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Weibull Analysis with Sequential Order Statistics Under a Power Trend Model for Hazard Rates with Application in Aircraft Data Analysis

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Abstract. In engineering systems, it is usually assumed that the lifetimes of components are independent and identically distributed (iid). But, the failure of a component results in a higher load on the remaining components and hence causes the distribution of the surviving components to change. For modelling this kind of system, the theory of sequential order statistics (SOS) can be used.

Assuming Weibull distribution for lifetimes of components and conditionally proportional hazard rates model as a special case of the SOS theory, the maximum likelihood estimates of the unknown parameters are obtained in different cases. A new model, denoted by PTCPHM, as a generalization of the iid case is proposed, and then statistical inferential methods including point and interval estimations as well as hypothesis tests under PTCPHM are developed. Finally, real data on failure times of aircraft components, due to Mann and Fertig (1973), are analysed to illustrate the model and inferential methods developed here.

Keywords. Censored data; estimation; hazard function; reliability; sequen-

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tial order statistics.

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1 Introduction

A system consisting of n components is said to be a r-out-of-n F-system if it fails when at least r failures occur smith (2002) and Billinton and Allan (1992). For $r = 1$ and $r = n$, it reduces to series and parallel systems, respectively. Let $X_1, \cdots, X_n$ be the lifetimes of the components in the system. Then, the lifetime of a r-out-of-n F-system is $X_{(r)}$, the r-th order statistic among $X_1, \cdots, X_n$. Thus, order statistics play an important role in the analysis of these systems.

In the literature, it is usually assumed that the random variables $X_1, \cdots, X_n$ are independent and identically distributed (iid). For more details, see Arnold et al. (2008). This assumption is violated in many practical engineering systems since as the components fail sequentially, the stress on the remaining components would be increased (see, Cramer and Kamps (1996)). Balakrishnan et al. (2008) gave the following example: "..., the failure of a high-voltage transmission line will increase the load put on the remaining high-voltage transmission lines, thus violating the iid assumption." A method for modelling these systems is through the theory of sequential order statistics (SOS) Kamps (1995). Under this model, the distribution of the remaining components are changed when some components fail. Hashempour and Doostparast (2016) considered Bayesian inference on multiply sequential order statistics from heterogeneous exponential populations with GLR test for homogeneity; see also Schenk et al. (2011) considered Bayes estimation and prediction on the basis of multiply Type-II censored data arising from one- and two-parameter exponential distributions; Shafay et al. (2012) for some additional results in this regard. Since the Weibull distribution is more flexible than the exponential distribution for modelling failure times as it possesses both increasing failure rate (IFR) and decreasing failure rate (DFR) properties, we consider here the problem of estimating the parameters of the two-parameter Weibull model based on a Type-II censored sample of SOS.

The rest of this paper is organized as follows. In Section 2, the conditionally proportional hazard rate model is reviewed. Based on a Type II

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censored SOS coming from a general class of distribution functions, the respective likelihood function (LF) is also obtained. A new power trend model, which includes the iid case as a special case, is proposed in Subsection 2.3. The two-parameter Weibull distribution is considered in Section 3 in more detail. More specifically, point estimates as well as approximate confidence intervals are obtained for the parameters of the Weibull distribution based on SOS. Assuming Weibull distribution for the lifetimes of the components, the problem of hypothesis testing for the new model is discussed in Subsection 3.3. Also, a test of exponentially for the random variables $X_1, \cdots, X_n$ against the Weibull model is considered in Subsection 3.4. In Section 4, a real data on failure times of aircraft components, due to Smith (2002), is analysed to illustrate the model and inferential methods developed here. Finally, some concluding remarks are made in Section 5.

2 Description of the Model and the Likelihood Function

In this section, we first describe the conditionally proportional hazard rate model in the set-up of SOS. Then, we present the likelihood function for a Type-II censored sample consisting of the first r SOS in the general case, and then its explicit form for the special case of a general exponential family of distributions. Finally, we focus on a special case of the conditionally proportional hazard rate model for SOS called the power trend conditionally proportional hazard model and present the corresponding likelihood function, which is what is used in the subsequent sections to develop inferential methods for the model parameters.

2.1 The Model

Suppose an engineering system has n components with random lifetimes $X_1, \cdots, X_n$. We assume, to begin with, that these components function independently and have identical lifetimes with cumulative distribution function (cdf) $F_1(x)$, probability density function (pdf) $f_1(x)$, and hazard rate function (hf) $h_1(x) = f_1(x)/(1-F_1(x))$. Next, we assume that, following the first failure, all surviving units face an increased load and continue to function independently but identically distributed with cdf, pdf and hf changed to $F_2(x)$, $f_2(x)$ and $h_2(x)$, respectively, and so on. In general, immediately following the j-th failure, the remaining $n-j$ surviving units face an increased load and continue to function independently but distributed with cdf, pdf and hf changed to $F_{j+1}(x)$, $f_{j+1}(x)$ and $h_{j+1}(x)$, respectively, for $j = 2, 3, \cdots, r$.

In practice, it is quite reasonable to assume that (Balakrishnan et al. (2008))

$$h_1(t) < h_2(t) < \cdots < h_n(t). \tag{1}$$

One way of considering the model in (1) is through the proportional hazard rate model. Specifically, we take

$$h_j(t) = \alpha_j h_0(t), \quad \text{for } j = 1, 2, \cdots, n, \tag{2}$$

where $h_0(t)$ is a baseline hazard rate function and $\alpha_1, \cdots, \alpha_n$ are positive constants. In fact, under this setting, we assume that $F_j(t) = 1 - [1 - F_0(t)]^{\alpha_j}$ for $j = 1, \cdots, n$ where $F_0(t)$ is the cdf of the baseline distribution. Thus, under this model, it assumed that when the j-th failure occurs, the failure rates of the remaining components in the system are changed from $\alpha_j h_0(t)$ to $\alpha_{j+1} h_0(t)$. In the literature, this model has been termed as conditionally proportional hazard rate model.

Remark 1. If we choose the constants α_j in (2) such that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, then the restricted sequential order statistics are obtained. For more details, see Balakrishnan et al. (2008) and the references contained therein.

2.2 The Likelihood Function

Suppose the first r SOS, denoted by $\mathbf{x}_* = (x_1, \cdots, x_r)$, are observed from the model in 2 with pdf and cdf $f(x)$ and $F(x)$, respectively. Then, the joint pdf of $\mathbf{x}_*$ is (Cramer and Kamps (1996))

$$\begin{aligned} f(x_1, \cdots, x_r) &= \frac{n!}{(n-r)!} \left(\prod_{j=1}^{r} \alpha_j \right) \left(\prod_{j=1}^{r-1} [1 - F(x_j)]^{m_j} f(x_j) \right) \\ &\quad \times [1 - F(x_r)]^{\alpha_r (n-r+1)-1} f(x_r), \end{aligned} \tag{3}$$

where $\alpha = (\alpha_1, \cdots, \alpha_r)$ and $m_j = (n - j + 1)\alpha_j - (n - j)\alpha_{j+1} - 1$ for $j = 1, \cdots, r - 1$. Let $\mathcal{C}$ be the class of all absolutely continuous distribution functions $F(x; \theta)$ of the form

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$$F(x;\theta) \quad = \quad 1 - \exp\left\{-k_\theta(x)\right\}, \qquad x > 0, \tag{4}$$

and hence with pdf $f(x;\theta) = k'_\theta(x)\exp\{-k_\theta(x)\}$, where $k_\theta(x)$ is the cumulative hazard function, increasing in x, $k'_\theta(x) = \frac{d}{d\theta}k_\theta(x)$, and θ is a vector of parameters. Substituting (4) into (3), the likelihood function (LF) of $\mathbf{x}_*$ becomes

$$\begin{aligned} L(\theta;\mathbf{x}_*) \quad &= \quad \frac{n!}{(n-r)!}\left(\prod_{j=1}^{r}\alpha_j\right)\left(\prod_{j=1}^{r}k'_\theta(x_j)\right) \\ &\quad \times \exp\left[-\left(\sum_{j=1}^{r-1}(m_j+1)k_\theta(x_j) + k_\theta(x_r)\alpha_r(n-r+1)\right)\right]. \end{aligned} \tag{5}$$

The class of distributions $\mathcal{C}$ in (4) includes many different lifetime distributions such as the exponential, Weibull and Pareto models (see AL-Hussaini (1999)). In what follows, we consider the Weibull distribution under this framework as the working model and develop different inferential methods for the model parameters. First, we propose a new alternative model here called the *power trend conditionally proportional hazard model*, denoted by PTCPHM.

2.3 Description of PTCPHM

A special case of the conditionally proportional hazard rate model for SOS in (2) is when $\alpha_j = a^j$ for $j = 1, \cdots, n$ and $a > 0$. We refer to this model as the *power trend conditionally proportional hazard model*, since in this case

$$h_j(t) = a h_{j-1}(t) = \cdots = a^j h_0(t), \quad \forall\ t > 0. \tag{6}$$

Then, the LF in (5) reduces to

$$L(\theta;\mathbf{x}_*) = \frac{n!}{(n-r)!}\left(a^{\frac{r(r+1)}{2}}\right)\left(\prod_{j=1}^{r} k'_\theta(x_j)\right) \times \exp\left[-\left(\sum_{j=1}^{r-1}((n-j+1)a^j - (n-j)a^{j+1})k_\theta(x_j) + k_\theta(x_r)a^r(n-r+1)\right)\right]. \quad (7)$$

For $a > 1$, PTCPHM reduces to the restricted SOS, while for $a = 1$, it corresponds to the usual order statistics based on a random sample. For this reason, we will treat the problem of testing the hypothesis $H : a = 1$ against the alternative $K : a > 1$ later in Subsection 3.3.

3 Weibull Analysis

The Weibull distribution has been used extensively in life-testing and reliability studies as it is a flexible lifetime model that includes increasing hazard rate and decreasing hazard rate in addition to the constant hazard rate corresponding to the exponential distribution; see, for example, Johnson et al. (1994). From the cdf of the Weibull distribution, it is evident that the Weibull distribution is a member of the class $\mathcal{C}$ in 4, with

$$k_\theta(x) = \left(\frac{x}{\sigma}\right)^\beta, \quad (8)$$

where $\theta = (\beta, \sigma)$, β is the shape parameter and σ is the scale parameter. We shall denote this Weibull distribution by $W(\beta, \sigma)$ from now on. In what follows, we develop inferential methods for the unknown parameters of the Weibull model based on SOS.

3.1 Point Estimation

Statistical inference on the basis of SOS has been discussed extensively in the literature; for example, see Cramer and Kamps (1996, 1998, 2001, 2003), Balakrishnan et al. (2008), Bedbur (2010).

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Schenk et al. (2011) considered Bayes estimation and prediction on the basis of multiply Type-II censored data arising from one- and two-parameter exponential distributions; see also Shafay et al. (2012). Since the Weibull distribution is more flexible than the exponential distribution for modelling failure times as mentioned above, we discuss here the estimation of parameters of the two-parameter Weibull model based on a Type-II censored sample of SOS.

From (5) and (8), the LF associated with $\mathbf{x}_*$ simplifies to

$$L(\alpha,\beta,\sigma;\mathbf{x}_*) = \left(\frac{n!}{(n-r)!}\right)\left(\prod_{j=1}^{r}\alpha_j\right)\left(\frac{\beta^r}{\sigma^{r\beta}}\right)[\eta(\mathbf{x}_*)]^{\beta-1} \times \exp\left[-\frac{1}{\sigma^\beta}\left(\sum_{j=1}^{r-1}(m_j+1)x_j^\beta + \alpha_r(n-r+1)x_r^\beta\right)\right], \tag{9}$$

and the log-likelihood function (LLF) becomes

$$l(\alpha,\beta,\sigma;\mathbf{x}_*) = \log\left(\frac{n!}{(n-r)!}\right) + \sum_{j=1}^{r}\log\alpha_j + r\log\beta - r\beta\log\sigma + (\beta-1)\log\eta(\mathbf{x}_*) - \frac{1}{\sigma^\beta}\left[\sum_{j=1}^{r-1}(m_j+1)x_j^\beta + \alpha_r(n-r+1)x_r^\beta\right], \tag{10}$$

where $\eta_r(\mathbf{x}_*) = \prod_{j=1}^{r} x_j$. We shall now develop the maximum likelihood (ML) estimates of the unknown parameters separately in two cases depending on whether α is known or unknown.

Case (i): α Known and (β, σ) Unknown

Suppose the constants α_j $(1 \le j \le r)$ in (9) are all known. From (10), the likelihood equations for β and σ are derived in this case as follows:

$$\begin{cases} \frac{r}{\beta} - r\log\sigma + \log\eta(\mathbf{x}_*) + \frac{\log\sigma}{\sigma^\beta}\left[\sum_{j=1}^{r-1}(m_j+1)x_j^\beta + \alpha_r(n-j+1)x_r^\beta\right] \\ -\frac{1}{\sigma^\beta}\left[\sum_{j=1}^{r-1}(m_j+1)(\log x_j)(x_j^\beta) + \alpha_r(n-r+1)(\log x_r)x_r^\beta\right] = 0, \\ \\ -\frac{r\beta}{\sigma} + \frac{\beta}{\sigma^{\beta+1}}\left[(\sum_{j=1}^{r-1}(m_j+1)x_j^\beta + \alpha_r(n-r+1)x_r^\beta)\right] = 0. \end{cases} \tag{11}$$

After some algebraic calculations that not reported here for the sake of brevity, the ML estimate of the scale parameter σ is obtained as

$$\widehat{\sigma}_1 \quad = \quad \left(\frac{(\sum_{j=1}^{r-1}(m_j+1)x_j^{\widehat{\beta}_1} + \alpha_r(n-r+1)x_r^{\widehat{\beta}_1})}{r}\right)^{\frac{1}{\widehat{\beta}_1}},$$

where $\widehat{\beta}_1$ is the ML estimate of the shape parameter β obtained numerically by solving the following equation:

$$\frac{1}{\beta} = \frac{\sum_{j=1}^{r-1}(m_j+1)(\log x_j)(x_j^\beta) + \alpha_r(n-r+1)(\log x_r)x_r^\beta}{\sum_{j=1}^{r-1}(m_j+1)x_j^\beta + \alpha_r(n-r+1)x_r^\beta} - \frac{1}{r}\sum_{j=1}^{r}\log x_j. \tag{12}$$

For the special case when $r = n$ and $\alpha_1 = \cdots = \alpha_n = 1$, SOS reduces to the usual complete sample. In this case, the ML estimates of the parameters σ and β simplify as follows:

$$\widehat{\sigma} = \sqrt[\widehat{\beta}]{\sum_{j=1}^{n}\frac{x_j^{\widehat{\beta}}}{n}}$$

and $\widehat{\beta}$ as the solution of the equation

$$\frac{1}{\widehat{\beta}} + \frac{1}{n}\sum_{j=1}^{n}\log x_j = \frac{\sum_{j=1}^{n}(\log x_j)x_j^{\widehat{\beta}}}{\sum_{j=1}^{n}x_j^{\widehat{\beta}}}, \tag{13}$$

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respectively, as given in Lehmann and Cassella (1998, p.468). They proved that Eq. (13) has a unique solution. It should be mentioned here that Balakrishnan and Kateri (2008) extended this result to different forms of censored Weibull data. In an analogous manner, we shall show here that Eq. (12) has a unique solution. To end this, let

$$h(\beta) = \frac{\sum_{j=1}^{r-1}(m_j+1)(\log x_j)(x_j^{\beta}) + \alpha_r(n-r+1)(\log x_r)x_r^{\beta}}{\sum_{j=1}^{r-1}(m_j+1)x_j^{\beta} + \alpha_r(n-r+1){x_r}^{\beta}} - \frac{1}{r}\sum_{j=1}^{r}\log x_j. \tag{14}$$

Then, Eq. (12) is equivalent to the equation $h(\beta) = 1/\beta$. Notice that

$$\frac{\partial h(\beta)}{\partial \beta} = \frac{\sum_{j=1}^{r-1}(m_j+1)(\log x_j)^2(x_j^{\beta}) + \alpha_r(n-r+1)(\log x_r)^2 x_r^{\beta}}{\sum_{j=1}^{r-1}(m_j+1)x_j^{\beta} + \alpha_r(n-r+1){x_r}^{\beta}} - \left(\frac{\sum_{j=1}^{r-1}(m_j+1)(\log x_j)(x_j^{\beta}) + \alpha_r(n-r+1)(\log x_r)x_r^{\beta}}{\sum_{j=1}^{r-1}(m_j+1)x_j^{\beta} + \alpha_r(n-r+1){x_r}^{\beta}}\right)^2 \tag{15}$$

For $a < n/(n-1)$, we prove that Equation (15) is positive. To do this, for $j = 1, \cdots, r-1$, let

$$\begin{aligned} a_j &= \frac{\sqrt{(m_j+1)}(\log x_j)(x_j^{\beta/2})}{\sqrt{\sum_{j=1}^{r-1}(m_j+1)x_j^{\beta} + \alpha_r(n-r+1){x_r}^{\beta}}} \\ b_j &= a_j \\ a_r &= \frac{\sqrt{\alpha_r(n-r+1)}(\log x_r)(x_r^{\beta/2})}{\sqrt{\sum_{j=1}^{r-1}(m_j+1)x_j^{\beta} + \alpha_r(n-r+1){x_r}^{\beta}}} \\ b_r &= a_r. \end{aligned}$$

Therefore, $m_j+1 = (n-j+1)\alpha_j - (n-j)\alpha_{j+1} > 0$. Applying the well known Cauchy-Schwartz inequality in (15), we conclude that the function $h(\beta)$ is increasing in β. Also, it is easy to verify that

$$-\infty \quad = \quad \lim_{\beta \to 0} h(\beta) < \frac{1}{r} \sum_{j=1}^{r} \log x_j < \log x_{(r)} = \lim_{\beta \to \infty} h(\beta).$$

Hence, the equation $h(\beta) = 1/\beta$ has a unique solution, as required. For $a \geq n/(n-1)$, the problem of uniqueness MLE remains open.

Case (ii): $\boldsymbol{\alpha}$ and $(\boldsymbol{\beta}, \boldsymbol{\sigma})$ are Unknown

In this case, we assume that the vector $\alpha = (\alpha_1, \cdots, \alpha_r)$ and the parameters β and σ in (9) are all unknown. Since there would be $r + 2$ unknown parameters in this case (Cramer and Kamps (1996)), we restrict our attention to the subclass with power trend in proportionality for the hazard rate function, defined by $\alpha_j = a^j, \;\; 1 \leq j \leq r$, where $a > 1$ is an unknown parameter. Earlier in Subsection 2.3, we referred to this subclass as PTCPHM. This assumption implies that $\alpha_1 < \alpha_2 < \cdots < \alpha_r$, as supposed by Balakrishnan et al. (2008). Thus, the hazard function of lifetimes of surviving components will increase. This case has been considered in the literature and is known as order restricted sequential order statistics; see, for example, Balakrishnan et al. (2008). Under PTCPHM, the LF in (9) becomes

$$\begin{aligned} L(\beta, \sigma, a; \mathbf{x}_*) = & \frac{n!}{(n-r)!} \left(a^{\frac{r(r+1)}{2}} \right) \frac{\beta^r}{\sigma^{r\beta}} \left[\eta(\mathbf{x}_*) \right]^{\beta - 1} \\ & \times \exp \Bigg[-\frac{1}{\sigma^\beta} \Bigg(\sum_{j=1}^{r-1} \Big[(n-j+1)a^j - (n-j)a^{j+1} \Big] x_j^\beta \\ & + a^r (n-r+1) x_r^\beta \Bigg) \Bigg], \end{aligned} \tag{16}$$

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with the corresponding LLF as

$$\begin{aligned} l(\beta,\sigma,a;\mathbf{x}_*) =& \log\left(\frac{n!}{(n-r)!}\right)+\frac{r(r+1)}{2}\log a \\ &+ r\log\beta - r\beta\log\sigma + (\beta-1)\log\eta(\mathbf{x}_*) \\ &- \frac{1}{\sigma^\beta}\left[\sum_{j=1}^{r-1}\left[(n-j+1)a^j-(n-j)a^{j+1}\right]x_j^\beta + a^r(n-r+1)x_r^\beta\right]. \end{aligned} \quad (17)$$

Thus, the ML estimates of the unknown parameters β, σ and a need to be obtained by solving the likelihood equations

$$\partial l/\partial\sigma = \partial l/\partial\beta = \partial l/\partial a = 0. \quad (18)$$

Explicit expressions for the partial derivatives in Eq. (18) are presented in the Appendix. From Eq. (18) and after some algebraic manipulations, the ML estimate of the parameter σ is

$$\widehat{\sigma}_2 = \left(\frac{\sum_{j=1}^{r-1}(m_j+1)x_j^{\widehat{\beta}_2} + \widehat{a}_2^r(n-r+1)x_r^{\widehat{\beta}_2}}{r}\right)^{\frac{1}{\widehat{\beta}_2}}, \quad (19)$$

where $\widehat{a}_2$ and $\widehat{\beta}_2$ are the ML estimates of the parameters a and β, obtained by solving the equations given in the Appendix B.

3.2 Approximate Interval Estimation

Since Eq. (12) has a unique solution, the ML estimates are consistent, asymptotically normal, and efficient. Therefore, from Lehmann and Casella (1998, p. 463, Theorem 5.1), the random vector $\sqrt{n}(\hat{\beta}_2-\beta,\hat{\sigma}_2-\sigma,\hat{a}_2-a)^T$ converges to the multivariate normal $\mathbf{N}_3(\mathbf{0}^T,[\mathbf{I}(\beta,\sigma,a)]^{-1})$ as n goes to infinity, where $\mathbf{0}^T=(0,0,0)$ and $\mathbf{I}(\beta,\sigma,a)$ is the Fisher information matrix given by

$$\mathbf{I}(\beta,\sigma,a) = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix}, \quad (20)$$

where $w_{11} = -E\left[\partial^2 \log L/\partial\beta^2\right]$, $w_{12} = w_{21} = -E\left[\partial^2 \log L/\partial\beta\partial\sigma\right]$, $w_{13} = w_{31} = -E\left[\partial^2 \log L/\partial\beta\partial a\right]$, $w_{22} = -E\left[\partial^2 \log L/\partial\sigma^2\right]$, $w_{23} = w_{32} = -E\left[\partial^2 \log L/\partial\sigma\partial a\right]$, and $w_{33} = -E\left[\partial^2 \log L/\partial a^2\right]$. Explicit expressions for w_{ij}, $i,j = 1,2,3$, are presented in the Appendix A. Since it is not possible to obtain the expectations involved in w_{ij}, we will use the observed Fisher information, denoted by $\hat{I}(\beta,\sigma,a)$, obtained by replacing β, σ and a by the corresponding ML estimates based on SOS into (20). Hence, the approximate $100(1-\gamma)\%$ equi-tailed confidence intervals for β, σ and a are, respectively, given by

$$\left(\hat{\beta} - z_{1-\gamma/2}\sqrt{\frac{1}{|\widehat{\mathbf{I}}(\beta,\sigma,a)|}\hat{b}_{11}}, \hat{\beta} + z_{1-\gamma/2}\sqrt{\frac{1}{|\widehat{\mathbf{I}}(\beta,\sigma,a)|}\hat{b}_{11}}\right), \qquad (21)$$

$$\left(\hat{\sigma} - z_{1-\gamma/2}\sqrt{\frac{1}{\left|\widehat{\mathbf{I}}(\beta,\sigma,a)\right|}\hat{b}_{22}}, \hat{\sigma} + z_{1-\gamma/2}\sqrt{\frac{1}{|\widehat{\mathbf{I}}(\beta,\sigma,a)|}\hat{b}_{22}}\right), \qquad (22)$$

and

$$\left(\hat{a} - z_{1-\gamma/2}\sqrt{\frac{1}{|\widehat{\mathbf{I}}(\beta,\sigma,a)|}\hat{b}_{33}}, \hat{a} + z_{1-\gamma/2}\sqrt{\frac{1}{|\widehat{\mathbf{I}}(\beta,\sigma,a)|}\hat{b}_{33}}\right), \qquad (23)$$

where z_γ is the γ-percentile of the standard normal distribution, $b_{11} = w_{22}w_{33} - w_{23}w_{32}$, $b_{22} = w_{11}w_{33} - w_{13}w_{31}$, $b_{33} = w_{11}w_{22} - w_{12}w_{21}$, and $\hat{b}_{ij}$ is obtained by replacing w_{ij} by the corresponding ML estimate, denoted by $\hat{w}_{ij}$. For simultaneous confidence intervals, there are various methods. For example, the Bonferroni simultaneous confidence intervals for the unknown parameters are obtained from 21, (22) and 23 by replacing $z_{1-\gamma/2}$ by $z_{1-\gamma/6}$. One can also use the recent approach of Casella and Hwang (2012) to obtain more accurate simultaneous confidence sets.

Remark 2. Let $g(\beta,\sigma,a)$ be an arbitrary measurable function of the three parameters β,σ and a. By the use of multivariate version of delta method, we obtain

$$\sqrt{n}\left(g(\hat{\beta}_2,\hat{\sigma}_2,\hat{a}_2) - g(\beta,\sigma,a)\right) \xrightarrow{D} \mathbf{N}(\mathbf{0}^T, \nabla g(\beta,\sigma,a)\left[\mathbf{I}(\beta,\sigma,a)\right]^{-1}\nabla g(\beta,\sigma,c)^T) \qquad (24)$$

as $n \to \infty$, where $\nabla g(\beta,\sigma,a)$ denotes the gradient of the function $g(\beta,\sigma,a)$.

As an example, suppose we want to estimate the baseline survival function of the lifetime at a fixed time (say, t_0), i.e., $S(t_0;\beta,\sigma) = \exp\left\{-(t_0/\sigma)^\beta\right\}$.

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Then, from (24), we conclude that, as $n \to \infty$, $\sqrt{n}(S(t_0; \hat{\beta}_2, \hat{\sigma}_2) - S(t_0; \beta, \sigma))$ tends to a normal distribution with mean zero and variance

$$\frac{e^{-2(t_0/\sigma)^\beta}\,(t_0/\sigma)^{2\beta}}{|\widehat{\mathbf{I}}(\beta,\sigma,a)|}\left[b_{11}\log^2\left(\frac{t_0}{\sigma}\right) - 2b_{21}\left(\frac{\beta}{\sigma}\right)\log\left(\frac{t_0}{\sigma}\right) + b_{22}\left(\frac{\beta}{\sigma}\right)^2\right],$$

which readily yields an approximate $100(1-\gamma)\%$ equi-tailed confidence interval for $S(t_0; \beta, \sigma)$ as

$$S(t_0; \hat{\beta}_2, \hat{\sigma}_2) \pm z_{1-\gamma/2}\sqrt{\frac{e^{-2(t_0/\sigma)^\beta}\,(t_0/\sigma)^{2\beta}}{|\widehat{\mathbf{I}}(\beta,\sigma,a)|}\left[\hat{b}_{11}\log^2\left(\frac{t_0}{\sigma}\right) - 2\hat{b}_{21}\left(\frac{\beta}{\sigma}\right)\log\left(\frac{t_0}{\sigma}\right) + \hat{b}_{22}\left(\frac{\beta}{\sigma}\right)^2\right]},$$

where $b_{12} = b_{21} = -(w_{12}w_{33} - w_{32}w_{13})$, $b_{13} = b_{31} = w_{12}w_{23} - w_{22}w_{13}$, $b_{23} = b_{32} = -(w_{11}w_{23} - w_{13}w_{21})$, $b_{33} = w_{11}w_{22} - w_{12}w_{21}$, and $\hat{b}_{ij}$ is obtained by replacing w_{ij} by the corresponding ML estimate, denoted by $\hat{w}_{ij}$.

3.3 Hypothesis Testing for PTCPHM

In the preceding section, we assumed a power trend for α_j as $\alpha_j = a^j$ $(1 \le j \le r)$. Here, we assume that $a \ge 1$. Under the null hypothesis $H : a = 1$, we assume that the failure of a component does not effect the distribution of the remaining components, while under the alternative $K : a > 1$, we have an increase in the hazard rates of the remaining components. Therefore, the problem of testing the null hypothesis $H : a = 1$ against the alternative $K : a > 1$ is of natural interest. From Lehmann and Romano (2005, p. 513), a generalized likelihood ratio (GLR) test has the rejection region as

$\{\mathbf{x}_* : \Lambda < k\}$, where

$$\Lambda = \frac{\sup_{\beta>0,\sigma>0} L(\beta,\sigma,1;\mathbf{x}_*)}{\sup_{\beta>0,\sigma>0,a\geq 1} L(\beta,\sigma,a;\mathbf{x}_\star)}$$
$$= \left(\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\right)^r \left(\frac{\sum_{j=1}^{r-1}\left[(n-j)\hat{a}_2^j - (n-j+1)\hat{a}_2^{j+1}\right] x_j^{\widehat{\beta}_2} + \widehat{a}_2^r (n-r+1) x_r^{\widehat{\beta}_2}}{\sum_{j=1}^{r-1} x_j^{\widehat{\beta}_1} + (n-r+1) x_r^{\widehat{\beta}_1}}\right)^r$$
$$\left(\widehat{a}_2^{-\left[\frac{r(r+1)}{2}\right]}\right) \times [\eta(\mathbf{x}_\star)]^{\left(\widehat{\beta}_1 - \widehat{\beta}_2\right)}.$$

Under the null hypothesis H and the usual regularity conditions (see Lehmann and Cassella (1998)), $-2\log\Lambda$ has asymptotically the chi-square distribution with 1 degree of freedom. Thus, for large n, the rejection region of the GLR test of size γ is

$$-2\log\Lambda > \chi^2_{1,1-\gamma}, \tag{25}$$

where $\chi^2_{\upsilon,1-\gamma}$ is the γ-th percentile of the chi-square distribution with υ degrees of freedom. Also, the actual level of the GLR test may be obtained by means of a Monte Carlo (MC) simulation study for given a, β and σ.

3.4 Exponential Baseline Distribution

When $\beta = 1$, the $W(\beta,\sigma)$ -distribution reduces to the exponential distribution, denoted by $Exp(\sigma)$. Now, suppose the baseline distribution is $Exp(\sigma)$. Then, the LF in (9) simplifies to

$$L(\sigma,\alpha;\mathbf{x}^*) = \frac{n!}{(n-r)!}\left(\prod_{j=1}^{r}\alpha_j\right)\sigma^{-r} \times \exp\left[-\frac{1}{\sigma}\left(\sum_{j=1}^{r-1}(m_j+1)x_j + \alpha_r(n-r+1)x_r\right)\right]. \tag{26}$$

For α known, the ML estimate of σ is

$$\widehat{\sigma}_{1,E} = \frac{\sum_{j=1}^{r-1}(m_j+1)x_j + \alpha_r(n-r+1)x_r}{r}.$$

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For α unknown and $\alpha_j = a^j$ $(1 \leq j \leq r)$, we have

$$\begin{aligned} L(\sigma, a; \mathbf{x}^*) &= \frac{n!}{(n-r)!} a^{\frac{r(r+1)}{2}} \sigma^{-r} \\ & \times \exp\left[-\frac{1}{\sigma}\left(\sum_{j=1}^{r-1}(m_j+1)x_j + a^r(n-r+1)x_r\right)\right]. \quad (27) \end{aligned}$$

Thus, the ML estimates of σ and a based on SOS are derived by solving the following equations:

$$\sigma = \frac{\sum_{j=1}^{r}(n-j+1)a^j x_j - \sum_{j=1}^{r-1}(n-j)a^{j+1}x_j}{r} \quad (28)$$

$$a = \frac{r(r+1)\sigma}{2\left[\sum_{j=1}^{r}(n-j+1)ja^{j-1}x_j - \sum_{j=1}^{r-1}(n-j)(j+1)a^j x_j\right]}. \quad (29)$$

Similarly, a GLR test of size γ for the null hypothesis $H: a = 1$ against the alternative $K: a > 1$ has its critical region as

$$-2\log\Lambda > \chi^2_{1,\gamma}, \quad (30)$$

where

$$\Lambda = \left(\frac{\sum_{j=1}^{r-1}(m_j+1)x_j + \widehat{a}^r_{2,E}(n-r+1)x_r}{\sum_{j=1}^{r-1}x_j + (n-r+1)x_r}\right)^r \left(\widehat{a}_{2,E}^{-\left[\frac{r(r+1)}{2}\right]}\right),$$

where $\widehat{a}_{2,E}$ is the ML estimate of a based on SOS are derived by solving Equations (28) and (29).

Remark 3. Schenk *et al.* (2011) and Shafay et al. (2012) discussed Bayesian estimation and prediction based on sequential order statistics from the exponential distribution.

4 Aircraft Data Set

To demonstrate the performance of the results obtained in Section 3, we present an illustrative example in this section.
Smith (2002, p. 130) gave failure times of aircraft components for a life-test, originally due to Mann and Fertig (1973). In the test, $n = 13$ components

Table 1. Fitted models for failure time of aircraft components

Model	ML estimates			Log-likelihood	AIC
	$\hat{\beta}$	$\hat{\sigma}$	$\hat{a}$		
$a = 1, \beta = 1$	-	2.3050	-	-18.3508	38.7016
$\beta = 1$	-	2.9704	1.04936	-18.2372	40.4743
$a = 1$	1.41746	2.27315	-	-17.6335	39.2670
$a \neq 1, \beta > 0, \sigma > 0$	2.02392	1.25749	0.823473	-16.7801	39.5602

were placed in a Type-II censored life test in which the failure times of first 10 components to fail were observed (in hours) as

$$0.22, 0.50, 0.88, 1.00, 1.32, 1.33, 1.54, 1.76, 2.50, 3.00.$$

Assuming that the lifetimes of the components are iid with an exponential distribution (i.e., $\beta = 1$ and $a = 1$ in Eq. 16), the ML estimate of the mean is obtained to be $\hat{\sigma} = 2.305$.

We analysed these data with different models, and the results of the fitted models are presented in Table 1. Adopting Akaike information criterion (AIC), we conclude that the exponential model is still supported for the failure times. But, a PTCPHM is also found to be suitable. The estimate hazard rate ratio function under the PTCPHM with the weibull baseline cdf is dervied from Equations (6), (8) and Table 1 as

$$\begin{aligned}\hat{h}_j(t) &= \hat{a}^j \hat{h}_0(t) \\ &= \hat{a}^j \left(\frac{t}{\hat{\sigma}}\right)^{\hat{\beta}} \\ &= 0.823473^j \left(\frac{t}{1.25749}\right)^{2.02392}, \quad \forall t > 0, \quad j = 1, \cdots, 13.\end{aligned}$$

The inverse of observed Fisher information is obtained from (20) as

$$I^{-1} = \begin{bmatrix} 0.520823 & -0.155695 & -0.0674688 \\ -0.155695 & 0.16624 & 0.0404666 \\ -0.0674688 & 0.0404666 & 0.0139039 \end{bmatrix}. \tag{31}$$

Thus, the approximate 95% confidence intervals for the unknown param-

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eters β, σ and a in (10) are obtained from (21), (22), (23) and (31) to be $(0.609424, 3.43842)$, $(0.458347, 2.05663)$ and $(0.592359, 1.05459)$, respectively. We also obtain an approximate simultaneous 95% confidence region for the three parameters from (21), (22) and (23) by replacing $z_{1-(0.05/2)}$ by $z_{1-(0.05/6)} = 2.39398$ to be

$$(\beta, \sigma, a) \in (0.299101, 3.74874) \times (0.283025, 2.23196) \times (0.541656, 1.10529).$$

5 Concluding Remarks

In this paper, by considering an engineering system with n components, we have considered the situation when failures of components increase the load on surviving components thus changing their lifetime distribution. For modelling this situation, we have described the conditionally proportional hazard rates model in the framework of sequential order statistics. We have then focused on a special case of this model called the power trend conditionally proportional hazard model, and developed inferential methods based on a Type-II censored data from this model under the assumption of Weibull distributed lifetimes. Using a well-known data of Mann and Fertig (1973) on failure times of aircraft components, we have illustrated the developed inferential results. Since the framework presented here is applicable for the general exponential family of distributions of the form in (4), inferential results analogous to those for the Weibull distribution here can be developed for some other lifetime distributions of interest such as Pareto. It will also be of interest to develop point and interval prediction methods under this general framework. Work on these problems is currently under progress and we hope to report these findings in a future paper.

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Appendix A

From Eq. (17), explicit expressions for the partial derivatives in Eq. (18) and for w_{ij}, $i, j, = 1, 2, 3$, in (20) are obtained as follows:

$$\frac{\partial l}{\partial \sigma} = -\frac{r\beta}{\sigma} + \frac{\beta}{\sigma^{\beta+1}}\left[\sum_{j=1}^{r-1}\left\{(n-j+1)a^j - (n-j)a^{j+1}\right\}x_j^\beta + a^r(n-r+1)x_r^\beta\right],$$

$$\begin{aligned}\frac{\partial l}{\partial \beta} =& \frac{r}{\beta} - r\log\sigma + \log\eta(\mathbf{x}_\star) \\ &+ \frac{\log\sigma}{\sigma^\beta}\left[\sum_{j=1}^{r-1}\left\{(n-j+1)a^j - (n-j)a^{j+1}\right\}x_j^\beta + a^r(n-j+1)x_r^\beta\right] \\ &- \frac{1}{\sigma^\beta}\left[\sum_{j=1}^{r-1}\left\{(n-j+1)a^j - (n-j)a^{j+1}\right\}(\log x_j)x_j^\beta \right. \\ &\left. + a^r(n-r+1)(\log x_r)x_r^\beta\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial l}{\partial a} =& \frac{r(r+1)}{2a} \\ &- \frac{1}{\sigma^\beta}\left[\sum_{j=1}^{r-1}\left\{(n-j+1)\ ja^{j-1} - (n-j)(j+1)a^j\right\}x_j^\beta \right. \\ &\left. + (n-r+1)\ ra^{r-1}x_r^\beta\right];\end{aligned}$$

$$\begin{aligned}w_{11} =& -\frac{\partial^2 l}{\partial \beta^2} \\ =& \frac{r}{\beta^2} + \frac{(\log\sigma)^2}{\sigma^\beta}\left[\sum_{j=1}^{r-1}(m_j+1)x_j^\beta + a^r(n-r+1)x_r^\beta\right] \\ &- \frac{2\log\sigma}{\sigma^\beta}\left[\sum_{j=1}^{r-1}(m_j+1)(\log x_j)x_j^\beta + a^r(n-r+1)(\log x_r)x_r^\beta\right] \\ &+ \frac{1}{\sigma^\beta}\left[\sum_{j=1}^{r-1}(m_j+1)(\log x_j)^2x_j^\beta + a^r(n-r+1)(\log x_r)^2x_r^\beta\right],\end{aligned}$$

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$$\begin{aligned} w_{12} &= -\frac{\partial^2 l}{\partial\beta\partial\sigma} \\ &= \frac{r}{\sigma} + \frac{\beta\log\sigma - 1}{\sigma^{\beta+1}}\left[\sum_{j=1}^{r-1}(m_j+1)x_j^{\beta} + a^r(n-r+1)x_r^{\beta}\right] \\ &\quad -\frac{\beta}{\sigma^{\beta+1}}\left[\sum_{j=1}^{r-1}(m_j+1)(\log x_j)x_j^{\beta} + a^r(n-r+1)(\log x_r)x_r^{\beta}\right], \end{aligned}$$

$$\begin{aligned} w_{13} =& -\frac{\partial^2 l}{\partial\beta\partial a} \\ =& -\frac{\log\sigma}{\sigma^{\beta}}\left[\sum_{j=1}^{r-1}\left\{(n-j+1)ja^{j-1} - (n-j)(j+1)a^j\right\}x_j^{\beta}\right. \\ &\left. + (n-r+1)ra^{r-1}x_r^{\beta}\right] \\ &+ \frac{1}{\sigma^{\beta}}\left[\sum_{j=1}^{r-1}\left\{(n-j+1)ja^{j-1} - (n-j)(j+1)a^j\right\}(\log x_j)x_j^{\beta}\right. \\ &\left. + (n-r+1)ra^{r-1}(\log x_r)x_r^{\beta}\right], \end{aligned}$$

$$\begin{aligned} w_{22} &= -\frac{\partial^2 l}{\partial\sigma^2} \\ &= \frac{-r\beta}{\sigma^2} + \beta(\beta+1)\sigma^{-(\beta+2)}\left[\sum_{j=1}^{r-1}(m_j+1)x_j^{\beta} + a^r(n-r+1)x_r^{\beta}\right], \end{aligned}$$

$$\begin{aligned} w_{23} &= -\frac{\partial^2 l}{\partial\sigma\partial a} \\ &= -\frac{\beta}{\sigma^{\beta+1}}\left[\sum_{j=1}^{r-1}\left\{(n-j+1)ja^{j-1} - (n-j)(j+1)a^j\right\}(\log x_j)x_j^{\beta}\right. \\ &\quad \left. +(n-r+1)ra^{r-1}(\log x_r)x_r^{\beta}\right], \end{aligned}$$

$$
\begin{aligned}
w_{33} =& -\frac{\partial^2 l}{\partial a^2} \\
=& \frac{r(r+1)}{2a^2} \\
& + \sigma^{-\beta}\Bigg[\sum_{j=1}^{r-1}\Big\{(n-j+1)j(j-1)a^{j-2} - (n-j)(j+1)ja^{j-1}\Big\}x_j^{\beta} \\
& + (n-r+1)r(r-1)a^{r-2}x_r^{\beta}\Bigg].
\end{aligned}
$$

Appendix B

The ML estimates of the parameters a and β in Equation (19) is obtained by solving the following equations:

$$
a = \frac{(r+1)\left(\sum_{j=1}^{r-1}(m_j+1)x_j^{\beta_2} + a_2^r(n-r+1)x_r^{\beta_2}\right)}{2\left[\sum_{j=1}^{r}(n-j+1)\ ja^{j-1}x_j^{\beta_2} - \sum_{j=1}^{r-1}(n-j)(j+1)a^j x_j^{\beta_2}\right]} \tag{32}
$$

and

$$
\frac{1}{\beta} = \frac{\sum_{j=1}^{r-1}(m_j+1)(\log x_j)(x_j^{\beta}) + a^r(n-r+1)(\log x_r)x_r^{\beta}}{\sum_{j=1}^{r-1}(m_j+1)x_j^{\beta} + a^r(n-r+1)x_r^{\beta}} - \frac{1}{r}\sum_{j=1}^{r}\log x_j, \tag{33}
$$

where $m_j = (n-j+1)a^j - (n-j)a^{j+1} - 1$ for $j = 1, \cdots, r-1$.

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