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The Rate of Entropy for Gaussian Processes

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Abstract. In this paper, we show that in order to obtain the Tsallis entropy rate for stochastic processes, we can use the limit of conditional entropy, as it was done for the case of Shannon and Renyi entropy rates. Using that we can obtain Tsallis entropy rate for stationary Gaussian processes. Finally, we derive the relation between Renyi, Shannon and Tsallis entropy rates for stationary Gaussian processes.

Keywords. Tsallis entropy; Renyi entropy; Shannon entropy; Gaussian process; entropy rate.

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1 Introduction

In 1948, Shannon used some axioms to introduce Shannon entropy. Then Renyi (1961) and Tsallis (1988) generalized this idea and obtained different entropic forms, which in special cases reduce to Shannon entropy.

By the introduction of entropy in the probability theory, entropy and stochastic processes became linked, and the entropy rate was defined for stochastic processes after Shannon proved that the Shannon entropy rate exists for stationary stochastic processes (Shannon, 1948). The rate of Shannon entropy is widely studied for stochastic processes, especially for stationary processes with discrete or continuous time, (see Girardin and Limnios, 2004, and references therein). For example, the rate of Shannon entropy for a stationary Gaussian process was obtained by Kolmogorov (1958). The rate of Renyi entropy for stochastic processes was obtained by Rached et al. (1999).

Up to now, the Renyi entropy rate for some stochastic processes have been studied, for example, by Rached et al. (1999, 2004) and Golshani and Pasha (2010). But in the case of the Tsallis entropy, the rate of entropy for stochastic processes has not been obtained for any processes yet. In this paper, we have obtained the rate of Tsallis entropy for a stationary Gaussian process with discrete-time.

The application of the rate of entropy can be found in many areas such as economics (Bentes, et al, 2008), medical sciences (Scalassara et al., 2008) statistical mechanics (Kirchnov, 2008), stochastic processes (Peres and Quas, 2011; Rached et al., 2004), statistics and related fields (Andai, 2009), (Jenssen and Eltoft, 2008) and biomedical engineering (Lake, 2006).

This paper is organized as follows. In Section 2, we propose some properties of measures of information that are common for Shannon, Renyi and Tsallis entropies. In Section 3, we show that in order to obtain the Tsallis entropy rate for stochastic processes, we can use the limit of conditional entropy, as it was done for the case of Shannon and Renyi entropy rates; and from this relation we obtain the rate of Tsallis entropy for stationary Gaussian processes and show that this quantity is related to a spectral density function. We also show that for autoregressive and moving average processes, the rate of Tsallis entropy is independent of their representations. Finally, in Section 4 we derive the relation between Renyi, Shannon and Tsallis entropy rates for stationary Gaussian processes.

2 Shannon, Renyi and Tsallis Entropies

Let X be a random variable having an absolutely continuous distribution with density function $f(x)$. The Shannon entropy is defined as (Shannon, 1948),

$$h_1(X) = -\int_R f(x) \ln f(x) dx,$$

and the Renyi entropy is given by (Renyi, 1961)

$$h_\alpha(X) = \frac{1}{1-\alpha} \ln \int_R f^\alpha(x) dx \qquad \alpha > 0\ ,\ \alpha \neq 1,$$

and the Tsallis entropy is given by (Tsallis, 1988)

$$S_\alpha(X) = \frac{1}{1-\alpha} \left(\int_R f^\alpha(x) dx - 1 \right) \qquad \alpha > 0\ ,\ \alpha \neq 1.$$

The limit of Tsallis entropy and Renyi entropy, as $\alpha \to 1$, is

$$h_1(X) = \lim_{\alpha\to 1} h_\alpha(X) = \lim_{\alpha\to 1} S_\alpha(X) = -\int_R f(x)\ln f(x)dx.$$

Remark 1. For a random variable, with normal distribution, the Renyi entropy is $h_\alpha(X) = \frac{1}{2}\ln 2\pi\sigma^2\alpha^{\frac{1}{\alpha-1}}$ and the Shannon entropy is $h_1(X) = \frac{1}{2}\ln 2\pi e\sigma^2$.

In the following example, we obtain Tsallis entropy for normal distribution.

Example 1. The Tsallis entropy for the random variable with normal distribution is:

$$\begin{aligned} S_\alpha(X) &= \frac{1}{1-\alpha}\left[\int_R \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^\alpha e^{\frac{-\alpha}{2\sigma^2}(X-\mu)^2}dx - 1\right] \\ &= \frac{1}{1-\alpha}\left[(2\pi\sigma^2)^{\frac{1-\alpha}{2}}\alpha^{-\frac{1}{2}}\int_R \frac{\alpha^{\frac{1}{2}}}{\sqrt{2\pi\sigma^2}}e^{\frac{-\alpha}{2\sigma^2}(X-\mu)^2}dx - 1\right] \\ &= \frac{1}{1-\alpha}\left[\left(\frac{(2\pi\sigma^2)^{1-\alpha}}{\alpha}\right)^{\frac{1}{2}} - 1\right]. \end{aligned}$$

Now, we review the definition of conditional entropy for three measures of information. The conditional Shannon entropy is (Cover and Thomas, 2006)

$$h_1(Y|X) = -\int_{R^2} f(x)f(y|x)\ln f(y|x)dxdy,$$

if the integral exist. For the conditional Renyi entropy we have (Golshani and Pasha, 2010)

$$h_\alpha(Y|X) = \frac{1}{1-\alpha}\ln\frac{\int_{R^2} f^\alpha(x,y)dxdy}{\int_R f^\alpha(x)dx}, \qquad \alpha > 0,\ \alpha \neq 1,$$

if the integrals exist.

Now, we propose definitions of the conditional Tsallis entropy and the joint Tsallis entropy. In order to define these for continuous random variables, we propose definitions similar to the case of discrete random variables, which was introduced by Abe (2000).

Definition 1. For two random variables X and Y, with the joint probability density function $f(x,y)$ and conditional probability density function $f(y|x)$, the joint Tsallis entropy is defined as

$$S_\alpha(X,Y) = \frac{1}{1-\alpha}\left(\int_{R^2} f^\alpha(x,y)dxdy - 1\right) \qquad \alpha > 0\ ,\ \alpha \neq 1,$$

and the conditional Tsallis entropy is defined as

$$S_\alpha(Y|X) = \frac{1}{1-\alpha}\left(\frac{\displaystyle\int_{R^2} f^\alpha(x,y)dxdy}{\displaystyle\int_R f^\alpha(x)dx} - 1\right) \qquad \alpha > 0\ ,\ \alpha \neq 1, \tag{1}$$

if the integrals exist.

Now, we propose some properties of measures of information that are common for Shannon, Renyi and Tsallis entropies which were respectively proved for Shannon (Cover and Thomas, 2006) and Renyi (Golshani and Pasha, 2010) entropies. Now, we prove the following for Tsallis entropy:

Proposition 1. $S_\alpha(X) \geq 0$ for $\alpha \geq 1$.

Proposition 2. The Tsallis entropy is invariant under a location transformation of the random variable. i.e., for $m \in R$, we have: $S_\alpha(X+m) = S_\alpha(X)$.

Proposition 3. The Tsallis entropy is not invariant under a scale transformation of the random variable. More explicitly, for $m \in R$, we have:

$$S_\alpha(mX) = |m|^{1-\alpha}\left(S_\alpha(X) + \frac{1}{1-\alpha}\right) - \frac{1}{1-\alpha}.$$

Proof. let $Y = mX$, so $f_Y(y) = \frac{1}{|m|} f_X(\frac{y}{m})$ and

$$\begin{aligned}
S_\alpha(Y) &= \frac{1}{1-\alpha}\left(\int_R f_Y^\alpha(y)dy - 1\right) \\
&= \frac{1}{1-\alpha}\left\{\int_R \left(\frac{1}{|m|}\right)^\alpha f_X^\alpha\left(\frac{y}{m}\right)dy - 1\right\} \\
&= \frac{1}{1-\alpha}\left\{\int_R \left(\frac{1}{|m|}\right)^{\alpha-1} \frac{1}{|m|} f_X^\alpha\left(\frac{y}{m}\right)dy - 1\right\} \\
&= \frac{1}{1-\alpha}\left(|m|^{1-\alpha}\int_R f_X^\alpha(x)dx - 1\right) \\
&= \frac{1}{1-\alpha}\left[|m|^{1-\alpha}\{(1-\alpha)S^\alpha(X)+1\} - 1\right] \\
&= |m|^{1-\alpha}\left(S^\alpha(X) + \frac{1}{1-\alpha}\right) - \frac{1}{1-\alpha}.
\end{aligned}$$

□

The following property is a relation between the conditional entropy and the joint entropy which was proved for Shannon (Cover and Thomas, 2006) and Renyi (Golshani and Pasha, 2010) entropies.

Proposition 4. $S_\alpha(X, Y) = S_\alpha(X) + S_\alpha(Y|X)$.

The complete proof of this relation, for the case of discrete random variables is given by Furuichi (2006). For the case of continuous random variables, the proof is done in a similar way.

Remark 2. For two independent random variables, $f(x, y) = f(x).f(y)$, if we use relation (1), we get: $S_\alpha(Y|X) = S_\alpha(Y)$.

Theorem 1. (Chain rule) *Let $(X_1, \ldots, X_n)$ be a random vector with probability density function $f(x_1, \ldots, x_n)$ and having finite Tsallis entropy for every n, then, (Furuichi, 2006):*

$$S_\alpha(X_1, \ldots, X_n) = \sum_{i=1}^{n} S_\alpha(X_i|X_{i-1}, \cdots, X_1), \qquad \forall\alpha.$$

3 Entropy Rate

In this section, we propose definitions for obtaining entropy rate for stochastic processes without considering the measure of information.

For a discrete-time process $X = (X_n)_{n\geqslant 1}$, the entropy at time n is defined as the entropy of the n-dimensional random vector of X, and the entropy rate is defined by the limit of the entropy at time n divided by n, when the limit exists, namely: $\overline{H}(X) =: \lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots, X_n)$ This definition is a regular definition for the entropy rate. By using the chain rule, another relation for obtaining the rate of entropy is obtained, based on the limit of conditional entropy, namely, $\overline{H}(X) = \lim_{n\to\infty} H(X_n|X_{n-1}, \ldots, X_1)$. This relation is obtained in Cover and Thomas (2006) and Golshani and Pasha (2010) for Shannon and Renyi entropies, respectively. Also, for Tsallis entropy, we can get this relation by using the chain rule (Theorem 1). So we conclude that when the chain rule holds for the measure of entropy, then we can use the relation based on the limit of conditional entropy for obtaining the rate of entropy.

We know that a discrete-time process is stationary if the distribution of $(X_{n_1+h}, \ldots, X_{n_k+h})$ is independent of h for any positive integer k and $n_1, \ldots, n_k$. Then, for conditional entropy, we get:

$$H(X_{n_k}|X_{n_1}, \cdots, X_{n_{k-1}}) = H(X_{n_k+h}|X_{n_1+h}, \ldots, X_{n_{k-1}+h})$$

and the rate of entropy for a stationary process is equal to

$$\overline{H}(X) = \lim_{n\to\infty} H(X_1|X_0, \cdots, X_{2-n}) = H(X_1|X_0, \cdots). \tag{2}$$

This relation constitutes a way for obtaining the rate of entropy for stationary processes.

In this paper we denote the rate of Shannon entropy by $\overline{h_1}(X)$, the rate of Renyi entropy by $\overline{h_\alpha}(X)$ and the rate of Tsallis entropy by $\overline{S}_\alpha(X)$.

Now, similar to the cases of the rates of Renyi entropy and Shannon entropy for a stationary Gaussian process, we use relation (2) to obtain the rate of Tsallis entropy for this process and show that this quantity is related to a spectral density function and also show that for autoregressive and moving average processes, the rate of Tsallis entropy is independent of their representations.

Let stationary processes $(X_n)_{n\in Z}$, where each X_n is a real random variable defined on a probability space (Ω, β, P) be regular (or purely nondeterministic) with its mean being zero and with autocovariance function

$\gamma(k) = E(X_n X_{n+k})$, its Fourier transform is the spectral density function $f(\lambda)$, i.e.:

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-i\lambda k} \qquad -\pi \leqslant \lambda \leqslant \pi.$$

For stationary Gaussian processes, we have the following representation (Ihara, 1993),

$$X_n = \sum_{j=0}^{\infty} \varphi_j \xi_{n-j} \qquad n \in Z, \tag{3}$$

where φ_j, $j \geqslant 0$ are constants, so that $\sum_{j=0}^{\infty} \varphi_j^2 < \infty$ and $\{\xi_n\}$ is a sequence of independent normal random variables with identical distribution $N(0,1)$, and $\beta_n(\xi) = \beta_n(X)$, ($\beta_n(X)$ is the σ-field generated by $(X_k, k \leqslant n)$). This expression is known as the moving average representation of the process.

Now, for these processes, we have the following theorems.

Theorem 2. *For stationary Gaussian processes, the rate of Shannon entropy is* $\overline{h_1}(X) = \frac{1}{2}\ln 2\pi e + \frac{1}{4\pi}\int_{-\pi}^{\pi} \log 2\pi f(\lambda) d\lambda$, *(Cover and Thomas, 2006) and the rate of Renyi entropy is equal to* $\overline{h_\alpha}(X) = \frac{1}{2}\ln 2\pi\alpha^{\frac{1}{\alpha-1}} + \frac{1}{4\pi}\int_{-\pi}^{\pi} \log 2\pi f(\lambda) d\lambda$, *(Golshani and Pasha, 2010).*

By the following theorem, we obtain the rate of Tsallis entropy for these processes.

Theorem 3. *For stationary Gaussian processes, the rate of Tsallis entropy is equal to:*

$$\overline{S_\alpha}(X) = \frac{1}{1-\alpha}\left(\left|2\pi \exp \frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\lambda) d\lambda\right|^{\frac{1-\alpha}{2}} \sqrt{\frac{(2\pi)^{1-\alpha}}{\alpha}} - 1\right). \tag{4}$$

Proof. Using relations (2) and (3), we have:

$$\begin{aligned} \overline{S_\alpha}(X) &= S_\alpha(X_1|X_0,\dots) = S_\alpha\left(\sum_{j=0}^{\infty} \varphi_j \xi_{1-j} \middle| X_0, \dots\right) \\ &= S_\alpha\left(\varphi_0 \xi_1 + \sum_{j=1}^{\infty} \varphi_j \xi_{1-j} \middle| X_0, \dots\right). \end{aligned}$$

Since the σ-field generated by $(\xi_k, k \leqslant 1)$ is equal to the σ-field generated by $(X_k, k \leqslant 1)$, and ξ_1 is independent of $(\xi_k, k \leqslant 0)$, thus ξ_1 is independent of $(X_k, k \leqslant 0)$. On the other hand since $\sum_{j=1}^{\infty} \varphi_j \xi_{1-j} \in \sigma(X_k, k \leqslant 0)$, it is a function of $(X_k, k \leqslant 0)$ and by Proposition 2, Proposition 3 and Remark 2 we get:

$$\begin{aligned}\overline{S_\alpha}(X) &= S_\alpha(\varphi_0 \xi_1) \\ &= |\varphi_0^2|^{\frac{1-\alpha}{2}} \left(S_\alpha(\xi_1) + \frac{1}{1-\alpha} \right) - \frac{1}{1-\alpha}.\end{aligned}$$

Since ξ_1 has a normal distribution by Example 1, we have:

$$\overline{S_\alpha}(X) = \frac{1}{1-\alpha} \left(|\varphi_0^2|^{\frac{1-\alpha}{2}} \sqrt{\frac{(2\pi)^{1-\alpha}}{\alpha}} - 1 \right).$$

Also, from the relationship between the spectral density function of the process and the coefficients φ_j in relation (3), we have

$$\varphi_0 = \sqrt{2\pi} \exp \frac{1}{4\pi} \int_{-\pi}^{\pi} \log f(\lambda) d\lambda,$$

(Ihara, 1993). So the Tsallis entropy rate is

$$\overline{S_\alpha}(X) = \frac{1}{1-\alpha} \left(\left| 2\pi \exp \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda) d\lambda \right|^{\frac{1-\alpha}{2}} \sqrt{\frac{(2\pi)^{1-\alpha}}{\alpha}} - 1 \right)$$

where f is a spectral density function of process. □

Proposition 5. The rate of Tsallis entropy for autoregressive-moving average processes (ARMA(p,q)) is:

$$\overline{S_\alpha}(X) = \frac{1}{1-\alpha} \left(\sqrt{\frac{(2\pi)^{1-\alpha}}{\alpha}} - 1 \right)$$

Proof. From (Ihara, 1993), the spectral density functions of these processes is:

$$f(\lambda) = \frac{1}{2\pi} \frac{\prod_{k=1}^{q} |e^{i\lambda} - \beta_k|^2}{\prod_{j=1}^{p} |e^{i\lambda} - \alpha_j|^2},$$

where $|\alpha_j| < 1$, $|\beta_k| \leqslant 1$. So, from relation (4), we have:

$$\begin{aligned}\overline{S_\alpha}(X) &= \frac{1}{1-\alpha}\Big(\Big|2\pi \exp \frac{1}{2\pi}\Big(\int_{-\pi}^{\pi} \log \frac{1}{2\pi} d\lambda + \sum_{k=1}^{q} \int_{-\pi}^{\pi} \log |e^{i\lambda} - \beta_k|^2 d\lambda \\ &\quad - \sum_{j=1}^{p} \int_{-\pi}^{\pi} \log |e^{i\lambda} - \alpha_j|^2 d\lambda\Big)\Big|^{\frac{1-\alpha}{2}} \sqrt{\frac{(2\pi)^{1-\alpha}}{\alpha}} - 1\Big).\end{aligned}$$

Now by using the fact that for $|\theta| \leqslant 1$, $\int_{-\pi}^{\pi} \log |e^{i\lambda} - \theta|^2 d\lambda = 0$ (Ihara, 1993), the result is obtained. □

Remark 3. For autoregressive-moving average processes (ARMA(p,q)) the rate of Shannon entropy is $\overline{h_1}(X) = \frac{1}{2} \ln 2\pi e$ and the rate of Renyi entropy is $\overline{h_\alpha}(X) = \frac{1}{2} \ln 2\pi\alpha^{\frac{1}{\alpha-1}}$

4 Relation between Renyi, Shannon and Tallis Entropy Rates

In this section, we derive the relation between Renyi, Shannon and Tsallis entropy rates. For this purpose, we first consider the following property for the Renyi entropy.

Remark 4. For $\alpha_1 < \alpha_2$, $h_{\alpha_1}(X) \geqslant h_{\alpha_2}(X)$, for all X; and equality holds if and only if X is a uniform random variable.

Using this remark, we have the following inequalities between Renyi and Shannon entropies:

1. For $\alpha < 1$, $h_1(\cdot) < h_\alpha(\cdot)$, (5)
2. For $\alpha > 1$, $h_\alpha(\cdot) < h_1(\cdot)$. (6)

Now, we obtain the relation between Renyi and Tsallis entropies.

Proposition 6. For Renyi and Tsallis entropies the following inequalities hold.

1. For $\alpha < 1$, $h_\alpha(\cdot) < S_\alpha(\cdot)$, (7)
2. For $\alpha > 1$, $h_\alpha(\cdot) > S_\alpha(\cdot)$. (8)

Proof. Suppose $a = \int_R f^\alpha(x)dx$ $(a > 0)$. Then, we have $\ln \int_R f^\alpha(x)dx < \int_R f^\alpha(x)dx - 1$. Now, let $\alpha < 1$, then multiplying both sides of the relation by $\frac{1}{1-\alpha}$, we get $h_\alpha(\cdot) < S_\alpha(\cdot)$. For $\alpha > 1$, the relation (8) is obtained in a similar way.

Now, by relations (5), (6), (7) and (8) the following inequalities between Renyi, Shannon and Tsallis entropies are obtained.

1. For $\alpha < 1$, $h_1(\cdot) < h_\alpha(\cdot) < S_\alpha(\cdot)$, (9)
2. For $\alpha > 1$, $h_1(\cdot) > h_\alpha(\cdot) > S_\alpha(\cdot)$. (10)

Now, we obtain relation between Renyi, Shannon and Tallis entropy rates for a stationary Gaussian process, using (9) and (10).

For a random vector $(X_1, \ldots, X_n)$, the inequality (9) becomes:

$$h_1(X_1, \ldots, X_n) < h_\alpha(X_1, \ldots, X_n) < S_\alpha(X_1, \ldots, X_n),$$

and

$$\frac{1}{n} h_1(X_1, \ldots, X_n) < \frac{1}{n} h_\alpha(X_1, \ldots, X_n) < \frac{1}{n} S_\alpha(X_1, \ldots, X_n)$$

Then, taking the limit of the entropy as $n \to \infty$ and considering that the rate of Renyi entropy (Golshani and Pasha, 2010) and the rate of Shannon entropy (Kolmogorov, 1958; Ihara, 1993) exist for autoregressive-moving average processes, and using Proposition 5 for Tsallis entropy rate, we have:

$$\overline{h_1}(X) \leqslant \overline{h_\alpha}(X) \leqslant \overline{S_\alpha}(X),$$

then

$$\frac{1}{2} \ln 2\pi e \leqslant \frac{1}{2} \ln 2\pi\alpha^{\frac{1}{\alpha-1}} \leqslant \frac{1}{1-\alpha} \left(\sqrt{\frac{(2\pi)^{1-\alpha}}{\alpha}} - 1 \right).$$

Similarly, for $\alpha > 1$, we get $\overline{S_\alpha}(X) \leqslant \overline{h_\alpha}(X) \leqslant \overline{h_1}(X)$. So

$$\frac{1}{2} \ln 2\pi e \geqslant \frac{1}{2} \ln 2\pi\alpha^{\frac{1}{\alpha-1}} \geqslant \frac{1}{1-\alpha} \left(\sqrt{\frac{(2\pi)^{1-\alpha}}{\alpha}} - 1 \right).$$

□

5 Conclusion

In this paper, we obtained the rate of Tsallis entropy for stationary Gaussian processes. To do this, we showed that one can use the limit of conditional

Tsallis entropy, similar to what was done for the cases of Renyi and Shannon entropies. Then, we showed that the rate of Tsallis entropy is related to a spectral density function, and then we proved that the rate of Tsallis entropy is independent of their representations, for autoregressive and moving average processes. Finally, we obtained the relation between Renyi, Shannon and Tsallis entropy rates, for stationary Gaussian processes. Further work can be done to obtain the rate of Tsallis entropy for Gaussian processes with continuous time.

References

Abe, S. (2000). Axioms and Uniqueness Theorem for Tsallis Entropy, *Phys. Lett. A.* **271** 74-79.

Andai, A. (2009). On the Geometry of Generalized Gaussian Distributions, *Journal of Multivariate Analysis.* **100** 777-793.

Bentes, S.R., Menezes, R. and Mendes, D.A. (2008). Long Memory and Volatility Clustering: Is the Empirical Evidence Consistent Across Stock Markets? *Physica A.* **387** 3826-3830.

Cover, T.M. and Thomas, J. (2006). *The Elements of Information Theory.* John Wiley and Sons, Second edition.

Furuichi, S. (2006). Information Theoretical Properties of Tsallis Entropies. *Journal of Mathematical Physics.* **47.**

Girardin, V. and Limnios, N. (2004). Entropy Rate and Maximum Entropy Methods for Countable Semi-Markov Chains, *Communications in Statistics: Theory and Methods.* **33**, 609-622.

Golshani, L. and Pasha, E. (2010). Renyi Entropy Rate for Gaussian Processes, *Information Sciences.* **180**, 1486-1491.

Ihara, S. (1993). *Information Theory for Continuous Systems,* world scientific, Singapore.

Jenssen, R. and Eltoft, T. (2008). A New Information Theoretic Analysis of Sum-of-Squared-Error Kernel Clustering, *Neurocomputing.* **72**, 23-31.

Kirchanov, V.S. (2008). Using the Renyi Entropy to Describe Quantum Dissipative Systems in Statistical Mechanics, *Theoretical and Mathematical Physics.* **156**, 1347-1355.

Kolmogorov, A.N. (1958). A New Invariant for Transitive Dynamical Systems, *Dokl. An. SSR.*, **119**, 861-864.

Lake, D.E. (2006). Renyi Entropy Measures of Heart Rate Gaussianity, *IEEE Transactions on Biomedical Engineering*, **53**, 21-27.

Peres, Y. and Quas, A. (2011). Entropy Rate for Hidden Markov Chains with Rare Transitions. Entropy of Hidden Markov Processes and Connections to Dynamical Systems, London Mathematical Society Lecture Note Series, vol. **385**, 172-178.

Rached, Z., Fady, A. and Campbell, L.L. (1999). Renyi's Entropy Rate for Discrete Markov Sources, in Proc. CISS'99, Baltimore, MD, mar. 17-19.

Rached, Z., Fady, A. and Campbell, L.L. (2004). The Kullback-Leibler Divergence Rate between Markov Sources, *IEEE Trans. Inform.Theory.*, **50**, 917-921.

Renyi, A. (1961).On Measures of Entropy and Information, in Proc .4th Berkeley Symp. Mathematical Statistics Probability. Berkeley, CA: Univ. Calif. Press., **1**, 547-561.

Scalassara, P.R., Dajer, M., Marrara, J.L., Maciel, C.D. and Pereira, J. (2008). Analysis of Voice Pathology Evolution Using Entropy Rate. *International Symposium on Multimedia.*, 580-585.

Shannon, C.E. (1948). A Mathematical Theory of Communication. *Bell Syst .Techn. J.*, **27**, 379-423, 623-656.

Tsallis, C. (1988). Possible Generalization of Bolzmann-Gibbs Statistics. *J. Stat. Phys.,* **52**, 479-487.

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