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Stress-strength Reliability of Exponential Distribution based on Type-I Progressively Hybrid Censored Samples

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Abstract. This paper considers the estimation of the stress-strength parameter, say R, based on two independent Type-I progressively hybrid censored samples from exponential populations with different parameters. The maximum likelihood estimator and asymptotic confidence interval for R are obtained. Bayes estimator of R is also derived under the assumption of independent gamma priors. A Monte Carlo simulation study is used to evaluate the performance of maximum likelihood estimator, Bayes estimator and asymptotic confidence interval. Finally, a pair of real data sets is analyzed for illustrative purposes.

Keywords. Stress-strength model; Type-I progressive Hybrid censoring; Bayes estimator; Maximum likelihood estimator; Monte Carlo simulation.

MSC 2010: 62N01; 62N02.

1 Introduction

Hybrid censoring scheme that is a combination of the usual Type-I and Type-II censoring scenarios has been discussed extensively in the literature. Kundu

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and Joarder (2006) proposed Type-I progressive hybrid censoring procedure by introducing a stopping time $\min(X_{m:m:n}, T)$. This approach is based on progressively Type-II censored order statistics $X_{1:m:n} \leqslant \cdots \leqslant X_{m:m:n}$, and it guarantees that the life test would not last beyond time T. T is prefixed time and is named threshold time. Suppose $X_{1:m:n}, \ldots, X_{m:m:n}$ are the progressively Type-II right censored order statistics corresponding to n units on a life-test with the progressive censoring scheme $\boldsymbol{r} = (r_1, \ldots, r_m)$. Then, the Type-I progressive hybrid censoring arises if the termination time of the life-test is chosen to be $\min(X_{m:m:n}, T)$. We denote this censoring scheme by $(\boldsymbol{r}, T)$. Under this censoring scheme, we have one of the two following types of observations:

$$\begin{aligned} &\text{Case I}: \quad \{X_{1:m:n} \leqslant \cdots \leqslant X_{m:m:n}\} \quad \text{if} \quad X_{m:m:n} \leqslant T, \\ &\text{Case II}: \quad \{X_{1:m:n} \leqslant \cdots \leqslant X_{d:m:n}\} \quad \text{if} \quad d < m, \quad X_{d:m:n} < T < X_{d:m:n}, \end{aligned}$$

where $X_{1:m:n} \leqslant X_{2:m:n} \leqslant \ldots$ denote the observed ordered failure times of the experimental units and d is the number of failures before the time T, where d is a random variable with support $\{0, 1, \ldots, m\}$.

In the stress-strength modelling, $R = P(Y < X)$ is a measure of component reliability when it is subjected to random stress Y and has strength X. It has many applications in engineering and medical sciences. For more details and applications, see Kotz et al. (2003). Much work has been done on the problem of estimating R of some distributions. Such as, Kundu and Gupta (2005), Kundu and Gupta (2006), Saracoglu et al. (2012), Asgharzadeh et al. (2011; 2013), Asgharzadeh and Kazemi (2014), Govidarajulu (1967), Enis and Geisser (1971), Downtown (1973), Awad et al. (1981), Sathe and Shah (1981), Awad and Gharraf (1986), Constantine et al. (1986), Gupta and Gupta (1990), McCool (1991), Nandi and Aich (1996), Surles and Padgett (1998), Gupta et al. (1999), Gupta and Brown (2001), Raqab and Kundu (2005), Mokhlis (2005), Raqab et al. (2008), Kundu and Raqab (2009), Gupta et al. (2010), Rao et al. (2013) and Rao et al. (2016).

The exponential distribution is a simple and fundamental lifetime model and is a member of such important families of reliability distributions as Weibull and gamma; see Balakrishnan and Basu (1995). The exponential distribution denoted by $\text{Exp}(\theta)$ has the probability density function (pdf)

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \qquad x > 0, \ \theta > 0, \tag{1}$$

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and the cumulative distribution function (cdf)

$$F(x;\theta) = 1 - e^{-\frac{x}{\theta}}, \qquad x > 0,\ \theta > 0. \tag{2}$$

Here θ is a scale parameter.

Based on Type-I progressively hybrid censored samples, we consider the estimation of $R = P(Y < X)$ when X and Y have two independent exponential distributions with different parameters.

2 Maximum Likelihood Estimator of R

Let $X \sim \text{Exp}(\theta_1)$ and $Y \sim \text{Exp}(\theta_2)$ be independent random variables. Then it can be easily seen that

$$R = P(Y < X) = \frac{\theta_1}{\theta_1 + \theta_2}. \tag{3}$$

Our interest is estimating R based on Type-I progressively hybrid censored data on both variables. To derive the MLE of R, first we obtain the MLEs of θ_1 and θ_2. Suppose $\boldsymbol{X} = (X_{1:m_1:n_1}, \ldots, X_{d_1:m_1:n_1})$ is a Type-I progressively hybrid censored sample from $\text{Exp}(\theta_1)$ with censored scheme $(\boldsymbol{r}_1, T_1)$ and $\boldsymbol{Y} = (Y_{1:m_2:n_2}, \ldots, Y_{d_2:m_2:n_2})$ is a Type-I progressively hybrid censored sample from $\text{Exp}(\theta_2)$ with censored scheme $(\boldsymbol{r}_2, T_2)$, where $\boldsymbol{r}_i = (r_{i1}, r_{i2}, \ldots, r_{im_i})$ and $\sum_{j=1}^{m_i} r_{ij} = n_i$, for $i = 1, 2$. For convenience, we will write $(X_1, \ldots, X_{d_1})$ instead of $(X_{1:m_1:n_1}, \ldots, X_{d_1:m_1:n_1})$, and $(Y_1, \ldots, Y_{d_2})$ instead of $(Y_{1:m_1:n_1}, \ldots, Y_{d_2:m_2:n_2})$. According to Cramér and Balakrishnan (2013), we have

$$\begin{aligned} L(\theta_1, \theta_2) = {} & c_1 c_2 \theta_1^{-d_1} \theta_2^{-d_2} \\ & \times \exp\Bigg\{ -\frac{1}{\theta_1} \left[\sum_{j=1}^{d_1} (1 + r_{1j}) X_j + \gamma_{d_1+1} T_1 \right] \\ & - \frac{1}{\theta_2} \left[\sum_{j=1}^{d_2} (1 + r_{2j}) Y_j + \gamma'_{d_2+1} T_2 \right] \Bigg\}, \end{aligned} \tag{4}$$

where $c_1 = \prod_{j=1}^{d_1} \gamma_j$ and $c_2 = \prod_{j=1}^{d_2} \gamma'_j$, such that $\gamma_j = \sum_{k=j}^{m_1} (1 + r_{1j})$ and $\gamma'_j = \sum_{k=j}^{m_2} (1 + r_{2j})$.

From (4), the log-likelihood function for the Type-I progressively hybrid censored data without the normalizing constant is

$$\begin{aligned} l(\theta_1, \theta_2) &= -d_1 \log(\theta_1) - d_2 \log(\theta_2) - \frac{1}{\theta_1} \left[\sum_{j=1}^{d_1} (1 + r_{1j}) X_j + \gamma_{d_1+1} T_1 \right] \\ &\quad - \frac{1}{\theta_2} \left[\sum_{j=1}^{d_2} (1 + r_{2j}) Y_j + \gamma'_{d_2+1} T_2 \right]. \end{aligned} \tag{5}$$

We denoted the MLEs of θ_1 and θ_2 by $\hat{\theta}_1$ and $\hat{\theta}_2$ respectively, where can be obtained as the solution of

$$\frac{\partial l}{\partial \theta_1} = -\frac{d_1}{\theta_1} + \frac{1}{\theta_1^2} \left[\sum_{j=1}^{d_1} (1 + r_{1j}) X_j + \gamma_{d_1+1} T_1 \right] = 0, \tag{6}$$

$$\frac{\partial l}{\partial \theta_2} = -\frac{d_2}{\theta_2} + \frac{1}{\theta_2^2} \left[\sum_{j=1}^{d_2} (1 + r_{2j}) Y_j + \gamma'_{d_2+1} T_2 \right] = 0. \tag{7}$$

If $d_1 > 0$ and $d_2 > 0$, from (6) and (7), we obtain

$$\hat{\theta}_1 = \frac{1}{d_1} \left[\sum_{j=1}^{d_1} (1 + r_{1j}) X_j + \gamma_{d_1+1} T_1 \right], \tag{8}$$

$$\hat{\theta}_2 = \frac{1}{d_2} \left[\sum_{j=1}^{d_2} (1 + r_{2j}) Y_j + \gamma'_{d_2+1} T_2 \right]. \tag{9}$$

Therefore, we compute the MLE of R as

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$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}. \tag{10}$$

But using the likelihood function (4), if $d_1 = 0$ or $d_2 = 0$, then $\hat{\theta}_1$ or $\hat{\theta}_2$ does not exist and therefore $\hat{R}$ does not exist.

3 Asymptotic Confidence Interval of R

In this section, we propose the asymptotic confidence interval of R, through computing the inverse of the observed Fisher information matrix $I(\theta_1, \theta_2)$ and using the delta method.

From the log-likelihood function in (5), we obtain the observed Fisher information matrix of (θ_1, θ_2) as

$$I(\theta_1, \theta_2) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \theta_1^2} & \frac{\partial^2 l}{\partial \theta_1 \partial \theta_2} \\ \\ \frac{\partial^2 l}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 l}{\partial \theta_2^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix},$$

where

$$\begin{aligned} -I_{11} &= \frac{d_1}{\theta_1^2} - \frac{2}{\theta_1^3}\left[\sum_{j=1}^{d_1}(1 + r_{1j})X_j + \gamma_{d_1+1}T_1\right], \\ -I_{22} &= \frac{d_2}{\theta_2^2} - \frac{2}{\theta_2^3}\left[\sum_{j=1}^{d_2}(1 + r_{2j})Y_j + \gamma_{d_2+1}T_2\right], \\ I_{12} &= I_{21} = 0. \end{aligned}$$

Now, the asymptotic variance-covariance matrix, $A = [a_{ij}]$, for the MLEs is obtained by inverting the Fisher information matrix as

$$A = I^{-1}(\theta_1, \theta_2) = \frac{1}{I_{11}I_{22}}\begin{pmatrix} I_{22} & 0 \\ \\ 0 & I_{11} \end{pmatrix} = \begin{pmatrix} I_{11}^{-1} & 0 \\ \\ 0 & I_{22}^{-1} \end{pmatrix}.$$

Now, the variance of $\hat{R}$, denoted by B, can be obtained use the delta method.

We have $\hat{R} = g(\hat{\theta}_1, \hat{\theta}_2)$, where

$$g(\theta_1, \theta_2) = \frac{\theta_1}{\theta_1 + \theta_2}.$$

Therefore, $B = \boldsymbol{b}^t A \boldsymbol{b}$, where

$$\boldsymbol{b} = \begin{pmatrix} \frac{\partial g}{\partial \theta_1} \\ \\ \frac{\partial g}{\partial \theta_2} \end{pmatrix} = \frac{1}{(\theta_1 + \theta_2)^2} \begin{pmatrix} \theta_2 \\ -\theta_1 \end{pmatrix}.$$

It can be easily verified that

$$B = \boldsymbol{b}^t A \boldsymbol{b} = \frac{1}{(\theta_1 + \theta_2)^4} \left[\theta_2^2 I_{11}^{-1} + \theta_1^2 I_{22}^{-1}\right].$$

To compute the confidence interval of R, it is enough to estimate B. We can using the MLEs of θ_1 and θ_2, to estimate B. It can be easily verified that

$$\hat{B} = \frac{d_1 + d_2}{d_1 d_2} \cdot \frac{\hat{\theta}_1^2 \, \hat{\theta}_2^2}{(\hat{\theta}_1^2 + \hat{\theta}_2^2)^4}.$$

As a consequence, a $100(1-\alpha)\%$ asymptotic confidence interval of R is

$$(\hat{R} - Z_{1-\alpha/2}\sqrt{\hat{B}}, \hat{R} + Z_{1-\alpha/2}\sqrt{\hat{B}}),$$

where Z_α is $100\alpha-$th percentile of standard normal distribution.

4 Bayes Estimator of *R*

In this section, we obtain the Bayes estimator of R under Type-I progressive hybrid censoring. For this purpose, we have the following theorem.

***Theorem* 1.** *Let $\lambda_1 = \frac{1}{\theta_1} \sim Gamma(a_1, b_1)$ and $\lambda_2 = \frac{1}{\theta_2} \sim Gamma(a_2, b_2)$ are independent, where a_1 and a_2 are the shape parameters and b_1 and b_2 are the rate parameters. Under the squared error loss function, the Bayes estimation of R denoted by $\hat{R}_B$ as*

$$\hat{R}_B = \frac{a_2 + d_2}{s} \left(\frac{b_2 + w_2}{b_1 + w_1}\right)^{a_2 + d_2} F_{2,1}\left(s, a_2 + d_2 + 1, s + 1, 1 - \frac{b_2 + w_2}{b_1 + w_1}\right),$$

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such that

$$
\begin{aligned}
s &= a_1 + d_1 + a_2 + d_2, \\
w_1 &= \sum_{j=1}^{d_1} (1 + r_{1j}) x_j + \gamma_{d_1+1} T_1, \\
w_2 &= \sum_{j=1}^{d_2} (1 + r_{2j}) y_j + \gamma'_{d_2+1} T_2,
\end{aligned}
$$

and $F_{2,1}$ *is the hypergeometric function given by (see Saracoglu et al. (2012));*

$$
\begin{aligned}
F_{2,1}(a, b, c, z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} dt \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(c)\Gamma(a+k)\Gamma(b+k) z^k}{\Gamma(a)\Gamma(b)\Gamma(c+k) k!}.
\end{aligned}
$$

Proof. To obtain the posterior distributions of λ_1 and λ_2, we have

$$
\begin{aligned}
\pi(\lambda_1 | data) &\propto \lambda_1^{a_1+d_1-1} e^{-(b_1+w_1)\lambda_1}, \\
\pi(\lambda_2 | data) &\propto \lambda_2^{a_2+d_2-1} e^{-(b_2+w_2)\lambda_2}.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\lambda_1 | data &\sim Gamma(a_1 + d_1, b_1 + w_1), \\
\lambda_2 | data &\sim Gamma(a_2 + d_2, b_2 + w_2).
\end{aligned}
$$

Using the independency of λ_1 and λ_2, when $\lambda_1 > 0$ and $0 < r < 1$, we compute the joint pdf of $R = \frac{\lambda_2}{\lambda_1+\lambda_2}$ and λ_1 given the data as follows:

$$
\begin{aligned}
f_{(R,\lambda_1)|data}(r, \lambda_1) &= \frac{\lambda_1}{(1-r)^2} \, f_{(\lambda_2,\lambda_1)|data}\left(\frac{r\lambda_1}{1-r}, \lambda_1\right) \\
&= \frac{\lambda_1}{(1-r)^2} \cdot \frac{(b_2+w_2)^{a_2+d_2}}{\Gamma(a_2+d_2)} \left(\frac{r\lambda_1}{1-r}\right)^{a_2+d_2-1} e^{-(b_2+w_2)\frac{r\lambda_1}{1-r}} \\
&\quad \times \frac{(b_1+w_1)^{a_1+d_1}}{\Gamma(a_1+d_1)} \lambda_1^{a_1+d_1-1} e^{-(b_1+w_1)\lambda_1}
\end{aligned}
$$

$$= \frac{(b_1+w_1)^{a_1+d_1}}{\Gamma(a_1+d_1)} \cdot \frac{(b_2+w_2)^{a_2+d_2}}{\Gamma(a_2+d_2)} \cdot \frac{r^{a_2+d_2-1}}{(1-r)^{a_2+d_2+1}} \times \lambda_1^{s-1} e^{-\left[b_1+w_1+\frac{r}{1-r}(b_2+w_2)\right]\lambda_1}.$$

Now, we can obtain pdf of R given the data by integrating with respect to λ_1. So for $0 < r < 1$,

$$\begin{aligned} f_{R|data}(r) &= \frac{(b_1+w_1)^{a_1+d_1}}{\Gamma(a_1+d_1)} \cdot \frac{(b_2+w_2)^{a_2+d_2}}{\Gamma(a_2+d_2)} \cdot \frac{r^{a_2+d_2-1}}{(1-r)^{a_2+d_2+1}} \\ &\quad \times \int_0^\infty \lambda_1^{s-1} e^{-\left[b_1+w_1+\frac{r}{1-r}(b_2+w_2)\right]\lambda_1} d\lambda_1 \\ &= \frac{(b_1+w_1)^{a_1+d_1}}{\Gamma(a_1+d_1)} \cdot \frac{(b_2+w_2)^{a_2+d_2}}{\Gamma(a_2+d_2)} \cdot \frac{r^{a_2+d_2-1}}{(1-r)^{a_2+d_2+1}} \\ &\quad \times \frac{\Gamma(s)}{\left(b_1+w_1+\frac{r}{1-r}(b_2+w_2)\right)^s} \\ &= \frac{(b_1+w_1)^{a_1+d_1}}{\Gamma(a_1+d_1)} \cdot \frac{(b_2+w_2)^{a_2+d_2}}{\Gamma(a_2+d_2)} \Gamma(s) \\ &\quad \times \frac{r^{a_2+d_2-1}(1-r)^{a_1+d_1-1}}{\left((1-r)(b_1+w_1)+r(b_2+w_2)\right)^s}. \end{aligned}$$

Now, under the squared error loss function, we obtain $\hat{R}_B$ as follows:

$$\begin{aligned} \hat{R}_B &= \int_0^1 r f_{R|data}(r) dr \\ &= \frac{(b_1+w_1)^{a_1+d_1}}{\Gamma(a_1+d_1)} \cdot \frac{(b_2+w_2)^{a_2+d_2}}{\Gamma(a_2+d_2)} \Gamma(s) \\ &\quad \times \int_0^1 \frac{r^{a_2+d_2}(1-r)^{a_1+d_1-1}}{\left((1-r)(b_1+w_1)+r(b_2+w_2)\right)^s} dr \\ &= \frac{(b_1+w_1)^{a_1+d_1}}{\Gamma(a_1+d_1)} \cdot \frac{(b_2+w_2)^{a_2+d_2}}{\Gamma(a_2+d_2)} \cdot \frac{\Gamma(s)}{(b_1+w_1)^s} \end{aligned}$$

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$$
\begin{aligned}
& \times \int_0^1 \frac{r^{a_2+d_2}(1-r)^{a_1+d_1-1}}{\left(1-\frac{b_1+w_1-(b_2+w_2)}{b_1+w_1} r\right)^s} dr \\
= \quad & \frac{(b_1+w_1)^{a_1+d_1}}{\Gamma(a_1+d_1)} \cdot \frac{(b_2+w_2)^{a_2+d_2}}{\Gamma(a_2+d_2)} \cdot \frac{\Gamma(s)}{(b_1+w_1)^s} \\
& \times \frac{\Gamma(a_2+d_2+1)\Gamma(a_1+d_1)}{\Gamma(s+1)} \\
& \times F_{2,1}\left(s, a_2+d_2+1, s+1, 1-\frac{b_2+w_2}{b_1+w_1}\right) \\
= \quad & \frac{a_2+d_2}{s} \left(\frac{b_2+w_2}{b_1+w_1}\right)^{a_2+d_2} \\
& \times F_{2,1}\left(s, a_2+d_2+1, s+1, 1-\frac{b_2+w_2}{b_1+w_1}\right). \qquad (11)
\end{aligned}
$$

□

5 Simulation Study

In this section, Monte Carlo simulations are carried out to evaluate the performance of the MLEs, Bayes estimators and asymptotic confidence intervals for different censoring schemes. The computations are performed using `R` software and `Mathematica` software. We mainly evaluate the performances of the MLEs and Bayes estimators in terms of average estimators and mean of square errors (MSE). Also we evaluate the performances of the asymptotic confidence intervals in terms of average lengths and coverage probabilities.

From the sample, we compute the MLE and Bayes estimator of R using (10) and (11), respectively. We compute the MLE and Bayes estimator of R for different Type-I progressive hybrid censoring schemes $(\boldsymbol{r}_1, T_1)$ and $(\boldsymbol{r}_2, T_2)$, and we report the average estimator and MSE of the MLEs and Bayes estimators of R by 20,000 replications. The results are represented in the Table 2. The Type-II progressive censoring schemes that are employed in computations, have represented in the Table 1. Also the priors are as follows:

$$
\begin{aligned}
\text{Prior I}: &\quad a_1=0,\ b_1=0,\ a_2=0,\ b_2=0, \\
\text{Prior II}: &\quad a_1=2,\ b_1=1,\ a_2=3,\ b_2=5, \\
\text{Prior III}: &\quad a_1=0.3,\ b_1=1,\ a_2=0.7,\ b_2=5,
\end{aligned}
$$

Table 1. Type-II progressive censoring schemes.

n	m	$\boldsymbol{r}$	scheme number
30	5	(5,5,5,5,5)	[1]
	10	(4,0,4,0,...,4,0)	[2]
	15	(15,0,0,...,0)	[3]
	20	(0,...,0,10)	[4]
40	5	(7,7,7,7,7)	[5]
	10	(6,0,6,0,...,6,0)	[6]
	15	(25,0,...,0)	[7]
	20	(0,...,0,20)	[8]

where the Prior I is a non-informative gamma prior and the Priors II and III are informative gamma priors.

From Table 2, in the most considered cases, we observe that for fixed n_1 and n_2, as m_1, m_2, T_1 or T_2 increases, the MSE decreases. Also we observe that the results of MLEs and Bayes estimators under Prior I are so similar, that is reasonable, since Prior I is a non-informative prior. Furthermore Bayes estimators under Prior II and Prior III are better than the MLEs and Bayes estimators under Prior I, that is reasonable too, since the priors II and III are informative priors.

Also we obtain 95% asymptotic confidence intervals of R by simulating 20,000 samples under different Type-I progressive hybrid censoring schemes, and obtain the average lengths and coverage probabilities of asymptotic confidence intervals. The results are represented in the Table 3.

From Table 3, in the all considered cases, we observe that coverage probabilities are almost equal and less than 0.95, where this is reasonable, since the variance of $\hat{R}$ is estimated using one sample alone. Also in all situations, for fixed n_1 and n_2, as m_1, m_2, T_1 or T_2 increases, the average length decreases. This is reasonable too.

6 Data Analysis

In this section, the analysis of a pair of real data sets is presented for illustrative purposes. Tables 4 and 5 show the breaking strengths of jute fiber at two different gauge lengths. These two data sets were used by Xia et al. (2009). Both data sets in Tables 4 and 5 have the exponential distribution

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Table 2. The average estimators (A.E) and MSE of the MLEs and Bayes estimators of R when $(\theta_1, \theta_2) = (1, 2)$.

$(\boldsymbol{r}_1, T_1)$	$(\boldsymbol{r}_2, T_2)$		MLE	Bayes		
				Prior I	Prior II	Prior III
([1], 1)	([5], 1)	A.E	0.34505	0.35667	0.32606	0.31695
		MSE	0.01954	0.01793	0.00904	0.00893
	([6], 1)	A.E	0.32929	0.34646	0.32157	0.33553
		MSE	0.01478	0.01433	0.00852	0.00941
	([7], 2)	A.E	0.32284	0.34168	0.31659	0.33692
		MSE	0.01371	0.01328	0.00822	0.00944
	([8], 2)	A.E	0.32717	0.34982	0.31924	0.36278
		MSE	0.01165	0.01142	0.00764	0.00888
([2], 1)	([5], 1)	A.E	0.35541	0.35379	0.33847	0.30148
		MSE	0.01715	0.01505	0.00768	0.00803
	([6], 1)	A.E	0.33905	0.34402	0.33119	0.32056
		MSE	0.01158	0.01121	0.00712	0.00760
	([7], 2)	A.E	0.33069	0.33881	0.32709	0.32098
		MSE	0.01014	0.00969	0.00674	0.00777
	([8], 2)	A.E	0.33593	0.34703	0.32813	0.34329
		MSE	0.00790	0.00784	0.00579	0.00630
([3], 1.5)	([5], 1)	A.E	0.36232	0.35786	0.34425	0.30165
		MSE	0.01637	0.01489	0.00714	0.00772
	([6], 1)	A.E	0.34488	0.34759	0.33683	0.31762
		MSE	0.01067	0.01030	0.00669	0.00713
	([7], 2)	A.E	0.33740	0.34160	0.33374	0.32108
		MSE	0.00921	0.00908	0.00616	0.00734
	([8], 2)	A.E	0.34228	0.34761	0.33690	0.34354
		MSE	0.00695	0.00712	0.00532	0.00578
([4], 1.5)	([5], 1)	A.E	0.35738	0.34880	0.34312	0.28544
		MSE	0.01433	0.01253	0.00601	0.00744
	([6], 1)	A.E	0.33987	0.33800	0.33671	0.30400
		MSE	0.00885	0.00831	0.00548	0.00625
	([7], 2)	A.E	0.33289	0.33258	0.33185	0.30543
		MSE	0.00756	0.00712	0.00505	0.00615
	([8], 2)	A.E	0.33786	0.33910	0.33914	0.33353
		MSE	0.00508	0.00484	0.00424	0.00424

Table 3. The average lengths (A.L) and coverage probabilities (CP) of asymptotic confidence intervals of R when $(\theta_1, \theta_2) = (1, 2)$.

$(\boldsymbol{r}_1, T_1)$	$(\boldsymbol{r}_2, T_2)$		CI
	$([5], 1)$	A.L	0.51428
		CP	0.88000
	$([6], 1)$	A.L	0.45777
		CP	0.88290
$([1], 1)$	$([7], 2)$	A.L	0.44387
		CP	0.88555
	$([8], 2)$	A.L	0.40900
		CP	0.89410
	$([5], 1)$	A.L	0.46572
		CP	0.89890
	$([6], 1)$	A.L	0.40160
		CP	0.91035
$([2], 1)$	$([7], 2)$	A.L	0.38451
		CP	0.91125
	$([8], 2)$	A.L	0.33854
		CP	0.91715
	$([5], 1)$	A.L	0.45436
		CP	0.90575
	$([6], 1)$	A.L	0.38475
		CP	0.91445
$([3], 1.5)$	$([7], 2)$	A.L	0.36776
		CP	0.92275
	$([8], 2)$	A.L	0.31670
		CP	0.92740
	$([5], 1)$	A.L	0.42741
		CP	0.90805
	$([6], 1)$	A.L	0.34971
		CP	0.92280
$([4], 1.5)$	$([7], 2)$	A.L	0.33015
		CP	0.92655
	$([8], 2)$	A.L	0.27147
		CP	0.92875

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Table 4. Data Set 1 (Breaking strength of jute fiber of gauge length 10 mm).

693.73	704.66	323.83	778.17	123.06	637.66	383.43	151.48	108.94	50.16
671.49	183.16	257.44	727.23	291.27	101.15	376.42	163.40	141.38	700.74
262.90	353.24	422.11	43.93	590.48	212.13	303.90	506.60	530.55	177.25

Table 5. Data Set 2 (Breaking strength of jute fiber of gauge length 20 mm).

71.46	419.02	284.64	585.57	456.60	113.85	187.85	688.16	662.66	45.58
578.62	756.70	594.29	166.49	99.72	707.36	765.14	187.13	145.96	350.70
547.44	116.99	375.81	581.60	119.86	48.01	200.16	36.75	244.53	83.55

and based on the complete data set, $\hat{R} = 0.51768$. See Asgharzadeh and Kazemi (2014).

For illustrative purposes, we have considered two different Type-I progressive hybrid censoring schemes.

Scheme 1:

$$\begin{aligned} \boldsymbol{r}_1 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1), \qquad T_1 = 300, \\ \boldsymbol{r}_2 &= (1,3,4,1,1,3,1,2,2,2), \qquad T_2 = 200. \end{aligned}$$

Applying these censoring scheme, we have obtained

$$\begin{aligned} \boldsymbol{x} &= (43.93, 50.16, 101.15, 108.94, 123.06, 141.38, 151.48, 177.25 \\ &\quad , 183.16, 212.13, 257.4), \\ \boldsymbol{y} &= (36.75, 45.58, 48.01, 71.46, 99.72, 113.85, 119.86, 116.49, 187.85). \end{aligned}$$

In this case, we have $\hat{R} = 0.5847$, $\hat{R}_B^{(I)} = 0.5897$, $\hat{R}_B^{(II)} = 0.6195$, $\hat{R}_B^{(III)} = 0.6012$, and 95% asymptotic confidence interval of R is $(0.3708, 0.7986)$.

Scheme 2:

$$\begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{r}_2 = (5,5,5,5,5), \\ T_1 &= T_2 = 150. \end{aligned}$$

Applying these censoring scheme, we have obtained

$$\begin{aligned} \boldsymbol{x} &= (43.93, 50.16, 108.94), \\ \boldsymbol{y} &= (36.75, 45.58, 71.46, 83.55). \end{aligned}$$

In this case, we have $\hat{R} = 0.5713$, $\hat{R}_B^{(I)} = 0.6264$, $\hat{R}_B^{(II)} = 0.6402$, $\hat{R}_B^{(III)} = 0.6422$, and 95% asymptotic confidence interval for R is $(0.2047, 0.9379)$.

7 Conclusion

In this paper, we considered the estimation of $R = P(Y < X)$ based on Type-I progressive hybrid censored samples. It is assumed that X and Y are two independent exponential distributions with different parameters. We derived the maximum likelihood and Bayes estimators of R and obtained the asymptotic confidence intervals of R. We evaluated performance of MLEs, Bayes estimators and asymptotic confidence intervals via simulation. Finally, we consider a pair of real data sets and computed the MLEs, Bayes estimators and asymptotic confidence intervals under two different Type-I progressive hybrid censoring schemes.

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