



Modified Signed Log-Likelihood Ratio Test for Comparing the Correlation Coefficients of Two Independent Bivariate Normal Distributions

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Abstract. In this paper, we use the method of modified signed log-likelihood ratio test for the problem of testing the equality of correlation coefficients in two independent bivariate normal distributions. We compare this method with two other approaches, Fisher's Z-transform and generalized test variable, using a Monte Carlo simulation. It indicates that the proposed method is better than the other approaches, in terms of the actual sizes and powers especially when the sample sizes are unequal. We illustrate performance of the proposed approach, using a real data set.

Keywords. Bivariate normal distribution; actual size; correlation coefficient; maximum likelihood estimator; power.

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1 Introduction

The linear association between two normal variables is usually measured by correlation coefficient. Statistical inferences for this parameter are divided to a single bivariate sample and several bivariate samples problems. In the case of a single sample, Fisher (1915) for the first time and then Hotelling (1953) provided the exact density function of product moment correlation coefficient. In testing and constructing the confidence interval for the correlation coefficient, Fisher (1921) introduced the well-known Fisher Z-transform, Sun and Wong (2007) proposed a likelihood-based higher-order asymptotic method, and Krishnamoorthy and Xia (2007) proposed a generalized pivotal approach. Kazemi and Jafari (2015) compared some confidence intervals for the correlation coefficient.

The problem of equality of two correlations arises practically, for example in comparing the correlations between the laterality of blood flow in each brain region and verbal memory score across gender (see Bilker et al., 2004). For inference about this problem, Zar (1999) used the Fisher Z-transform to test that whether all samples came from populations having common correlation coefficient, and Olkin and Finn (1995) obtained an asymptotic distribution of the difference between two sample correlation coefficients. Krishnamoorthy and Xia (2007) proposed a generalized test variable and studied the performances of this test, Fisher Z-transform test and Olkin and Finn's method. They concluded that Olkin and Finn's method is satisfactory for large sample sizes, and Fisher Z-transform test is conservative (i.e. its actual size is very smaller than the nominal level) when the samples are small. In addition, the actual size of generalized test variable is close to the nominal level for moderate samples.

The aim of this paper is to develop a modified signed log-likelihood ratio (MSLR) method for testing the equality of two correlation coefficients in two independent bivariate normal distributions. We used the test statistic proposed by DiCiccio et al. (2001) which has a simple form and then applied the traditional signed log-likelihood ratio (SLR) test in its form. We propose a parametric bootstrap method to approximate the distribution of SLR statistic and then use it to compute the MSLR statistic. Our simulation results show that MSLR test always are satisfactory regardless of the sample sizes and values of the common correlation coefficient.

This paper is organized as follows: Some preliminaries are given in Section 2. The MSLR is explained for testing the equality of two correlation coeffi-

cients in Section 3. In Section 4, a simulation study is performed to evaluate and compare the actual sizes and powers of MSLR, Fisher's Z-transform and generalized variable approaches. Also, the approaches are illustrated using a real example.

2 Preliminaries

Let (X_{ij}, Y_{ij}) , $i = 1, 2$ and $j = 1, 2, \dots, n_i$ be a random sample from the bivariate normal distribution with mean vector $\boldsymbol{\mu}_i = (\mu_{1i}, \mu_{2i})'$ and variance covariance matrix

$$\Sigma_i = \begin{bmatrix} \sigma_{1i}^2 & \rho_i \sigma_{1i} \sigma_{2i} \\ \rho_i \sigma_{1i} \sigma_{2i} & \sigma_{2i}^2 \end{bmatrix}, \quad i = 1, 2.$$

Our goal is to test the hypothesis

$$H_0 : \rho_1 = \rho_2 = \rho \quad \text{vs.} \quad H_1 : \rho_1 \neq \rho_2, \quad (1)$$

where ρ is the common correlation coefficient. We use the method of SLR for this problem. To apply this method, we need to find the full and constrained maximum likelihood estimators (MLE) of the unknown model parameters. Considering $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, where $\boldsymbol{\theta}_i = (\mu_{1i}, \mu_{2i}, \sigma_{1i}, \sigma_{2i}, \rho_i)$, it can be shown that the log-likelihood function can be written as

$$\ell(\boldsymbol{\theta}) = c + \ell_1(\boldsymbol{\theta}_1) + \ell_2(\boldsymbol{\theta}_2), \quad (2)$$

where

$$\begin{aligned} \ell_i(\boldsymbol{\theta}_i) = & -n_i \log(\sigma_{1i}) - n_i \log(\sigma_{2i}) - \frac{n_i}{2} \log(1 - \rho_i^2) \\ & - \frac{n_i \mu_{1i}^2}{2(1 - \rho_i^2) \sigma_{1i}^2} - \frac{n_i \mu_{2i}^2}{2(1 - \rho_i^2) \sigma_{2i}^2} \\ & + \frac{n_i \rho_i \mu_{1i} \mu_{2i}}{2(1 - \rho_i^2) \sigma_{1i} \sigma_{2i}} - \frac{\rho_i \sum_{k=1}^{n_i} x_{ij}^2}{2(1 - \rho_i^2) \sigma_{1i}^2} \\ & - \frac{\rho_i \sum_{j=1}^{n_i} y_{ij}^2}{2(1 - \rho_i^2) \sigma_{2i}^2} + \frac{(\mu_{1i} \sigma_{2i} - \mu_{2i} \sigma_{1i}) \sum_{k=1}^{n_i} x_{ij}}{(1 - \rho_i^2) \sigma_{1i}^2 \sigma_{2i}} \end{aligned}$$

$$+ \frac{(\mu_{2i}\sigma_{1i} - \mu_{1i}\sigma_{2i}) \sum_{k=1}^{n_i} y_{ij}}{(1 - \rho_i^2) \sigma_{2i}^2 \sigma_{1i}} + \frac{\rho_i \sigma_{1i} \sum_{j=1}^{n_i} x_{ij} y_{ij}}{(1 - \rho_i^2) \sigma_{1i} \sigma_{2i}}.$$

It is known that, under the full model (without any constraint), the MLE's of parameters θ_i are $\hat{\theta}_i = (\bar{X}_i, \bar{Y}_i, S_{1i}, S_{2i}, R_i)$, $i = 1, 2$, where

$$\begin{aligned} \bar{X}_i &= \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}, & \bar{Y}_i &= \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}, & R_i &= \frac{S_{12(i)}}{\sqrt{S_{1i}^2 S_{2i}^2}}, \\ S_{1i}^2 &= \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2, & S_{2i}^2 &= \frac{1}{n_i} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_i)^2, \end{aligned}$$

and $S_{12(i)} = \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i) (Y_{ij} - \bar{Y}_i)$.

For the constrained model i.e. under the hypothesis in (1), Pearson (1933) showed that the MLE of the common correlation coefficient, $\tilde{\rho}$, is obtained by solving the following equation:

$$\frac{n_1 (r_1 - \tilde{\rho})}{(1 - \tilde{\rho} r_1)} + \frac{n_2 (r_2 - \tilde{\rho})}{(1 - \tilde{\rho} r_2)} = 0, \quad (3)$$

where r_i is the observed value of R_i (For more details, refer to Pearson, 1933; Donner and Rosner, 1980). Also, the constrained MLE's of parameters μ_{1i} , μ_{2i} , σ_{1i}^2 and σ_{2i}^2 are

$$\tilde{\mu}_{1i} = \bar{x}_i, \quad \tilde{\mu}_{2i} = \bar{y}_i, \quad \tilde{\sigma}_{1i}^2 = \frac{s_{1i}^2 (1 - \tilde{\rho} r_i)}{1 - \tilde{\rho}^2}, \quad \tilde{\sigma}_{2i}^2 = \frac{s_{2i}^2 (1 - \tilde{\rho} r_i)}{1 - \tilde{\rho}^2},$$

where $\bar{x}_i, \bar{y}_i, s_{1i}^2$ and s_{2i}^2 are the observed value of $\bar{X}_i, \bar{Y}_i, S_{1i}^2$ and S_{2i}^2 . In this case, the MLE of parameter θ_i is $\tilde{\theta}_i = (\tilde{\mu}_{1i}, \tilde{\mu}_{2i}, \tilde{\sigma}_{1i}^2, \tilde{\sigma}_{2i}^2, \tilde{\rho})$.

Donner and Rosner (1980) defined

$$R_F = \frac{\exp(2\bar{Z}) - 1}{\exp(2\bar{Z}) + 1} = \tanh(\bar{Z}), \quad (4)$$

where $\bar{Z} = \frac{(n_1-3)Z_1 + (n_2-3)Z_2}{n_1 + n_2 - 6}$ and $Z_i = \frac{1}{2} \log\left(\frac{1+R_i}{1-R_i}\right) = \tanh^{-1}(R_i)$, $i = 1, 2$. They showed that the estimators $\tilde{\rho}$ and R_F are close when the samples sizes are equal, i.e. $n_1 = n_2$. Simulation studies (not reported here) show that the results for MSLR method based on the estimators $\tilde{\rho}$ and R_F are close to

each other. But the estimator R_F decrease the execution time. Therefore, R_F can be used instead of $\tilde{\rho}$ to estimate the common correlation coefficient ρ .

The following lemma helps us to generate the sample correlation coefficient from a random sample of a bivariate normal distribution. It is notable that this formula is different from formula (16) of Krishnamoorthy and Xia (2007) and also it cannot be used as the generalized pivotal quantity.

Lemma 1. *Let R_i be the sample correlation coefficient from a bivariate normal distribution with mean vector $\boldsymbol{\mu}_i$ and variance-covariance matrix Σ_i . Then*

$$R_i \stackrel{d}{=} \frac{\rho_i^* V_i + N_i}{\sqrt{(\rho_i^* V_i + N_i)^2 + W_i^2}}, \quad (5)$$

where $\rho_i^* = \frac{\rho_i}{\sqrt{1-\rho_i^2}}$, and V_i^2 , W_i^2 , and N_i are independent random variables with $\chi_{(n-1)}^2$, $\chi_{(n-2)}^2$ and $N(0, 1)$ distributions, respectively.

Proof. Let $S_i = \begin{bmatrix} S_{1i}^2 & S_{12(i)} \\ S_{12(i)} & S_{2i}^2 \end{bmatrix}$. It is well-known that $A_i = n_i S_i \sim W(n_i - 1, \Sigma_i)$, i.e. it has a Wishart distribution with $n_i - 1$ degrees of freedom and parameter Σ_i . Since Σ_i is a positive definite matrix, there is a unique lower triangular matrix, L_i , such that $L_i L_i' = \Sigma_i$ (Cholesky decomposition) and it is easily verified that

$$P_i = L_i^{-1} A_i L_i'^{-1} \sim W(n_i - 1, I),$$

where I is the identity matrix. Put $P_i = C_i C_i'$. From Theorem 3.2.14 of Muirhead (1982), the elements of matrix $C_i = \begin{bmatrix} V_i & 0 \\ N_i & W_i \end{bmatrix}$ are independent and distributed as

$$V_i^2 \sim \chi_{(n_i-1)}^2, \quad W_i^2 \sim \chi_{(n_i-1)}^2 \quad \text{and} \quad N_i \sim N(0, 1).$$

It can be shown that matrix L_i has the following form:

$$L_i = \begin{bmatrix} \sigma_{1i} & 0 \\ \rho_i \sigma_{2i} & \sigma_{2i} \sqrt{1 - \rho_i^2} \end{bmatrix}.$$

Therefore, we have

$$A_i = \left(A_{kl}^{(i)} \right) \stackrel{d}{=} L_i C_i C_i' L_i' \\ = \begin{bmatrix} \sigma_{1i}^2 V_i^2 & \sigma_{1i} \sigma_{2i} \sqrt{1 - \rho_i^2} (\tilde{\rho}_i V_i^2 + N_i V_i) \\ \sigma_{1i} \sigma_{2i} \sqrt{1 - \rho_i^2} (\tilde{\rho}_i V_i^2 + N_i V_i) & \sigma_{2i}^2 (1 - \rho_i^2) [(\tilde{\rho}_i V_i + N_i)^2 + W_i^2] \end{bmatrix}.$$

Since $R_i = \frac{A_{12}^{(i)}}{\sqrt{A_{11}^{(i)} A_{22}^{(i)}}}$, the proof is completed. \square

3 Testing the Equality of Two Correlation Coefficients

In this section, we consider the problem of testing the equality of two independent correlation coefficients. At first, we propose the method of MSLR and give an algorithm that can be used for this problem. Then, we review two other existing approaches.

3.1 Modified Signed Log-Likelihood Ratio Test

It is easily verified that the SLR test statistic to test the hypothesis in (1) has the following form

$$\begin{aligned} \text{SLR} &= \sqrt{2} \text{sign}(r_1 - r_2) \sqrt{2(\ell(\hat{\boldsymbol{\theta}}) - \ell(\tilde{\boldsymbol{\theta}}))} \\ &= \text{sign}(r_1 - r_2) \left(\sum_{i=1}^2 n_i \log \left(\frac{(1 - \tilde{\rho}_i r_i)^2}{(1 - r_i^2)(1 - \tilde{\rho}_i^2)} \right) \right)^{\frac{1}{2}}, \end{aligned} \quad (6)$$

where $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2)$ and $\tilde{\boldsymbol{\theta}} = (\tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2)$, and $\text{sign}(x) = 1$, if $x > 0$ and $\text{sign}(x) = -1$, if $x < 0$.

It is well known that SLR is asymptotically distributed as a standard normal distribution (Cox and Hinkley, 1979), and a p-value for testing the hypothesis in (1) is

$$p = 2(1 - \Phi(|\text{SLR}_0|)), \quad (7)$$

where SLR_0 is the observed value of the statistic SLR and $\Phi(t)$ is the standard normal distribution function.

If we use the estimator R_F instead of $\tilde{\rho}$, the SLR statistic in (6) is rewritten as

$$\text{SLR} = \text{sign}(r_1 - r_2) \left(\sum_{i=1}^k n_i \log \left\{ \frac{(1 - R_F r_i)^2}{(1 - r_i^2)(1 - R_F^2)} \right\} \right)^{\frac{1}{2}}. \quad (8)$$

Pierce and Peters (1992) showed the SLR test is not very accurate, and some modifications are needed to increase the accuracy of the SLR. There exist various ways to improve the accuracy of this approximation by adjusting the SLR statistic. For the various ways to improve the accuracy of SLR method, refer to the works of Barndorff-Nielsen (1986, 1991), Skovgaard (2001), and DiCiccio et al. (2001). We used the method proposed by DiCiccio et al. (2001), which has the following form

$$\text{MSLR} = \frac{\text{SLR} - m(\text{SLR})}{\sqrt{v(\text{SLR})}}, \quad (9)$$

where $m(\text{SLR})$ and $v(\text{SLR})$ are the mean and variance of the SLR statistic evaluated at the constrained MLE's of the model parameters and is asymptotically distributed as a standard normal distribution.

Krishnamoorthy and Lee (2014) used the parametric bootstrap approach to approximate the mean and variance of the MSLR test statistic for the problem of testing the equality of normal coefficients of variation. We use this approach for the problem of testing the equality of two normal correlation coefficients. In Section 4, using Monte Carlo simulation, we will show that this new method is more accurate than the other competing methods. This approach is given in the following algorithm:

Algorithm 1. Given r_1 and r_2 ,

1. Compute r_F , the observed value of estimator R_F in (4).
2. Generate $V_i^2 \sim \chi_{(n_i-1)}^2$, $W_i^2 \sim \chi_{(n_i-2)}^2$, $N_i \sim N(0, 1)$, $i = 1, 2$.
3. Compute $r_F^* = r_F / \sqrt{1 - r_F^2}$.
4. Compute r_i^* by substituting r_F instead of ρ in (5) as

$$r_i^* = \frac{r_F^* V_i + N_i}{\sqrt{(r_F^* V_i + N_i)^2 + W_i^2}}.$$

5. Compute the test statistic SLR in (8).
6. Repeat steps 3-5 for a large number of times (say $M = 10,000$).
7. Compute the sample mean and sample variance of SLR and compute the MSLR in (9).
8. Determine the p-value for testing $H_0 : \rho_1 = \rho_2$ vs $H_1 : \rho_1 \neq \rho_2$ as

$$\text{p-value} = 2(1 - \Phi(|\text{MSLR}|)). \quad (10)$$

3.2 Fisher's Z-transform

It is well-known that

$$Z_i = \frac{1}{2} \log\left(\frac{1 + R_i}{1 - R_i}\right) = \tanh^{-1}(R_i),$$

has asymptotic normal distribution with mean $\tanh^{-1}(\rho_i)$ and variance $\frac{1}{n_i-3}$. Therefore, a test statistic for testing $H_0 : \rho_1 = \rho_2$, vs $H_1 : \rho_1 \neq \rho_2$ can be given by extending the one-sample Fisher's Z-transformation to the two-sample case. Consider the following test statistic

$$FZ = \frac{Z_1 - Z_2}{\sqrt{\frac{1}{n_1-3} + \frac{1}{n_2-3}}}. \quad (11)$$

Then, FZ has asymptotic standard normal distribution, and the null hypothesis is rejected if $|FZ| > z_{\alpha/2}$. For more details, refer to Zar (1999) and Krishnamoorthy and Xia (2007).

3.3 Generalized Test Variable

Krishnamoorthy and Xia (2007) proposed a generalized pivotal variable for ρ_i as

$$G_{\rho_i} = \frac{r_i^* W_i - Z_i}{\sqrt{(r_i^* W_i - Z_i)^2 + V_i^2}}, \quad i = 1, 2, \quad (12)$$

where $r_i^* = \frac{r_i}{\sqrt{1-r_i^2}}$, and V_i , W_i , and Z_i are independent random variables with

$$V_i^2 \sim \chi_{(n_i-1)}^2, \quad W_i^2 \sim \chi_{(n_i-2)}^2, \quad Z_i \sim N(0, 1).$$

Therefore, a generalized pivotal variable for $\rho_1 - \rho_2$ is given as

$$G_\rho = G_{\rho_1} - G_{\rho_2}.$$

So, the generalized p-value for testing $H_0 : \rho_1 = \rho_2$ vs $H_1 : \rho_1 \neq \rho_2$ is given by

$$p = 2 \min \{P(G_\rho < 0), P(G_\rho > 0)\}. \quad (13)$$

4 Numerical Study

4.1 Simulation Study

We performed a simulation study with 10,000 replications to evaluate and compare the actual sizes of three approaches: the modified signed likelihood ratio test (MSLR), Fisher's Z-transform (FZ) test, and generalized test variable (GV). We generate random samples of size n_1 and n_2 from two independent bivariate normal distributions for different values of common correlation $\rho = 0.0, 0.1, 0.2, \dots, 0.9$. We obtained the sample correlation coefficient and then the p -values of the MSLR, FZ and GV to test the hypothesis $H_0 : \rho_1 = \rho_2$ vs $H_1 : \rho_1 \neq \rho_2$. Here, we consider the nominal level $\alpha = 0.05$. The results are given in Table 1.

We can conclude that

- i. The actual size of MSLR test is satisfactory for all different values of common correlation coefficient and sample sizes.
- ii. The actual size of FZ test is smaller than the nominal level when the sample sizes are small,
- iii. The actual size of GV test is very smaller than the nominal level when n_1 is small and n_2 is large.

Since, the MSLR test is the only test that controls the correct frequency of rejected hypotheses in all cases, we recommend the MSLR for practical applications.

We also performed a simulation study to compare the powers of the considered approaches. The results are given in Tables 2 and 3. It can be concluded that the powers of the three tests MSLR, GV and Fisher Z-transform are close when the sample sizes are equal. But the power of MSLR is larger than powers of GV and Fisher Z-transform when the sample sizes are unequal.

Table 1. The actual sizes of the tests at nominal level $\alpha = 0.05$.

ρ	Method	n_1, n_2				
		5,5	5,10	10,10	5,15	5,25
0	MSLR	0.052	0.050	0.051	0.056	0.049
	FZ	0.046	0.049	0.048	0.045	0.052
	GV	0.052	0.049	0.053	0.047	0.040
0.1	MSLR	0.053	0.050	0.055	0.050	0.053
	FZ	0.043	0.047	0.050	0.046	0.047
	GV	0.051	0.048	0.054	0.044	0.036
0.2	MSLR	0.052	0.051	0.049	0.051	0.048
	FZ	0.045	0.045	0.050	0.051	0.046
	GV	0.052	0.050	0.051	0.047	0.038
0.3	MSLR	0.051	0.048	0.050	0.050	0.048
	FZ	0.043	0.045	0.048	0.046	0.050
	GV	0.051	0.051	0.051	0.045	0.033
0.4	MSLR	0.053	0.049	0.048	0.050	0.051
	FZ	0.041	0.043	0.049	0.047	0.045
	GV	0.050	0.051	0.052	0.044	0.036
0.5	MSLR	0.051	0.049	0.049	0.050	0.048
	FZ	0.047	0.045	0.049	0.044	0.045
	GV	0.051	0.049	0.051	0.047	0.037
0.6	MSLR	0.054	0.050	0.048	0.048	0.049
	FZ	0.045	0.046	0.049	0.042	0.045
	GV	0.051	0.049	0.051	0.048	0.035
0.7	MSLR	0.049	0.052	0.051	0.050	0.050
	FZ	0.040	0.046	0.052	0.044	0.045
	GV	0.053	0.050	0.051	0.045	0.036
0.8	MSLR	0.054	0.049	0.049	0.050	0.052
	FZ	0.038	0.045	0.049	0.046	0.044
	GV	0.051	0.051	0.051	0.047	0.037
0.9	MSLR	0.053	0.055	0.051	0.052	0.049
	FZ	0.041	0.044	0.045	0.044	0.046
	GV	0.051	0.048	0.053	0.045	0.039

Table 2. Empirical powers of the tests at nominal level $\alpha = 0.05$ with $\rho_1 = 0.05$.

ρ	Method	n_1, n_2							
		5,5	5,10	10,10	15,10	5,15	5,25	20,20	25,25
0.15	MSLR	0.052	0.055	0.057	0.063	0.061	0.069	0.073	0.078
	FZ	0.049	0.046	0.051	0.048	0.046	0.052	0.074	0.080
	GV	0.058	0.056	0.057	0.055	0.044	0.043	0.077	0.079
0.25	MSLR	0.059	0.064	0.067	0.082	0.063	0.076	0.111	0.123
	FZ	0.051	0.056	0.068	0.072	0.052	0.059	0.105	0.135
	GV	0.061	0.057	0.066	0.070	0.048	0.042	0.112	0.133
0.35	MSLR	0.069	0.071	0.088	0.111	0.088	0.089	0.172	0.201
	FZ	0.056	0.064	0.091	0.094	0.067	0.067	0.178	0.214
	GV	0.070	0.063	0.094	0.095	0.057	0.054	0.177	0.212
0.45	MSLR	0.076	0.085	0.097	0.164	0.099	0.108	0.251	0.324
	FZ	0.066	0.067	0.117	0.133	0.080	0.084	0.252	0.313
	GV	0.080	0.074	0.122	0.135	0.070	0.064	0.255	0.317
0.55	MSLR	0.087	0.105	0.151	0.226	0.115	0.123	0.377	0.442
	FZ	0.071	0.082	0.153	0.181	0.100	0.102	0.351	0.422
	GV	0.091	0.085	0.153	0.185	0.079	0.073	0.355	0.425
0.65	MSLR	0.102	0.131	0.201	0.290	0.136	0.171	0.456	0.549
	FZ	0.092	0.101	0.208	0.235	0.119	0.122	0.448	0.541
	GV	0.121	0.108	0.204	0.236	0.099	0.091	0.435	0.538
0.75	MSLR	0.113	0.151	0.245	0.372	0.192	0.189	0.528	0.638
	FZ	0.098	0.124	0.240	0.291	0.139	0.140	0.540	0.650
	GV	0.125	0.121	0.252	0.291	0.123	0.114	0.546	0.651
0.85	MSLR	0.131	0.175	0.290	0.448	0.214	0.235	0.652	0.776
	FZ	0.119	0.142	0.297	0.348	0.167	0.162	0.638	0.741
	GV	0.134	0.139	0.294	0.363	0.152	0.133	0.643	0.739
0.95	MSLR	0.143	0.208	0.356	0.523	0.253	0.264	0.709	0.823
	FZ	0.129	0.168	0.349	0.414	0.193	0.196	0.717	0.814
	GV	0.155	0.171	0.355	0.410	0.173	0.158	0.710	0.813

Table 3. Empirical powers of the tests at nominal level $\alpha = 0.05$ with $\rho_1 = 0.05$.

ρ	Method	n_1, n_2							
		5,5	5,10	10,10	15,10	5,15	5,25	20,20	25,25
-0.15	MSLR	0.054	0.058	0.066	0.072	0.058	0.061	0.063	0.064
	FZ	0.049	0.051	0.068	0.069	0.053	0.053	0.075	0.077
	GV	0.063	0.056	0.071	0.071	0.048	0.038	0.071	0.079
-0.25	MSLR	0.066	0.072	0.084	0.113	0.064	0.071	0.103	0.145
	FZ	0.058	0.068	0.090	0.099	0.058	0.057	0.119	0.137
	GV	0.071	0.061	0.089	0.100	0.055	0.044	0.114	0.133
-0.35	MSLR	0.074	0.089	0.116	0.121	0.080	0.083	0.183	0.207
	FZ	0.060	0.074	0.117	0.121	0.068	0.068	0.179	0.211
	GV	0.080	0.074	0.120	0.141	0.061	0.055	0.176	0.212
-0.45	MSLR	0.085	0.114	0.164	0.192	0.101	0.092	0.246	0.319
	FZ	0.075	0.092	0.162	0.188	0.078	0.078	0.249	0.319
	GV	0.090	0.092	0.164	0.186	0.077	0.065	0.257	0.314
-0.55	MSLR	0.106	0.133	0.205	0.247	0.116	0.118	0.366	0.418
	FZ	0.090	0.108	0.206	0.247	0.101	0.101	0.349	0.426
	GV	0.109	0.107	0.208	0.252	0.086	0.081	0.346	0.432
-0.65	MSLR	0.121	0.166	0.252	0.278	0.136	0.158	0.417	0.569
	FZ	0.105	0.130	0.248	0.304	0.122	0.122	0.439	0.541
	GV	0.119	0.126	0.257	0.299	0.100	0.098	0.448	0.542
-0.75	MSLR	0.144	0.194	0.308	0.385	0.179	0.196	0.565	0.620
	FZ	0.120	0.154	0.304	0.372	0.143	0.143	0.546	0.651
	GV	0.143	0.150	0.312	0.368	0.127	0.113	0.552	0.651
-0.85	MSLR	0.161	0.216	0.368	0.445	0.188	0.240	0.660	0.734
	FZ	0.142	0.174	0.364	0.440	0.166	0.166	0.630	0.750
	GV	0.166	0.181	0.368	0.442	0.147	0.135	0.638	0.735
-0.95	MSLR	0.179	0.253	0.430	0.505	0.254	0.268	0.709	0.819
	FZ	0.158	0.202	0.425	0.511	0.193	0.193	0.716	0.813
	GV	0.192	0.199	0.424	0.512	0.176	0.164	0.717	0.821

Table 4. correlations between laterality of blood flow in three brain regions and verbal memory score.

Gender	laterality of blood flow		
	Temporal	Subcortical	Frontal
Male	$r_{M,vt} = -0.340$	$r_{M,vs} = 0.641$	$r_{M,vf} = -0.032$
Female	$r_{F,vt} = 0.812$	$r_{F,vs} = 0.491$	$r_{F,vf} = -0.212$

Table 5. P-values of the tests for equality of correlations between laterality of blood flow in three brain regions and verbal memory score.

method	$\rho_{M,vt} = \rho_{F,vt}$	$\rho_{M,vs} = \rho_{F,vs}$	$\rho_{M,vf} = \rho_{F,vf}$
MSLR	0.0008	0.5978	0.6677
GV	0.0008	0.5948	0.6682
Fisher Z	0.0004	0.6018	0.6673

4.2 Real Data

In this example, we test the equality of two independent correlations in three groups of data. This data set is given by Bilker et al. (2004) and also have been analyzed by Krishnamoorthy and Xia (2007).

For each of two groups of 14 men and 14 women, the sample correlation between a verbal memory score (v) and laterality of blood flow in each of three brain regions, namely, temporal (t), frontal (f) and subcortical (s) are obtained in Table 4. It may be of interest to compare the correlations between the laterality of blood flow in each brain region and verbal memory score across gender. The results are given in Table 5. We can find that there is no significant difference between male and female correlations in frontal and subcortical cases.

5 Conclusion

Existing methods for comparing the correlation coefficients of two independent bivariate normal distributions do not perform well in a range of small-sample settings. Krishnamoorthy and Xia (2007) obtained a generalized pivotal quantity for difference of two correlation coefficients and they gave a method for testing the equality of two correlation coefficients using this generalized pivotal quantity. By using a simulation study, they showed the

test size of their method is greater than the nominal level for small sample sizes. In other words, they showed that their method is liberal.

Using the method of modified signed log likelihood, Krishnamoorthy and Lee (2014) considered the problem of equality of coefficients of variation for independent normal populations. This method is an exact method to test a hypothesis for unknown parameter. The accuracy of this method is very satisfactory such that the actual size of test is approximately close to nominal level even for small sample sizes. In this article, we used the MSLR method for testing the equality of two independent correlation coefficients because of the accuracy of this method for inference about the unknown model parameter and that the other competing methods have deviations that we cannot rely to them. In this paper, we explained the generating the sample correlation coefficients of a bivariate normal distribution (See Lemma 2.1). Then, we obtained the MSLR test statistics for the problem of the equality of two correlation coefficients. All the mathematical formulas obtained in our paper are different from that of used in chapter 4 of Krishnamoorthy and Xia (2007). As we see, for using the MSLR method, one should consider the MLEs of the unknown parameters but the MLE was not stated in Krishnamoorthy and Xia (2007) at all. We compared our method with the method of GV used by Krishnamoorthy and Xia (2007) and another competing method by simulation studies. We found that MSLR test always are satisfactory in terms of actual sizes regardless of the sample sizes and values of the common correlation coefficient in comparison with existing methods. Also, the power of MSLR is larger than powers of GV and Fisher Z-transform when the sample sizes are unequal.

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