, 11.0666, 0.2, -0.5)$	18.2063	1.0451	0.9521
$(10, 0.6, 11.0666, 0.3, -0.3)$	22.8243	0.9943	0.9459

with $\varepsilon \sim N(0, 0.24^2)$by Moran (1953). By assuming an initial sample (m) is available, we estimate the two-stage stopping variable. After determine it, if the initial sample might not enough, the difference $N_m - m$ or $N_m^d - m$ is generated at the second stage. In order to evaluate the performance, we compare the two-stage procedure with the modified two-stage procedure. The difference between the modified two-stage procedure and the two-stage procedure is that it provides a strategy for determining the initial sample size and this has encouraged us to examine the two procedures together. The modified two-stage variables are considered for point and region estimation based on the initial sample size respectively that in order to evaluate the performance, we have examined estimating of stopping variable. Due to the strategy related to the modified two-stage procedure $m_0 \geq 3$ and $\gamma \in (1/2, \infty)$ as follows.

$$m = \max\{m_0, \lfloor (Ap)^{1/2(1+\gamma)} \rfloor + 1\},$$
$$N_p = \max\{m, \lfloor (Ap)^{1/2} \hat{\sigma}_m \rfloor + 1\}.$$

© 2020, SRTC Iran

Table 3. Point estimation of two-stage and modified two-stage procedures.

(m,c)	N_m	N_p	$\widehat{\beta}_{1N_m}$	$\widehat{\beta}_{2N_m}$	$\widehat{\beta}_{1N_p}$	$\widehat{\beta}_{2N_p}$	$\widehat{\sigma}_{N_m}$	$\widehat{\sigma}_{N_p}$
$(2,3)$	8	9	1.0480	1.0293	1.0770	1.0259	15.5440	12.9871
$(8,10)$	18	18	1.0076	1.0076	0.9946	0.9946	10.2840	10.2840
$(15,30)$	26	26	1.0034	1.0034	0.9976	0.9976	9.4458	9.4458

Table 4. Region estimation of two-stage and modified two-stage procedures.

(m,d)	N_m^d	N_p^d	$\widehat{\beta}_{1N_m^d}$	$\widehat{\beta}_{2N_m^d}$	$\widehat{\beta}_{1N_p^d}$	$\widehat{\beta}_{2N_p^d}$	$\widehat{\sigma}_{N_m^d}$	$\widehat{\sigma}_{N_p^d}$
$(8,0.9)$	3975	1110	0.9709	0.9920	0.9712	0.9753	8.5829	8.7214
$(12,0.7)$	1835	1576	0.9917	0.9915	0.9743	0.9736	8.7160	8.7537
$(20,0.5)$	2882	2645	0.9911	0.9908	0.9719	0.9707	8.6979	8.8051

and

$$m = \max\{m_0, \lfloor (\chi_p^2(1-\alpha)/d)^{2/(1+\gamma)} \rfloor + 1\},$$

$$N_p^d = \max\{m, \lfloor d^{-2}\hat{\sigma}_m^2 \lambda_{min}^{-1} \left(m^{-1} \sum_{i=1}^{m} \boldsymbol{X}_{i-1}\boldsymbol{X}'_{i-1} \right) \chi_p^2(1-\alpha) \rfloor + 1\}.$$

The estimator based on N_m, N_p, N_m^d and N_p^d for different m, A, and d are reported in Table 3 and 4.

As can be seen in Table 3, for the different values of (m, A), the values of the stopping variable and the estimators are very close to each other. By increasing the A, the values of the estimators from the two procedures are very close together, which indicates the performance of the two procedures is the same. The estimators are also reported based on the stopping variables N_m^d and N_p^d in Table 4. As can be seen, the values of the stopping variables from both procedures are approximated by decreasing d. It is noteworthy that the values of these stopping variables are obtained according to the proposed stop strategy. Also, the values of the estimators for different values of (m, d) are not much different, which again emphasizes the same performance of both procedures.

Conclusion

The two-stage procedure is investigated for autoregressive parameters estimation in the p-order autoregressive model (AR(p)). The asymptotic properties of the two-stage procedure are established as A tends to ∞ which these properties include asymptotically first-order efficiency, asymptotically first-order risk efficiency, and asymptotically consistent. The performance of the two-stage procedure in terms of criteria is shown by Monte Carlo simulation studies which indicated the good performance of the procedure and the confirmation of the previous results. Also, the performance of the procedure using real data is very good compared to other sequential procedures. The simplicity and the operational savings are the advantages of this procedure are compared to other sequential procedures, despite the problem of over-estimation in the region estimating. We propose this process to investigate and determine the sample size due to the importance of research costs in analysis time series models.

References

Basawa, I.V., McCormick, W.P., and Sriram, T.N. (1990). Sequential Estimation for Dependent Observations with an Application to Non-Standard Autoregressive Processes. *Stochastic processes and their applications*, **35**, 149-168.

Basu, A.K., and Das, J.K. (1997). Sequential Estimation of the Autoregressive Parameters in Ar(p) Model. *Sequential Analysis*, **16**, 1-24.

Brockwell, P.J., and Davis, R.A. (1987). *Time Series: Theory and Methods*. 2nd ed., Springer Texts in Statistics.

Fakhre-Zakeri, I., and Lee, S. (1992). Sequential Estimation of the Mean of a Linear Process. *Sequential analysis*, **11**, 181-197.

Ghosh, M., Mukhopadhyay, N., and Sen, P.K. (1997). *Sequential Estimation*, New York, Wiley.

Gombay, E. (2010). Sequential Confidence Intervals for Time Series. *Periodica Mathematica Hungarica*, **61**, 183-193.

Hu, J., and Zhuang, Y. (2020). A broader class of modified two-stage minimum risk point estimation procedures for a normal mean. *Communications in Statistics-Simulation and Computation*, 1-15.

Karmakar, B., and Mukhopadhyay, I. (2018). Risk Efficient Estimation of Fully Dependent Random Coefficient Autoregressive Models of General Order. *Communications in Statistics-Theory and Methods*, **47**, 4242-4253.

Karmakar, B., and Mukhopadhyay, I. (2019). Risk-Efficient Sequential Estimation of Multivariate Random Coefficient Autoregressive Process. *Sequential Analysis*, **38**, 26-45.

© 2020, SRTC Iran

Kashkovsky, D.V., and Konev, V.V. (2008). Sequential Estimates of the Parameters in a Random Coefficient Autoregressive Process. *Optoelectronics, Instrumentation and Data Processing*, **44**, 52-61.

Khalifeh, A., Mahmoudi, E., and Chaturvedi, A. (2020). Sequential Fixed-Accuracy Confidence Intervals for the Stress–Strength Reliability Parameter for the Exponential Distribution: Two-Stage Sampling Procedure. *Computational Statistics*, **35**, 1553-1575.

Kusainov, M.I. (2015). Risk Efficiency of Adaptive One-Step Prediction of Autoregression with Parameter Drift. *Vestnik Tomskogo gosudarstvennogo universiteta. Upravlenie, vychislitelquotesinglenaya tekhnika i informatika*, **32**, 33-43.

Lai, T.L., and Wei, C.Z. (1982). Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems. *The Annals of Statistics*, **10**, 154-166.

Lee, S. (1994). Sequential Estimation for the Parameters of a Stationary Autoregressive Model. *Sequential Analysis*, **13**, 301-317.

Lee, S., and Sriram,T.N. (1999). Sequential Point Estimation of Parameters in a Threshold AR(1) Model. *Stochastic processes and their Applications*, **84**, 343-355.

Mahmoudi, E., Khalifeh, A., and Nekoukhou, V. (2019). Minimum Risk Sequential Point Estimation of the Stress-Strength Reliability Parameter for Exponential Distribution. *Sequential Analysis*, **38**, 279-300.

Merlevède, F., and Peligrad, M. (2013). Rosenthal-type inequalities for the maximum of partial sums of stationary processes and examples. *The Annals of Probability*, **41**, 914-960.

Moran, P.A.P. (1953). The statistical analysis of the Canadian lynx cycle. *Australian Journal of Zoology*, **1**, 291-298.

Mukhopadhyay, N. (1980). A Consistent and Asymptotically Efficient Two-Stage Procedure to Construct Fixed Width Confidence Intervals for the Mean. *Metrika*, **27**, 281-284.

Mukhopadhyay, N., and Duggan, W.T. (1997). Can a Two-Stage Procedure Enjoy Second-Order Properties. *Sankhyā: The Indian Journal of Statistics, Series A*, **59**, 435-448.

Mukhopadhyay, N., and Duggan, W.T. (1999). On a Two-Stage Procedure Having Second-Order Properties with Applications. *Annals of the Institute of Statistical Mathematics*, **51**, 621-636.

Mukhopadhyay, N., and Sriram, T.N. (1992). On Sequential Comparisons of Means of First-Order Autoregressive Models. *Metrika*, **39**, 155-164.

Mukhopadhyay, N., and Zacks, S. (2018). Modified Linex Two-Stage and Purely Sequential Estimation of the Variance in a Normal Distribution with Illustrations Using Horticultural Data. *Journal of Statistical Theory and Practice*, **12**, 111-135.

Sriram, T. N. (1987). Sequential Estimation of the Mean of a First-Order Stationary Autoregressive Process. *The Annals of Statistics*, **15**, 1079-1090.

Sriram, T.N. (1988). Sequential Estimation of the Autoregressive Parameter in a First Order Autoregressive Process. *Sequential Analysis*, **7**, 53-74.

Sriram, T.N. (2001). Fixed Size Confidence Regions for Parameters of Threshold AR(1) Models. *Journal of statistical planning and inference*, **97**, 293-304.

Sriram, T.N., and Samadi, S.Y. (2019). Second-Order Analysis of Regret for Sequential Estimation of the Autoregressive Parameter in a First-Order Autoregressive Model. *Sequential Analysis*, **38**, 411-435.

Stein, C. (1945). A Two-Sample Test for a Linear Hypothesis Whose Power Is Independent of the Variance. *The Annals of Mathematical Statistics*, **16**, 243–258.

Stein, C. (1949). Some Problems in Sequential Estimation (Abstract). *Econometrica*, **17**, 77-78.

Woodroofe, M. (1982). *Nonlinear renewal theory in sequential analysis.* Philadelphia: SIAM.

Eisa Mahmoudi
Department of Statistics,
Yazd University,
Yazd, Iran.
email: *emahmoudi@yazd.ac.ir*

Soudabe Sajjadipanah
Department of Statistics,
Yazd University,
Yazd, Iran.
email: *soodabesajadi@yahoo.com*

Mohsmmsdsadegh Zamani
Department of Statistics,
Yazd University,
Yazd, Iran.
email: *zamani@yazd.ac.ir*

© 2020, SRTC Iran