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Bayesian Method for Finding True Change Point when a Control Chart is used

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Abstract. The process personnel always seek the opportunity to improve the processes. One of the essential steps for process improvement is to quickly find the starting time or the change point of a process disturbance. To do this, after a control chart triggers an out-of-control signal, an order of points in time (known as a plan) should be identified such that if the process examined sequentially at them, the true change point is detected as soon as possible. A typical method is to start the examination of the process from the signal time of the control chart and proceed to neighbouring points. In this paper, we establish a Bayesian method to solve this problem, i.e. to find a plan for examining the process sequentially such that it minimizes the Bayes risk among all other possible plans. At last, our proposed Bayes method is applied to a normal process, and compared to a typical method which is usually used to find the true change point through a series of simulations.

Keywords. Bayesian method; Bayesian risk; change point; control chart.

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1 Introduction

Statistical process control (SPC) charts are commonly used for detecting the presence of disturbances in a process. The primary function of SPC charts is that an out-of-control signal will be triggered when the process disturbances have occurred in the process.

When the control chart signals that the process is out-of-control, the process personnel must initiate a search for the special cause of the process disturbance. It is worthwhile to distinguish the difference between the change point and the out-of-control signal time which is triggered by the SPC charts. The change point is the time that the disturbances affect the process and the SPC signal time is the time that the out-of-control state is detected by the SPC charts. Actually, the change point time occurs first, and then the SPC signal is subsequently triggered. If the change point time can be determined, the special cause can be identified more quickly, and appropriate actions need to improve quality can be implemented sooner.

To find the true change point we should identify a plan which is an order of the points in time, according to which the examination of the process is done sequentially. A typical plan is to search for the true change point starting from an estimate of the change point and proceed to neighbouring points in time until the true change point is found. Usually, the signal time T is chosen as an estimate of the change point, and the search for finding the change point starts from this point in time. If the root causes cannot be identified at the initial signal time T; the searching process may proceed at time $T-1$. This process should continue until the identification is made. Throughout this paper this plan is referred to as the Typical Plan.

In their paper, Samuel et al. (1998) addressed the issue of estimating the change point of a normal process. Pinatiello and Samuel (2001) used exponentially weighted moving average (EWMA) and Cusum charts and MLE to estimate the change point of a process. Shao and Hou (2004) provided some statistical properties for the change point estimators. In addition, Shao and Hou (2006) derived the change point estimators in the case of the S chart and MLE are used in a gamma process. Later Shao et al. (2006), used an $\bar{X}$ control chart and MLE to estimate the change point of a Gamma process. Shao and Hou (2013a) applied an integrated approach of neural network and analysis of variance to identify a change point in an industrial process. Shao and Hou (2013b) used a two-stage hybrid scheme to estimate a change point for a multivariate process. Hou et al. (2013) used a combined MLE and

Generalized P chart approach to estimate the change point of a multinomial process.

All the above studies discuss the problem of estimating the change point. Different form these papers, in this study, we consider the problem of finding the true change point, when a control chart is used, from Bayesian point of view. It will be shown that, when a control chart makes a signal, how one can find a plan for finding true change point which minimizes the Bayes risk among all the possible plans. The structure of this paper is as follows. In Section 2, the model is introduced. Bayesian inference is explained in Section 3, briefly. Section 4, our plan for finding true change point is introduced and two plans is compared in Section 5. Finally, in Section 6, conclusions are made.

2 The Model

This study assumes the process is initially in control, and the sample observations come from a known density function $f(x|\theta_0)$ where θ_0 is known. However, after an unknown point in time τ, a disturbance is introduced into the process and starting from the point in time $\tau+1$ (known as the process change point) it changes the process parameter from θ_0 to θ_1. It is also assumed that once the parameter θ_0 changed, it remains at the new level of θ_1 until the root causes of the disturbance have been identified and removed. Let X_{ij} denote the jth observation in subgroup i with distribution $f(x|\cdot)$. That is

$$X_{ij} \overset{iid}{\sim} f(x|\theta_0), \quad \begin{aligned} i &= 1,2,\ldots,\tau \\ j &= 1,2,\ldots,n, \end{aligned}$$

and

$$X_{ij} \overset{iid}{\sim} f(x|\theta_1), \quad \begin{aligned} i &= \tau+1,\ldots,T \\ j &= 1,2,\ldots,n, \end{aligned}$$

where n is the subgroup sample size and T is the signal time which is triggered by the control chart. $\overset{iid}{\sim}$ stands for independent and identically distributed, also T is a random variable. It is assumed that $T \sim g(t|\tau,\theta_1)$, where the distribution $g(\cdot|\tau, \theta_1)$ depends to the type of the control chart which is applied.

3 Bayesian Inference

In general, making inference about the parameters in a statistical problem by the Bayesian approach, necessitates specificating a prior distribution for the parameters which reflects our prior information about them. Assuming $\mathbf{x}_i = (x_{i1}, \ldots, x_{in}),\ i = 1, \cdots, T$, the joint distribution of the observation, i.e. $\mathbf{x}_i$ s and T, is

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_t, t|\tau, \theta_0, \theta_1) = \prod_{i=1}^{\tau}\prod_{j=1}^{n} f(x_{ij}|\theta_0) \prod_{i=\tau+1}^{t}\prod_{j=1}^{n} f(x_{ij}|\theta_1) g(t|\tau, \theta_1). \quad (1)$$

Let $\pi(\tau, \theta_1)$ be the joint prior distribution for the parameters τ and θ_1, then the posterior probability distribution of τ is

$$\pi(\tau|\mathbf{x}_1, \ldots, \mathbf{x}_t, t) = \frac{\int_{-\infty}^{+\infty} f(\mathbf{x}_1, \ldots, \mathbf{x}_t, t|\tau, \theta_1)\pi(\tau, \theta_1)d\theta_1}{\sum_{\tau}\int_{-\infty}^{+\infty} f(\mathbf{x}_1, \cdots, \mathbf{x}_t, t|\tau, \theta_1)\pi(\tau, \theta_1)d\theta_1}, \tau = 0, 1, \ldots, t-1. \quad (2)$$

4 Identifying a Plan for Finding True Change Point

When a control chart triggers a signal, one should find a plan according to which the process is examined sequentially until the true change point is found. Each plan is just an arrangement of the points in the set $\{1, 2, \ldots, T\}$ which can be shown as a vector. For example the vector $(i_1, i_2, \ldots, i_T)$, which is a permutation of the elements of the set $\{1, 2, \ldots, T\}$, is a plan according to which the examination of the process for finding the main reason of the disturbances is done sequentially at times $i_1, i_2, \ldots$ until it is found at time $\tau + 1$.

4.1 Finding True Change Point Using Bayesian Method

Our inferential problem is to find a suitable plan for identifying the true change point. That is, to find the vector $(i_1, i_2, \ldots, i_T)$, which is a permutation of the elements of the set $\{1, 2, \ldots, T\}$, such that the true change point can be found efficiently by examining the process sequentially at the points in time $i_1, i_2, \ldots$.

The efficiency of any plan from Bayesian point of view is based on its corresponding Bayes risk which is the expectation of the loss function. In our problem a natural loss function is the number of the points in time at which the process should be examined, until the main causes of disturbances is found at $\tau + 1$. That is

$$L(\tau, (i_1, i_2, \ldots, i_T)) = j, \qquad \text{if} \qquad i_j = \tau + 1, \quad j = 1, 2, \ldots, T. \tag{3}$$

4.2 Bayesian Plan for Finding the True Change Point

A Bayes action is a decision rule that minimize the Bayes risk or equivalently posterior expected value of the loss function. Here, our action can be any permutation of the elements of the set $\{1, 2, \ldots, T\}$ and the loss function is as (3). Thus, to find the Bayesian plan for finding the true change point one should find a permutation of the elements of the set $\{1, 2, \ldots, T\}$ such that it minimizes the posterior expected value of the loss function. To do this, we need the following theorem.

***Theorem* 1.** *Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ be real numbers such that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_m$ and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_m$ then $a_1 b_m + a_2 b_{m-1} + \cdots + a_m b_1 \leqslant a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_m b_{i_m}$ for all $(i_1, i_2, \ldots, i_m)$ which is an arbitrary permutation of $\{1, 2, \ldots, m\}$ $\;, \forall m \in \mathbb{N}$.*

Proof. By induction
for $m = 2$ the inequality $a_1(b_2 - b_1) \leqslant a_2(b_2 - b_1)$ is obtained easily. Thus, $a_1 b_2 + a_2 b_1 \leqslant a_1 b_1 + a_2 b_2$. If the inequality holds for $m = k$ we will prove the inequality for $m = k + 1$. Assuming $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{k+1}$ and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_{k+1}$ one can write the inequality for $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$ and $b_2 \leqslant b_3 \leqslant \cdots \leqslant b_{k+1}$. Thus the following inequality is obtained,

$$a_1 b_{k+1} + a_2 b_k + \cdots + a_k b_2 \leqslant a_1 b_{i_{k+1}} + a_2 b_{i_k} + \cdots + a_k b_{i_2}, \tag{4}$$

where $(i_2, i_3, \ldots, i_{k+1})$ is an arbitrary permutation of the elements of the set $\{2, 3, \ldots, k + 1\}$. Adding the term $a_{k+1} b_1$ to both side of the inequality (4) gives

$$a_1 b_{k+1} + a_2 b_k + \cdots + a_k b_2 + a_{k+1} b_1 \leqslant a_1 b_{i_{k+1}} + a_2 b_{i_k} + \cdots + a_k b_{i_2} + a_{k+1} b_1, \tag{5}$$

but the inequality holds for $m = 2$, $a_r \leqslant a_{k+1}$, $b_1 \leqslant b_{i_{k+2-r}}$ and gives the inequality $a_r b_{i_{k+2-r}} + a_{k+1} b_1 \leqslant a_r b_1 + a_{k+1} b_{i_{k+2-r}}$ for $r = 1, 2, \ldots, k$. Thus in the right side of inequality (5) one can swap b_1 and $b_{i_{k+2-r}}$ or equivalently the indices 1 and i_{k+2-r} for $r = 1, 2, \ldots, k$. In other words one can swap the index of b_1, i.e. 1, with each one of the indices in the set $\{i_2, i_3, \ldots, i_{k+1}\} = \{2, 3, \ldots, k+1\}$ and still the inequality holds. It means that for all $(i_1, i_2, \ldots, i_{k+1})$ which is an arbitrary permutation of the elements of the set $\{1, 2, \ldots, k+1\}$ the inequality holds. □

Theorem 2. *Assume the control chart triggers a signal at time T. Let $p_1, p_2, \ldots, p_T$ be the posterior probability of the change point at points in time $1, 2, \ldots, T$, i.e. $p_r = \pi(r-1|\mathbf{x}_1, \ldots, \mathbf{x}_T, T)$, and $p_{(1)}, p_{(2)}, \ldots, p_{(T)}$ be increasingly ordered values of p_r s. Let $i_{[r]}$ be the point in time whose posterior probability is $p_{(r)}$. Then the plan $(i_{[T]}, i_{[T-1]}, \ldots, i_{[1]})$ is the Bayes plan. That is, it has the smallest Bayes risk.*

Proof. Assume $\mathcal{D} = (\mathbf{x}_1, \ldots, \mathbf{x}_T, T)$, the posterior risk is obtained,

$$\begin{aligned}
&E[L\big(\tau, (i_{[T]}, i_{[T-1]}, \ldots, i_{[1]})\big)|\mathcal{D}] \\
&= 1 \times P(\tau = i_{[T]}|\mathcal{D}) + 2 \times P(\tau = i_{[T-1]}|\mathcal{D}) + \cdots + T \times P(\tau = i_{[1]}|\mathcal{D}) \\
&= 1 \times p_{(T)} + 2 \times p_{(T-1)} + \cdots + T \times p_{(1)},
\end{aligned}$$

by attention to the inequalities $1 \leqslant 2 \leqslant \cdots \leqslant T$, $p_{(1)} \leqslant p_{(2)} \leqslant \cdots \leqslant p_{(T)}$ and using Theorem 1, the following inequality is given,

$$\begin{aligned}
1 \times p_{(T)} + 2 \times p_{(2)} + \cdots + T \times p_{(1)} &\leqslant 1 \times p_{i_1} + 2 \times p_{i_2} + \cdots + T \times p_{i_T} \\
&= 1 \times P(\tau = r_{i_1}|\mathcal{D}) + 2 \\
&\quad \times P(\tau = r_{i_2}|\mathcal{D}) + \cdots + T \times P(\tau = r_{i_T}|\mathcal{D}) \\
&= E[L\big(\tau, (r_{i_1}, r_{i_2}, \ldots, r_{i_T})\big)|\mathcal{D}],
\end{aligned}$$

where $(r_{i_1}, r_{i_2}, \ldots, r_{i_T})$ is an arbitrary permutation of the elements of the set $\{1, 2, \ldots, T\}$ with posterior probabilities $(p_{i_1}, \ldots, p_{i_T})$. Thus

$$E[L\big(\tau, (i_{[T]}, i_{[T-1]}, \ldots, i_{[1]})\big)|\mathcal{D}] \leqslant E[L\big(\tau, (r_{i_1}, r_{i_2}, \ldots, r_{i_T})\big)|\mathcal{D}]$$

and $(i_{[T]}, i_{[T-1]}, \ldots, i_{[1]})$ is the Bayes plan. □

Therefore, when the control chart signals that some disturbances introduce into the process, the plan according to which the examination of the process is done sequentially at points of time in order of decreasing posterior probability, is the Bayes plan for finding the true change point.

5 Comparing Two Plans

In this section, we consider a normal process which is monitored by an $\bar{X}$ control chart, the most commonly used statistical process control chart in industry. The control limits for a Shewhart $\bar{X}$ control chart is as follows

$$\begin{aligned} UCL &= \mu_0 + k\sigma_0 \\ LCL &= \mu_0 - k\sigma_0, \end{aligned}$$

where μ_0 and σ_0 are the mean and standard deviation of the normal process, when it is under control. It is usually assumed that $k = 3$, although it can be identified in light of the specified type I error (i.e., α). To see a full discussion of $\bar{X}$ control chart see Montgomery (2009).
It is assumed after an unknown time τ, the mean of the process changes from μ_0 to $\mu_1 = \mu_0 + \delta\frac{\sigma_0}{\sqrt{n}}$, and remains at the new level until time T at which the control chart makes an out-of-control signal. when the control chart signals that the process is out-of-control, the process personnel must initiate a search for the special cause of the process disturbance.

For this type of control chart the distribution of the signal time T given the parameter τ is follows. After change point time, the parameter μ_0 changes to μ_1. From this time, the probability of observing a subgroup mean out of control for each subsample is

$$\alpha(\mu_1) = P(\bar{X}_i < LCL) + P(\bar{X}_i > UCL), \qquad i = \tau+1, \tau+2, \ldots$$

where LCL and UCL are lower and upper control limits, respectively. Then given τ and μ_1, the distribution of T is as follows

$$\begin{aligned} g(t|\tau,\mu_1) &= P(T = t|\tau,\mu_1) \\ &= \alpha(\mu_1)(1-\alpha(\mu_1))^{t-\tau-1}, \qquad t = \tau+1, \tau+2, \ldots, \end{aligned} \tag{6}$$

or equivalently $T - \tau$ is distributed as Geometric$(\alpha(\mu_1))$. Although the control chart is able to trigger a signal when the assignable causes have occurred in the process, it may still need some time to determine the root causes of the problem. Using the Typical Plan, starting from the signal time T, the number of points in time at which the process should be examined to find the change point $\tau+1$ is $T-\tau$ which is distributed as a Geometric distribution with parameter $\alpha(\mu_1)$. Therefore, to find the change point, starting from the signal time, in average the process should be examined at $E(T-\tau)=\frac{1}{\alpha(\mu_1)}$ points in time, which is in fact the out-of-control average run length. In the next subsection the Typical Plan is compared to the Bayes plan based on a reasonable criteria.

5.1 Simulation Studies

In this section the Typical Plan for finding true change point is compared to the Bayes plan. The one which helps us to find the true change point more quickly is a better plan. Therefore, a suitable criteria for comparing the two aforementioned plans is the average number of points in time at which the process should be examined when a specific plan is used to find the true change point. For simplicity this criteria is shown as ANPT. It is easy to see that for the Typical Plan we have ANPT$=E(T-\tau)=\frac{1}{\alpha(\mu_1)}$. Note that in this case ANPT does not depend on the value of τ. Now, we compute ANPT for the Bayes plan through a series of simulations.

When the $\bar{X}$ control chart signals that the mean of the process has changed, from Bayesian point of view, we assume that μ_1 and τ are independent, and have the probability density $\pi(\mu_1)=1,\ -\infty<\mu_1<+\infty$ and $\pi(\tau)=\frac{1}{t}, \tau=0,1,\ldots,t-1$ (t is the observed signal time) respectively. A Monte Carlo simulation study was conducted to study the performance of the Bayes plan for finding true change point. Suppose $n=4$, sample observations are randomly generated from $N(0,1)$ distribution for subgroups $1,2,\ldots,\tau$. Then, starting with subgroup $\tau+1$, observations were randomly generated from $N(\mu_1,1)$ where $\mu_1=\delta\frac{1}{\sqrt{n}}$ until at time T the $\bar{X}$ control chart triggers a signal. In this case, the probability density function of the observations is as (1) where $f(x_{ij}|\theta_0)$ and $f(x_{ij}|\theta_1)$ are respectively the probability density function of $N(\mu_0,1)$ and $N(\mu_1,1)$ with $\theta_0=\mu_0=0$ and $\theta_1=\mu_1$. By putting this probability density function and $g(t|\tau,\mu_1)$ in relation (6) into (2) the posterior mass function of τ can be computed easily.

For each of the values of $\tau=25,50,100$ and $\delta=0.3,0.4,\cdots,3$, this procedure is repeated a total of 1000 times. For each simulation run, the Bayes plan for finding true change point is applied and its ANPT is computed. Figure 1 shows $ANPT$s of Bayes plans, for a range of values of τ and δ.
We consider that for $\tau=25,50,100$, the performance of the Bayes plan is not

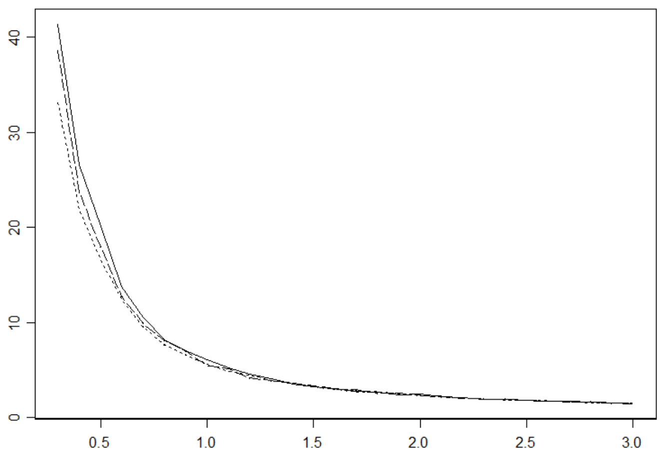

Figure 1. The ANPTs of Bayes plans for $\tau = 25$ (dotted curve), $\tau = 50$ (broken curve) and $\tau = 100$ (solid curve) (subgroup size $n = 4$).

much different. Although, it seems that the performance of the Bayes plan slightly decreases for small changes in the process mean when τ increases. In addition, based on ANPT (Figure 2) criteria, the Bayes plan for $\tau = 100$ (consequently for $\tau = 25, 50$) outperforms the Typical Plan for all values of δ (note that the ANPT of the Typical Plan is constant for all values of τ).

6 Conclusion

Our study presented a Bayesian approach for finding the true change point for an arbitrary process when implementing a control chart. After the control chart triggers an out of control signal, we should initiate a search for finding the true change point. To do this, a plan should be identified according to which the process is examined at points in time $1, 2, \ldots, T$. It was shown that if the process is examined sequentially at points in time in order of decreasing posterior probability of them, the Bayes risk is minimized. That is, this plan is the Bayes plan for finding the true change point. This result is true for every control chart and it is true even if more than one parameter of the process change after the change point. In addition, the proposed Bayes

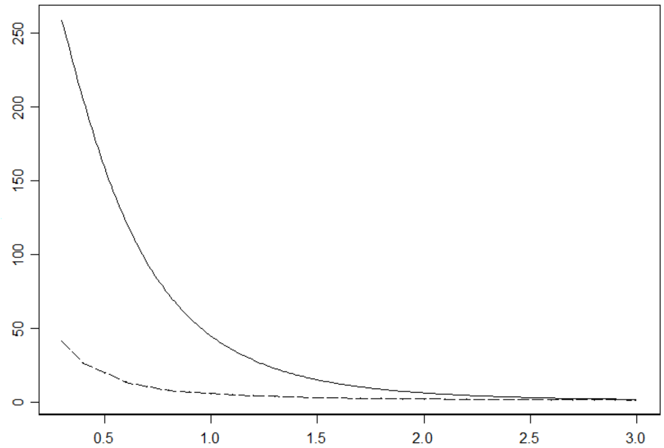

Figure 2. The ANPT of Bayes plan for $\tau = 100$ (broken curve) and ANPT of the Typical Plan (solid curve) (subgroup size $n = 4$).

method was used for a normal process which is monitored by an $\bar{X}$ control chart. Then it is compared to the typical method for finding true change point through a series of simulations. These simulations show that the Bayes plan out performs the typical plan for a range of parameters' values.

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