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On Properties of a Class of Bivariate FGM Type Distributions

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Abstract. In this paper, we consider a new class of bivariate copulas and study their measures of association. Specifically, we propose a bivariate copula based distribution and obtain explicit expressions for the corresponding marginal and joint distributions of concomitants of generalized order statistics. Using these results, we provide the minimum variance linear unbiased estimator for the location and scale parameters of the concomitants of order statistics of Burr and logistic distributions. Then, we introduce a class of absolutely continuous bivariate distributions whose univariate margins are exponential distributions. In addition, we discuss their properties such as moment generating function, stress-strength probability and reliability of two component systems. Monte Carlo simulations are performed to highlight properties of the parameters estimates. Finally, we analyze two data sets to illustrate the flexibility and potential of the proposed distribution compared to several competing models.

Keywords. Burr distribution, concomitants, exponential distribution, generalized order statistics, minimum variance, Monte Carlo simulation, reversed hazard rate.

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1 Introduction

Construction of bivariate distributions with given margins has been main interest of many researchers for many years. A popular approach is to use copula and the Sklar's theorem (Nelson, 2006). Copulas are very powerful tools to model multivariate distributions and to take into account the dependence structure among random variables, independent of the type of marginal distributions. Recently, various authors have introduced different classes of copulas to improve the performance of associated bivariate distributions, such as Amblard and Girard (2009), Amini et al. (2011), Fischer and Klein (2007), Klein and Christa (2011), Lallena and Úbeda-Flores (2010), Morillas (2005) and Sharifonnasabi et al. (2018). On the other hand, several authors such as Mirhosseini et al. (2014) and Shiau (2006) have used copula methods to construct bivariate distributions.
The univariate exponential distribution is useful to describe the lifetime of a single component. It is perhaps the most widely applied statistical distributions in reliability. Bivariate distributions with exponential marginals are similarly useful and highly helpful to describe the joint lifetime of systems with two components. In many reliability cases, bivariate lifetime data play an important role in the data analysis process.
One of the most popular bivariate exponential distributions is Marshal-Olkin distribution which contains a singular part, that is, $P(X = Y) > 0$ when the random vector (X, Y) is distributed according to the Marshal-Olkin model. Some properties of the bivariate exponential distribution and their applications to reliability are given by Basu (1988), Balakrishnan and Basu (1995), and Balakrishnan and Lai (2009).
Kamps (1995) introduced a class of generalized order statistics as a unified approach to a variety of models for ordered random variables. The class includes several subjects, such as order statistics, record values, k-record values and progressively Type-II censored order statistics, as special cases. If the pairs of random variables (X_i, Y_i), $i = 1, 2, ..., n$, have been ordered by the X_i values, then the Y variate associated with the r-th order statistics, $X_{(r)}$ of X, denoted by $Y_{[r]}$, $1 \leqslant r \leqslant n$, is so-called concomitants of the rth ordered statistic. Concomitants are of interest in selection and prediction problems. David and Nagaraja (1998) presented an excellent review on the concomitants of order statistics. Beg and Ahsanullah (2008) considered concomitants of generalized order statistics for Farlie-Gumbel-Morgenstern (FGM) bivariate distribution and studied relations between their moments.

Tahmasebi and Behboodian (2010) obtained Shannon entropy for the concomitants of ordinary order statistics for Generalized Morgenstern (GM) bivariate distribution. Mohie et al. (2015) provided the minimum variance linear unbiased estimator for the location and scale parameters of the concomitants of ordinary order statistics from certain distributions. In this paper, we shall introduce a new bivariate distribution using copula methods and explore its main properties. Then, we present another interesting bivariate exponential distribution, as its especial case, and provide the parameters estimate in practical applications.

The rest of the paper is organized as follows. In Section 2, we consider a special case of the bivariate copula class introduced by Sharifonnasabi et al. (2018) and explore its additional properties. In Section 3, we generate a bivariate distribution in a general form by applying our proposed copula and study their marginal and joint distributions of concomitants of generalized order statistics. Then we derive several explicit expressions and recurrence relations for the moments of concomitants. Further, we provide the minimum variance linear unbiased estimator for the location and scale parameters of the concomitants of ordinary order statistics from Burr and logistic distributions. In Section 4, we propose a bivariate distribution whose univariate margins are exponential and derive the moment generating function, conditional moment, stress-strength value, mean time of failure and reliability of parallel and series systems. Moreover, we provide some simulation results and analyze two real-life data sets to highlight the usefulness of our proposed distributions.

2 Copulas

Copula is a multivariate function that links univariate marginal distribution functions (df) to construct a multivariate df. The theoretical basis of copulas was first given by Sklar (1959). For a bivariate case, according to Sklar's Theorem, if two random variables X and Y follow arbitrary marginal dfs $F_X(x)$ and $G_Y(y)$, respectively, then there exists a copula, C, that combines these two marginals to give their joint df, $H_{X,Y}(x, y)$, as follows

$$H_{X,Y}(x, y) = C(F_X(x), G_Y(y)). \tag{1}$$

Conversely, if C is a copula and $F_X(x)$ and $G_Y(y)$ are dfs, then the function H defined by (1) is a joint df with margins $F_X(x)$ and $G_Y(y)$. Furthermore, if $F_X(x)$ and $G_Y(y)$ are absolutely continuous, then C is unique. Under the assumption that the marginal distributions are continuous with probability density functions $f_X(x)$ and $g_Y(y)$, the joint probability density function (pdf) then becomes

$$h_{X,Y}(x,y) = f_X(x)g_Y(y)c(F_X(x), G_Y(y)), \tag{2}$$

where c is the density function of C, defined by

$$c(u,v) = \frac{\partial^2}{\partial v \partial u} C(u,v). \tag{3}$$

Recently, Sharifonnasabi et al. (2018) show that the following bivariate function

$$C(u,v) = K(u,v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}}, \qquad \forall u,v \in I, \tag{4}$$

is a copula under certain conditions.
As a special case of (4), we can see that the bivariate function

$$C(u,v;\theta,\delta) = uv[1 + \delta(1-u)^2(1-v)^2][1 + \theta(1-u)(1-v)], \tag{5}$$

is a copula for $\theta \in (-1,1)$, $\delta \in [0,1]$. The density copula of (5) is given by

$$\begin{aligned} c(u,v;\theta,\delta) &= 1 + \theta + \delta + \delta\theta - (4\delta + 2\theta + 6\theta\delta)(u+v) - 4\theta\delta(u^3+v^3) \\ &\quad + (3\delta + 9\theta\delta)(u^2+v^2) - (12\delta + 54\theta\delta)(uv^2 + u^2v) \\ &\quad + 24\theta\delta(uv^3 + u^3v) - 36\theta\delta(u^2v^3 + u^3v^2) \\ &\quad + (16\delta + 4\theta + 36\theta\delta)uv + (9\delta + 81\theta\delta)u^2v^2 + 16\theta\delta u^3v^3 \\ &= \sum_{i=0}^{3}\sum_{j=0}^{3} a_{i,j}(\theta,\delta)u^i v^j, \end{aligned} \tag{6}$$

where $a_{i,j}(\theta,\delta)$'s are constant w.r.t. u and v. As it can be seen below, copula (5) contains certain appealing characteristics. First, we give some features of the new copula (cf. Shrifonnasabi et al., 2018). For the dependence measures used below, we refer to Nelsen (2006).

Remark 1. If r.v.'s X and Y are related by the new copula defined in (5) and $\theta \in [0,1]$, then we can show that
i. X and Y are PQD and $LCSD$ and Y is LTD in X.
ii. X and Y have zero tail dependence.

Table 1. τ and ρ_s values of copula (5) for different parameters setting.

δ	θ	τ	ρ_s	$\frac{\rho_s}{\tau}$	δ	θ	τ	ρ_s	$\frac{\rho_s}{\tau}$
0.0001	0.0001	28e-6	42e-6	1.4999	0.0001	-0.0001	-0.17e-6	-0.25e-6	1.4999
0.0001	0.4001	0.0889	0.1334	1.4999	0.0001	-0.4001	-0.0889	-0.1334	1.4999
0.5001	0.1001	0.0512	0.0765	1.4963	0.5001	-0.1001	0.0044	0.0068	1.5354
0.2001	0.1001	0.0338	0.0506	1.4978	0.2001	-0.1001	-0.0116	-0.0173	1.4947
0.4001	0.3001	0.0917	0.137	1.4942	0.4001	-0.3001	-0.0471	-0.0703	1.4938
0.3001	0.4001	0.1084	0.162	1.4947	0.3001	-0.4001	-0.0748	-0.1120	1.4967
0.7001	0.2001	0.0866	0.1292	1.4932	0.7001	-0.2001	-0.0086	-0.0126	1.4537
0.6001	0.9001	0.2465	0.3662	1.4858	0.6001	-0.9001	-0.1777	-0.2662	1.4985
0.9001	0.5001	0.1717	0.2552	1.4862	0.9001	-0.5001	-0.0707	-0.1052	1.4878
0.5001	0.6001	0.1682	0.2507	1.4902	0.5001	-0.6001	-0.1119	-0.1674	1.4961
0.8001	0.8001	0.2377	0.3526	1.4832	0.8001	-0.8001	-0.1465	-0.2192	1.4961
0.9001	0.9001	0.2698	0.3993	1.4802	0.9001	-0.9001	-0.1665	-0.2493	1.4972
1	0.99	0.2999	0.4430	1.4771	1	-0.99	-0.1844	-0.2764	1.4989

iii. C is positively ordered with respect to θ and δ and C is Schur-concave.
iv. C is not concave in u and thus Y is not stochastically increasing in X.
v. Kendall's τ and Spearman's ρ for copula in (5) are given by

$$\tau = \frac{2\theta}{9} + \frac{\delta}{18} + \frac{\delta\theta}{45} + \frac{\delta\theta^2}{450} + \frac{2\delta^2\theta}{11025}, \quad \rho_s = \frac{3\delta\theta}{100} + \frac{\delta}{12} + \frac{\theta}{3},$$

where PQD, LCSD and LTD denote positively quadrant dependent, left corner set decreasing, and left tail decreasing, respectively.

Values of τ and ρ_s for copula (5) are given in Table 1. It follows that $\frac{\rho_s}{\tau} \to \frac{3}{2}$ as $\theta \to 0$ and $\delta \to 0$. Since τ and ρ_s are increasing functions of θ and δ, the range of values of τ and ρ_s for copula (5) are

$$-0.2222 \leqslant \tau \leqslant 0.3024, \quad -0.3333 \leqslant \rho_s \leqslant 0.4467.$$

3 Concomitants of Generalized Order Statistics

In this section, we investigate concomitants of generalized order statistics for the related bivariate random variable (X, Y) according to the copula C in (5). Generalized order statistics provide a unified approach to a variety of

models of ordered random variables such as order statistics, record values, k-record values and progressively Type-II censored order statistics, as special cases. The most important applications of concomitants arise in selection procedures and prediction problems. For more details we refer to Kamps (1995), David and Nagaraja (1998) and Beg and Ahsanullah (2008).
Let (X_i, Y_i), $i = 1, 2, \ldots$ be a sequence of independent bivariate random variables identical to (X, Y) with an absolutely continuous joint df $H_{X,Y}$ and joint pdf $h_{X,Y}$. Let $Y_{[r,n,m,k]}$ denote the concomitant of the rth generalized order statistics associated with $X_{(r,n,m,k)}$. That is, $Y_{[r,n,m,k]} = Y_i$ if, and only if, $X_{(r,n,m,k)} = X_i$. The pdf and df of $Y_{[r,n,m,k]}$, $1 \leqslant r \leqslant n$ are respectively given by

$$g_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} h_{Y|X}(y|x) f_{r,n,m,k}(x)dx \tag{7}$$

and

$$G_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} H_{Y|X}(y|x) f_{r,n,m,k}(x)dx, \tag{8}$$

where $f_{r,n,m,k}(x)$ is th pdf of the r-th generalized order statistics.
Let $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ be concomitants of the rth and sth generalized order statistics $(1 \leqslant r < s \leqslant n)$, respectively. Then the joint df and pdf of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ are, respectively, given by

$$g_{[r,s,n,m,k]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} p_{(r,s,n,m,k)}(x_1, x_2, y_1, y_2) dx_1 dx_2, \tag{9}$$

$$G_{[r,s,n,m,k]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} P_{(r,s,n,m,k)}(x_1, x_2, y_1, y_2) dx_1 dx_2, \tag{10}$$

where

$$p_{(r,s,n,m,k)}(x_1, x_2, y_1, y_2) = h_{Y|X}(y_1|x_1) h_{Y|X}(y_2|x_2) f_{(r,s,n,m,k)}(x_1, x_2),$$
$$P_{(r,s,n,m,k)}(x_1, x_2, y_1, y_2) = H_{Y|X}(y_1|x_1) H_{Y|X}(y_2|x_2) f_{(r,s,n,m,k)}(x_1, x_2),$$

and $f_{(r,s,n,m,k)}(x_1, x_2)$ is the joint pdf of the rth and sth; $1 \leqslant r < s \leqslant n$; generalized order statistics.
Now, let the bivariate random variable (X, Y) be related to the copula (5) with bivariate df $H_{X,Y}(x, y)$ and bivariate pdf $h_{X,Y}(x, y)$, as follows

$$H_{X,Y}(x, y) = F_X(x) G_Y(y)[1 + \delta \overline{F}_X^2(x) \overline{G}_Y^2(y)][1 + \theta \overline{F}_X(x) \overline{G}_Y(y)], \tag{11}$$

and

$$h_{X,Y}(x,y) = f_X(x)g_Y(y)\sum_{i=0}^{3}\sum_{j=0}^{3} a_{i,j}(\theta,\delta)F_X(x)^i G_Y(y)^j, \qquad (12)$$

where $F_X(x)$ and $G_Y(y)$ are dfs of X and Y, $\overline{F}_X(x)$ and $\overline{G}_Y(y)$ are survival functions of X and Y, respectively, and $a_{i,j}(\theta,\delta)$ is a bivariate function depending on δ and θ. Then, their corresponding conditional df and conditional pdf are given by

$$\begin{aligned} H_{Y|X}(y|x) &= P(Y \leqslant y|X=x) = \frac{\int_{-\infty}^{y} h_{X,Y}(x,t)dt}{f_X(x)}, \\ &= \sum_{i=0}^{3}\sum_{j=0}^{3} \frac{a_{i,j}(\theta,\delta)}{j+1} F_X(x)^i G_Y(y)^{j+1}, \qquad (13) \end{aligned}$$

and

$$h_{Y|X}(y|x) = \frac{h_{X,Y}(x,y)}{f_X(x)} = g_Y(y)\sum_{i=0}^{3}\sum_{j=0}^{3} a_{i,j}(\theta,\delta)F_X(x)^i G_Y(y)^j.$$

We can rewrite $h_{Y|X}(y|x)$ as follows

$$h_{Y|X}(y|x) = g_Y(y)\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{l=0}^{i} a_{i,j}(\theta,\delta)\binom{i}{l}(-1)^l \overline{F}_X(x)^l G_Y(y)^j. \qquad (14)$$

***Theorem* 1.** *For the bivariate distribution (11), the pdf and cdf of* $Y_{[r,n,m,k]}$ *are given by*

$$\begin{aligned} g_{[r,n,m,k]}(y) &= \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{l=0}^{i} a_{i,,j}(\theta,\delta)g(y)G(y)^j K_{[r,n,m,k,i,l]}, \qquad (15) \\ G_{[r,n,m,k]}(y) &= \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{l=0}^{i} \frac{a_{i,j}(\theta,\delta)}{j+1} G(y)^{j+1} K_{[r,n,m,k,i,l]}, \qquad (16) \end{aligned}$$

where

$$K_{[r,n,m,k,i,l]} = \frac{c_{r-1,n,\gamma_{r+1,n}}}{(r-1)!(m+1)^r}(-1)^l\binom{i}{l}B(\frac{\gamma_{r+1,n}+l}{m+1}+1,r)$$

and

$$B(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Proof. Utilizing Equation (14), from (7) one obtains

$$\begin{aligned} g_{[r,n,m,k]}(y) &= \frac{c_{r-1,n,\gamma_{r+1,n}}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{l=0}^{i} a_{i,j}(\theta,\delta)\binom{i}{l}(-1)^l g(y)G(y)^j \\ &\quad\times\int_{-\infty}^{\infty}\overline{F}_X(x)^l(1-(\overline{F}(x))^{m+1})^{r-1}(\overline{F}(x))^{\gamma_{r,n}-1}f(x)dx. \end{aligned}$$

Thus, making transformation $u=\overline{F}(x)^{m+1}$, we have

$$\begin{aligned} g_{[r,n,m,k]}(y) &= \frac{c_{r-1,n,\gamma_{r+1,n}}}{(r-1)!(m+1)^{r}}\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{l=0}^{i}\binom{i}{l}(-1)^l a_{i,j}(\theta,\delta)g(y)G(y)^j \\ &\quad\times\int_{-\infty}^{\infty}u^{\frac{\gamma_{r+1,n}+l}{m+1}}(1-u)^{r-1}du, \\ &= \frac{c_{r-1,n,\gamma_{r+1,n}}}{(r-1)!(m+1)^{r}}\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{l=0}^{i}\binom{i}{l}(-1)^l a_{i,j}(\theta,\delta)g(y)G(y)^j \\ &\quad\times B(\frac{\gamma_{r+1,n}+l}{m+1}+1,r)=\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{l=0}^{i}a_{i,j}(\theta,\delta)g(y)G(y)^jK_{[r,n,m,k,i,l]}. \end{aligned}$$

We can similarly obtain cdf of $Y_{[r,n,m,k]}$. □

Theorem 2. *The pth moment of $Y_{[r,n,m,k]}$ is given by*

$$\mu^p_{[r,n,m,k]}=\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{l=0}^{i}\frac{a_{i,j}(\theta,\delta)}{j+1}K_{[r,n,m,k,i,l]}\mu^p_{(j+1,j+1)} \tag{17}$$

and its moment generating function (mgf) is

$$M_{[r,n,m,k]}(t)=\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{l=0}^{i}\frac{a_{i,j}(\theta,\delta)}{j+1}K_{[r,n,m,k,i,l]}M_{(j+1,j+1)}(t), \tag{18}$$

where $\mu^p_{(j+1,j+1)}$ and $M_{(j+1,j+1)}(t)$ are the pth moment and mgf of $Y_{(j+1,j+1)}$, *the $(j+1)$-th order statistics of $Y_1,\ Y_2,\ \ldots,\ Y_{j+1},\ j=0,1,2,3,$ respectively.*

Proof. The proof is obvious, thus omitted. □

To continue, we present the following Lemma.

***Lemma* 1.** *Let p and q be real numbers, then*

$$\begin{aligned} & \frac{c_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} (\overline{F}(x_1))^p(\overline{F}(x_2))^m \\ \times\ & [1-(\overline{F}(x_1))^{m+1}]^{r-1}[(\overline{F}(x_1))^{m+1}-(\overline{F}(x_2))^{m+1}]^{s-r-1} \\ \times\ & (\overline{F}(x_2))^{\gamma_s-1}(\overline{F}(x_2))^q f(x_1)f(x_2)dx_1dx_2, \\ =\ & \frac{\gamma_1\gamma_2\ldots\gamma_s}{(\gamma_1+p+q)(\gamma_2+p+q)\ldots(\gamma_r+p+q)(\gamma_{r+1}+q)\ldots(\gamma_s+q)} = I_{p,q}. \end{aligned}$$

Proof. See Beg and Ahsanullah (2008). □

Now utilizing (9), (10), and Lemma 1, we obtain

$$\begin{aligned} g_{[r,s,n,m,k]}(y_1,y_2) &= g(y_1)g(y_2)\sum_{i_1=0}^{3}\sum_{j_1=0}^{3}\sum_{i_2=0}^{3}\sum_{j_2=0}^{3}\sum_{l_1=0}^{i_1}\sum_{l_2=0}^{i_2}(-1)^{l_1+l_2}\binom{i_1}{l_1}\binom{i_2}{l_2} \\ &\times\ a_{i_1,j_1}(\theta,\delta)a_{i_2,j_2}(\theta,\delta)G(y_1)^{j_1}G(y_2)^{j_2}I_{l_1,l_2}, \end{aligned}$$

and

$$\begin{aligned} G_{[r,s,n,m,k]}(y_1,y_2) &= \sum_{i_1=0}^{3}\sum_{j_1=0}^{3}\sum_{i_2=0}^{3}\sum_{j_2=0}^{3}\sum_{l_1=0}^{i_1}\sum_{l_2=0}^{i_2}(-1)^{l_1+l_2}\binom{i_1}{l_1}\binom{i_2}{l_2} \\ &\times\ \frac{a_{i_1,j_1}(\theta,\delta)a_{i_2,j_2}(\theta,\delta)}{(j_1+1)(j_2+1)}G(y_1)^{j_1+1}G(y_2)^{j_2+1}I_{l_1,l_2}, \end{aligned}$$

where I_{l_1,l_2} is defined in Lemma 1.

***Theorem* 3.** *The (p,q)th product moments and the joint moment generating function of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ are, respectively, given by*

$$\begin{aligned} \mu^{p,q}_{[r,s,n,m,k]} &= \sum_{i_1=0}^{3}\sum_{j_1=0}^{3}\sum_{i_2=0}^{3}\sum_{j_2=0}^{3}\sum_{l_1=0}^{i_1}\sum_{l_2=0}^{i_2}\binom{i_1}{l_1}\binom{i_2}{l_2}(-1)^{l_1+l_2}\frac{a_{i_1,j_1}(\theta,\delta)}{(j_1+1)} \\ &\times\frac{a_{i_2,j_2}(\theta,\delta)}{(j_2+2)}\mu^{p}_{(j_1+1,j_1+1)}\mu^{q}_{(j_2+1,j_2+1)}I_{l_1,l_2} \end{aligned}$$

and

$$M_{[r,s,n,m,k]}(t_1,t_2) = \sum_{i_1=0}^{3}\sum_{j_1=0}^{3}\sum_{i_2=0}^{3}\sum_{j_2=0}^{3}\sum_{l_1=0}^{i_1}\sum_{l_2=0}^{i_2}\binom{i_1}{l_1}\binom{i_2}{l_2}(-1)^{l_1+l_2}I_{l_1,l_2} \times\frac{a_{i_1,j_1}(\theta,\delta)a_{i_2,j_2}(\theta,\delta)}{(j_1+1)(j_2+2)}M_{(j_1+1,j_1+1)}(t_1)M_{(j_2+1,j_2+1)}(t_2),$$

where $\mu^p_{(j_i+1,j_i+1)}$ *and* $M_{(j_i+1,j_i+1)}(t)$ *are pth moment and mgf of* $Y_{(j_i+1,j_i+1)}$, *the* (j_i+1)*th order statistics of* $Y_1,\ Y_2,\ ...,\ Y_{j_i+1},\ j=0,1,2,3$ *and* $i=1,2$.

Proof. The proof is obvious and thus omitted. □

Example 1. Consider a random sample (X_i,Y_i), $i=1,2,...,n$, from the bivariate distribution $H_{X,Y}$ in (11). Let concomitant of the rth order statistics $X_{r,n}$ be denoted by $Y_{[r,n]}$, $1\leqslant r\leqslant n$. The pdf of $Y_{[r,n]}$, denoted by $g_{[r,n]}$, is given by

$$g_{[r,n]}(y)=\sum_{i=0}^{3}\sum_{j=0}^{3}a_{i,j}(\theta,\delta)g(y)G(y)^jS_{[r,n,i]},$$

where

$$S_{[r,n,i]}=K_{[r,n,0,1,i,l]}=\frac{n!}{(r-1)!(n-r)!}B(n-r+1,r+i).$$

The pth moment and mgf of $Y_{[r,n]}$ can be deduced, respectively, as

$$\mu^p_{[r,n]} = \sum_{i=0}^{3}\sum_{j=0}^{3}\frac{a_{i,j}(\theta,\delta)}{j+1}S_{[r,n,i]}\mu^p_{(j+1,j+1)},$$

$$M_{[r,n]}(t) = \sum_{i=0}^{3}\sum_{j=0}^{3}\frac{a_{i,j}(\theta,\delta)}{j+1}S_{[r,n,i]}M_{(j+1,j+1)}(t),$$

where $\mu^p_{(j+1,j+1)}$ and $M_{(j+1,j+1)}(t)$ are pth moment and mgf of $Y_{(j+1,j+1)}$, the $(j+1)$th order statistics of $Y_1,\ Y_2,\ ...,\ Y_{j+1},\ j=0,1,2,3$, respectively.

3.1 Application

An interesting application of concomitants is the minimization of variance linear unbiased estimates (MVLUE) of the location and scale parameters when other parameters are known. Suppose that the random variables have location parameter ν and scale parameter ω. By using the method of David and Nagaraja (2003), the MVLUE of $\underline{\theta}$ equals

$$\widehat{\underline{\theta}} = (\mathbf{A}^{'}\mathbf{V}^{-1}\mathbf{A})^{-1}(\mathbf{A}^{'}\mathbf{V}^{-1}\mathbf{y}),$$

where $\mathbf{V} = (V_{i,j})$ is the variance of the ith and j-th concomitants, $\mathbf{V}^{-1}$ is the inverse of the matrix $\mathbf{V}$, $\mathbf{y}^{'}$ is the observed value of the vector

$$\mathbf{Y}^{'} = [Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, ..., Y_{[n,n,m,k]}]$$

and $\mathbf{A}$ and $\underline{\theta}$ are defined by:

$$\mathbf{A}^{'} = \begin{bmatrix} 1 & 1 & . & . & . & 1 \\ \mu_{[1,n,m,k]} & \mu_{[2,n,m,k]} & . & . & . & \mu_{[n,n,m,k]} \end{bmatrix} \quad and \quad \underline{\theta}^{'} = \begin{bmatrix} \nu & \omega \end{bmatrix}.$$

Numerical methods are used to obtain the coefficients of the location and scale parameters of the concomitants of order statistics ($\gamma_i = n-i+1$) when the marginal distributions are Burr and logistic. Results appear in Tables 2 − 6.

Recall that a random variable X has a Burr XII distribution with parameters β, λ, ν and ω if

$$G_Y(y) = 1 - \beta^{\lambda}(\beta + \frac{y-\nu}{\omega})^{-\lambda}, \qquad y > \nu, \beta > 0, \omega > 0, \nu \in R,$$

and it has a logistic distribution with parameters ν and ω if

$$G_Y(y) = [1 + \exp\{-\frac{y-\nu}{\omega}\}]^{-1}, \qquad y \in R, \omega > 0, \nu \in R.$$

4 A Bivariate Exponential Distribution

Exponential distribution plays an important role in all theoretical and applied fields of statistics, especially as a lifetime distribution. For this reason, different bivariate exponential distributions have been introduced in the lit-

Table 2. Coefficients of MVLUE of ν and ω for Burr distribution based on concomitants of order statistics with $n = 5$, λ=2.65, β=0.5.

Known parameters	Estimates	Coefficients				
$\theta = 0.1$, $\delta = 0.95$	$\hat{\nu}$	5.4565	1.6981	-0.8747	-2.3762	-2.9036
	$\hat{\omega}$	-17.3514	-4.9386	3.5542	8.5041	10.2318
$\theta = 0.3$, $\delta = 0.75$	$\hat{\nu}$	3.0736	1.1058	-0.2529	-1.1705	-1.7560
	$\hat{\omega}$	-9.4916	-2.9826	1.5044	4.5262	6.4436
$\theta = 0.5$, $\delta = 0.55$	$\hat{\nu}$	2.2427	0.7894	-0.1299	-0.7420	-1.1602
	$\hat{\omega}$	-6.7469	-1.9410	1.0948	3.1107	4.4824
$\theta = 0.7$, $\delta = 0.35$	$\hat{\nu}$	1.8436	0.5882	-0.0946	-0.5224	-0.8148
	$\hat{\omega}$	-5.4224	-1.2824	0.9713	2.3837	3.3498
$\theta = 0.9$, $\delta = 0.15$	$\hat{\nu}$	1.6268	0.4349	-0.0861	-0.3866	-0.5890
	$\hat{\omega}$	-4.6947	-0.7858	0.9333	1.9334	2.6139
$\theta = 0.2$, $\delta = 0.25$	$\hat{\nu}$	4.8720	2.0030	-0.2810	-2.0892	-3.5047
	$\hat{\omega}$	-15.4211	-5.9476	1.5912	7.5559	12.2217
$\theta = 0.4$, $\delta = 0.45$	$\hat{\nu}$	2.6812	1.0030	-0.1542	-0.9726	-1.5574
	$\hat{\omega}$	-8.1933	-2.6463	1.1744	3.8715	5.7938
$\theta = 0.6$, $\delta = 0.65$	$\hat{\nu}$	1.9528	0.6400	-0.1137	-0.5853	-0.8939
	$\hat{\omega}$	-5.7905	-1.4477	1.0413	2.5936	3.6033
$\theta = 0.8$, $\delta = 0.85$	$\hat{\nu}$	1.5996	0.4347	-0.0929	-0.3831	-0.5583
	$\hat{\omega}$	-4.6240	-0.7709	0.9721	1.9263	2.4964

Table 3. Coefficients of MVLUE of ν and ω for Burr distribution based on concomitants of order statistics with $n = 5$, λ=4.5, β=1.5.

Known parameters	Estimates	Coefficients				
$\theta = 0.1$, $\delta = 0.95$	$\hat{\nu}$	5.4412	1.6673	-0.9106	-2.3813	-2.8165
	$\hat{\omega}$	-12.2320	-3.4213	2.5959	6.0250	7.0323
$\theta = 0.3$, $\delta = 0.75$	$\hat{\nu}$	3.1946	1.1563	-0.2684	-1.2369	-1.8456
	$\hat{\omega}$	-6.9913	-2.2290	1.0980	3.3553	4.7669
$\theta = 0.5$, $\delta = 0.55$	$\hat{\nu}$	2.3471	0.8562	-0.1231	-0.8006	-1.2796
	$\hat{\omega}$	-5.0101	-1.5317	0.7548	2.3358	3.4512
$\theta = 0.7$, $\delta = 0.35$	$\hat{\nu}$	1.9256	0.6599	-0.0780	-0.5725	-0.9350
	$\hat{\omega}$	-4.0184	-1.0794	0.6419	1.8010	2.6549
$\theta = 0.9$, $\delta = 0.15$	$\hat{\nu}$	1.6898	0.5108	-0.0656	-0.4312	-0.7037
	$\hat{\omega}$	-3.4551	-0.7409	0.6026	1.4683	2.1251
$\theta = 0.2$, $\delta = 0.25$	$\hat{\nu}$	5.1774	2.1353	-0.3053	-2.2456	-3.7619
	$\hat{\omega}$	-11.6159	-4.5146	1.1811	5.7073	9.2421
$\theta = 0.4$, $\delta = 0.45$	$\hat{\nu}$	2.8199	1.0815	-0.1516	-1.0475	-1.7022
	$\hat{\omega}$	-6.1144	-2.0562	0.8221	2.9120	4.4365
$\theta = 0.6$, $\delta = 0.65$	$\hat{\nu}$	2.0346	0.6986	-0.1048	-0.6331	-0.9952
	$\hat{\omega}$	-4.2789	-1.1658	0.7106	1.9449	2.7892
$\theta = 0.8$, $\delta = 0.85$	$\hat{\nu}$	1.6542	0.4819	-0.0835	-0.4178	-0.6348
	$\hat{\omega}$	-3.3852	-0.6661	0.6563	1.4422	1.9528

Table 4. Coefficients of MVLUE of ν and ω for Burr distribution based on concomitants of order statistics with $n=5$, λ=3.5, β=1.

Known parameters	Estimates	Coefficients				
$\theta=-0.1$, $\delta=0.95$	$\hat{\nu}$	17.7638	-3.1961	-12.6796	-9.2825	8.3944
	$\hat{\omega}$	-43.9269	8.4745	32.1915	23.7146	-20.4536
$\theta=-0.3$, $\delta=0.75$	$\hat{\nu}$	-3.3478	-2.5611	-0.9370	1.8014	6.0444
	$\hat{\omega}$	8.8550	6.9060	2.8545	-3.9946	-14.6209
$\theta=-0.5$, $\delta=0.55$	$\hat{\nu}$	-1.7301	-1.0972	-0.2158	1.0565	2.9866
	$\hat{\omega}$	4.8221	3.2439	1.0425	-2.1391	-6.9695
$\theta=-0.7$, $\delta=0.35$	$\hat{\nu}$	-1.0611	-0.6555	-0.1066	0.7119	2.1112
	$\hat{\omega}$	3.1581	2.1371	0.7608	-1.2843	-4.7717
$\theta=-0.9$, $\delta=0.15$	$\hat{\nu}$	-0.7044	-0.4410	-0.0797	0.5025	1.7226
	$\hat{\omega}$	2.2756	1.5982	0.6839	-0.7701	-3.7876
$\theta=-0.2$, $\delta=0.25$	$\hat{\nu}$	-5.0783	-3.1280	-0.5552	2.7659	6.9956
	$\hat{\omega}$	13.1922	8.3213	1.8914	-6.4127	-16.9922
$\theta=-0.4$, $\delta=0.45$	$\hat{\nu}$	-2.2833	-1.4302	-0.2656	1.347	3.6320
	$\hat{\omega}$	6.2044	4.0767	1.1676	-2.865	-8.5837
$\theta=-0.6$, $\delta=0.65$	$\hat{\nu}$	-1.3623	-0.8751	-0.1843	0.8572	2.5645
	$\hat{\omega}$	3.9040	2.6883	0.9625	-1.6415	-5.9132
$\theta=-0.8$, $\delta=0.85$	$\hat{\nu}$	-0.9026	-0.5954	-0.1476	0.5913	2.0543
	$\hat{\omega}$	2.7588	1.9876	0.8668	-0.9800	-4.6333

Table 5. Coefficients of MVLUE of ν and ω for the logistic distribution based on concomitants of order statistics with $n=5$.

Known parameters	Estimates	Coefficients				
$\theta=0.1$, $\delta=0.95$	$\hat{\nu}$	0.2005	0.1987	0.2006	0.2015	0.1986
	$\hat{\omega}$	-1.9485	-0.6037	0.4388	1.0354	1.0781
$\theta=0.3$, $\delta=0.75$	$\hat{\nu}$	0.2005	0.1984	0.2006	0.2019	0.1985
	$\hat{\omega}$	-1.3004	-0.4907	0.1783	0.6714	0.9414
$\theta=0.5$, $\delta=0.55$	$\hat{\nu}$	0.2028	0.1969	0.1978	0.2007	0.2018
	$\hat{\omega}$	-0.9716	-0.4015	0.0806	0.4866	0.8059
$\theta=0.7$, $\delta=0.35$	$\hat{\nu}$	0.2073	0.1945	0.1928	0.1979	0.2074
	$\hat{\omega}$	-0.7784	-0.3364	0.0352	0.3755	0.7041
$\theta=0.9$, $\delta=0.15$	$\hat{\nu}$	0.2144	0.1915	0.1859	0.1934	0.2148
	$\hat{\omega}$	-0.6531	-0.2874	0.0110	0.3005	0.6290
$\theta=0.2$, $\delta=0.25$	$\hat{\nu}$	0.1995	0.2001	0.2007	0.2006	0.1991
	$\hat{\omega}$	-2.4276	-0.9997	0.2245	1.2239	1.9789
$\theta=0.4$, $\delta=0.45$	$\hat{\nu}$	0.2012	0.1985	0.1993	0.2007	0.2003
	$\hat{\omega}$	-1.2179	-0.5063	0.1036	0.6128	1.0079
$\theta=0.6$, $\delta=0.65$	$\hat{\nu}$	0.2050	0.1946	0.1957	0.2006	0.2040
	$\hat{\omega}$	-0.8071	-0.3295	0.0655	0.4013	0.6699
$\theta=0.8$, $\delta=0.85$	$\hat{\nu}$	0.2118	0.1879	0.1892	0.2002	0.2109
	$\hat{\omega}$	-0.6026	-0.2351	0.0468	0.2922	0.4987

Table 6. Coefficients of MVLUE of ν and ω for the logistic distribution based on concomitants of order statistics with $n = 5$.

Known parameters	Estimates	Coefficients				
$\theta = -0.1$, $\delta = 0.95$	$\hat{\nu}$	0.1996	0.1958	0.1964	0.2005	0.2077
	$\hat{\omega}$	-5.3028	-0.4786	2.4888	2.8940	0.3986
$\theta = -0.3$, $\delta = 0.75$	$\hat{\nu}$	0.1879	0.2006	0.2076	0.2067	0.1973
	$\hat{\omega}$	1.7026	2.3856	1.6929	-0.7312	-5.0499
$\theta = -0.5$, $\delta = 0.55$	$\hat{\nu}$	0.1994	0.2002	0.2006	0.2002	0.1996
	$\hat{\omega}$	1.3358	0.8297	0.1858	-0.6435	-1.7077
$\theta = -0.7$, $\delta = 0.35$	$\hat{\nu}$	0.2064	0.1971	0.1939	0.1965	0.2060
	$\hat{\omega}$	0.9083	0.4716	0.0407	-0.4294	-0.9912
$\theta = -0.9$, $\delta = 0.15$	$\hat{\nu}$	0.2144	0.1928	0.1862	0.1925	0.2141
	$\hat{\omega}$	0.6887	0.3225	0.0079	-0.3141	-0.7050
$\theta = -0.2$, $\delta = 0.25$	$\hat{\nu}$	0.1975	0.2008	0.2020	0.2012	0.1984
	$\hat{\omega}$	3.3481	2.3639	0.7102	-1.6594	-4.7628
$\theta = -0.4$, $\delta = 0.45$	$\hat{\nu}$	0.1983	0.2006	0.2015	0.2008	0.1988
	$\hat{\omega}$	1.6706	1.0739	0.2600	-0.8149	-2.1895
$\theta = -0.6$, $\delta = 0.65$	$\hat{\nu}$	0.2010	0.1996	0.1992	0.1993	0.2009
	$\hat{\omega}$	1.1136	0.6684	0.1388	-0.5282	-1.3926
$\theta = -0.8$, $\delta = 0.85$	$\hat{\nu}$	0.2061	0.1973	0.1948	0.1968	0.2051
	$\hat{\omega}$	0.8384	0.4666	0.0824	-0.3817	-1.0058

erature. However, it is necessary to propose new bivariate exponential distributions for cases when others can not be fitted well to the real data. Here, we introduce a new one whose marginals are alone exponential distributions. We later show that our bivariate model performs better than competitors in fitting real data. To operationalize it, we use (11) with marginal distributions $F_X(x) = 1 - e^{-\lambda_1 x}$ and $G_Y(y) = 1 - e^{-\lambda_2 y}$, and obtain

$$H_{X,Y}(x, y) = (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y})(1 + \delta e^{-2\lambda_1 x - 2\lambda_2 y})(1 + \theta e^{-\lambda_1 x - \lambda_2 y)}), \tag{19}$$

for $x, y \geq 0$ and $\lambda_1, \lambda_2 > 0, -1 < \theta < 1$ and $0 \leqslant \delta \leqslant 1$. The underlying joint density function is given by

$$h_{X,Y}(x, y) \quad = \quad \lambda_1 \lambda_2 e^{-\lambda_1 x - \lambda_2 y} \sum_{m=0}^{3} \sum_{n=0}^{3} a_{m,n}(\theta, \delta)(1 - e^{-\lambda_1 x})^m (1 - e^{-\lambda_2 y})^n. \tag{20}$$

If the random pair (X, Y) has the bivariate pdf (20), then

- The conditional pdfs are given by

$$h_{X|Y}(x|y) = \lambda_1 e^{-\lambda_1 x} \sum_{m=0}^{3}\sum_{n=0}^{3} a_{m,n}(\theta,\delta)(1-e^{-\lambda_1 x})^m (1-e^{-\lambda_2 y})^n,$$

and

$$h_{Y|X}(y|x) = \lambda_2 e^{-\lambda_2 y} \sum_{m=0}^{3}\sum_{n=0}^{3} a_{i,j}(\theta,\delta)(1-e^{-\lambda_1 x})^m (1-e^{-\lambda_2 y})^n.$$

- The joint mgf of (X, Y), for $|t| < 4\lambda_1$ and $|s| < 4\lambda_2$, is

$$M_{X,Y}(t,s) = \lambda_1\lambda_2 \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{i=0}^{m}\sum_{j=0}^{n} \frac{a_{n,m}(\theta,\delta)\binom{m}{i}\binom{n}{j}(-1)^{i+j}}{(t-(i+1)\lambda_1)(s-(j+1)\lambda_2)}.$$

- The Pearson's correlation coefficient of (X, Y) is given by

$$Corr(X,Y) = \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{i=0}^{m}\sum_{j=0}^{n} \frac{a_{n,m}(\theta,\delta)\binom{m}{i}\binom{n}{j}(-1)^{i+j}}{(i+1)^2(j+1)^2} - 1.$$

- The pth conditional moments are

$$E(Y^p|X=x) = \frac{\Gamma(p+1)}{\lambda_2^p} \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{j=0}^{n} \frac{a_{n,m}(\theta,\delta)(-1)^j\binom{n}{j}(1-e^{-\lambda_1 x})^m}{(j+1)^{p+1}}.$$

In particular, for the special case $p = 1$, we have

$$E(Y|X=x) = \int_0^\infty f_{Y|X=x}(y|x)dy$$
$$= \frac{6\theta + 12 + 4e^{-\lambda_1 x}\delta - 12e^{-\lambda_1 x}\theta + 3e^{-2\lambda_1 x}\delta\theta - 6e^{-2\lambda_1 x}\delta - 4e^{-3\lambda_1 x}\theta\delta}{12\lambda_1}$$

- The stress-strength value of (X,Y) is given by

$$R = P(X > Y) = \lambda_1 \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{i=0}^{m}\sum_{j=0}^{n+1} \frac{a_{n,m}(\theta,\delta)\binom{m}{i}\binom{n+1}{j}(-1)^{i+j}}{(n+1)(\lambda_1(i+1)+\lambda_2 j)}.$$

- The mean time of failure is given by

$$MTTF = \int_0^\infty \int_0^\infty xyf(x,y)dxdy = \frac{\delta\theta + 4\delta + 36\theta + 144}{144\lambda_1\lambda_2}.$$

In particular, the maximum of MTTF is $\frac{1.28}{\lambda_1\lambda_2}$ at $\theta = -1$ and $\delta = 0$ and minimum of MTTF is $\frac{0.73}{\lambda_1\lambda_2}$ at $\theta = \delta = 1$.

Remark 2. Note that $P(X > Y)$ is increasing in θ and δ. Also, if X and Y have the identical exponential distribution with mean $\frac{1}{\lambda}$, then we have $P(X > Y) = 0.5$.

Now we investigate some reliability features of parallel and series systems (see e.g., Yilmaz, 2011). Let $T_{1:2}$ be the lifetime of series system having two components. Then the survival, distribution and density of $T_{1:2}$ are given by

$$\begin{aligned}
\overline{F}_{T_{1:2}}(t) &= P(T_{1:2} \geqslant t) = P(\min(X,Y) \geqslant t)\\
&= P(X \geqslant t, Y \geqslant t) = H_{X,Y}(t,t) + 1 - F_X(t) - G_Y(t)\\
&= (1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})(1+\delta e^{-2\lambda_1 t - 2\lambda_2 t})(1+\theta e^{-\lambda_1 t - \lambda_2 t)})\\
&\quad + e^{-\lambda_1 t} + e^{-\lambda_2 t} - 1\\
F_{1:2}(t) &= P(T_{1;2} \leqslant t) = 1 - P(T_{1;2} \geqslant t) = 1 - P(X \geqslant t, Y \geqslant t) = 1 - \overline{H}(t,t)\\
&= 2 - (1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})(1+\delta e^{-2\lambda_1 t - 2\lambda_2 t})(1+\theta e^{-\lambda_1 t - \lambda_2 t)})\\
&\quad - e^{-\lambda_1 t} - e^{-\lambda_2 t}\\
f_{1:2}(t) &= f(t) + g(t) - \frac{d}{dt}H(t,t)\\
&= \{\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}\}[1+\delta e^{-2\lambda_1 t - 2\lambda_2 t}]\\
&\quad \times[1+\theta e^{-(\lambda_1+\lambda_2)t}] - (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})\\
&\quad \times\{3\theta\delta e^{-2(\lambda_1+\lambda_2)t} + 2\delta e^{-(\lambda_1+\lambda_2)t} + \theta\} + \lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}.
\end{aligned}$$

The hazard rate and reversed hazard rate function of $T_{1;2} = \min(X,Y)$ are

$$h_{1:2}(t) = -\frac{d}{dt}\ln \overline{F}_{1:2}(t) \qquad r_{1:2}(t) = \frac{d}{dt}\ln F_{1:2}(t) = f_{1:2}(t)(F_{1:2}(t))^{-1}.$$

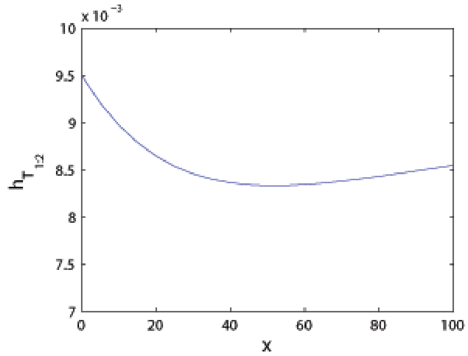

Figure 1. hazard rate of $T_{1:2}$

As seen in Figure 1, $T_{1:2}$ doesn't have a monotone hazard rate, but the reversed hazard rate of $T_{1:2}$ is decreasing for all of values of its parameters, because $f_{1:2}(t)$ and $(F_{1:2}(t))^{-1}$ are decreasing.

The mean residual of $T_{1:2}$ is given by:

$$\begin{aligned}
E(T_1 - t|T_1 > t) &= \int_0^{\infty} (x-t)\frac{f_{T_1}(t+x)}{\overline{F}_{T_1}(t)}dx \\
&= \frac{1}{\overline{F}_{T_1}(t)} \int_t^{\infty} (x-t)\{\lambda_1 e^{-\lambda_1(x+t)} + \lambda_2 e^{-\lambda_2(x+t)} \\
&- \lambda_1\lambda_2 e^{-(\lambda_1+\lambda_2)t} \\
&\times \sum_{m=0}^{3}\sum_{n=0}^{3} a_{n,m}(\theta,\delta)(1-e^{-\lambda_1(x+t)})^m(1-e^{-\lambda_2(x+t)})^n\}dx \\
&= \frac{1}{\overline{F}_{T_1}(t)} \int_0^{\infty} y\{\lambda_1 e^{-\lambda_1(y+2t)} + \lambda_2 e^{-\lambda_2(y+2t)} - \lambda_1\lambda_2 e^{-(\lambda_1+\lambda_2)t} \\
&\times \sum_{m=0}^{3}\sum_{n=0}^{3} a_{n,m}(\theta,\delta)(1-e^{-\lambda_1(y+2t)})^m(1-e^{-\lambda_2(y+2t)})^n\}dx \\
&= \frac{1}{\overline{F}_{T_1}(t)}\{\frac{e^{-2\lambda_1 t}}{\lambda_1} + \frac{e^{-2\lambda_2 t}}{\lambda_2} - \lambda_1\lambda_2 e^{-2(\lambda_1+\lambda_2)t} \\
&\times \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{i=0}^{m}\sum_{j=0}^{m} e^{-2\lambda_1 ti - 2\lambda_2 tj}\frac{a_{n,m}(\theta,\delta)\binom{m}{i}\binom{n}{j}(-1)^{i+j}}{[(i+1)\lambda_1 + (j+1)\lambda_2]^2}\}.
\end{aligned}$$

Now, let $T_{2:2}$ be the lifetime of a parallel system. Then the distribution, survival and density functions of $T_{2:2}$ are given by

$$\begin{aligned}
F_{T_{2:2}}(t) &= P(T_{2:2} \leqslant t) = P(\max(X,Y) \leqslant t) = P(X \leqslant t, Y \leqslant t) = H(t,t) \\
&= (1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})(1+\delta e^{-2\lambda_1 t-2\lambda_2 t})(1+\theta e^{-\lambda_1 t-\lambda_2 t)}), \\
\overline{F}_{T_{2:2}}(t) &= P(T_{2:2} \geq t) = 1 - P(X \leqslant t, Y \leqslant t) = 1 - H(t,t) \\
&= 1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})(1+\delta e^{-2\lambda_1 t-2\lambda_2 t})(1+\theta e^{-\lambda_1 t-\lambda_2 t)}),
\end{aligned}$$

$$\begin{aligned}
f_{T_{2:2}}(t) &= \frac{d}{dt}H(t,t) \qquad (21) \\
&= \{\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}\}[1+\delta e^{-2\lambda_1 t-2\lambda_2 t}] \\
&\quad \times[1+\theta e^{-(\lambda_1+\lambda_2)t}] - (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t}) \\
&\quad \times\{3\theta\delta e^{-2(\lambda_1+\lambda_2)t} + 2\delta e^{-(\lambda_1+\lambda_2)t} + \theta\}.
\end{aligned}$$

The hazard rate and reversed hazard rate functions of $T_{2;2}$ are given as follows

$$\begin{aligned}
h_{T_{2:2}}(t) &= \frac{f_{T_{2:2}}(t)}{1-H(t,t)} \\
r_{T_{2:2}}(t) &= \sum_{i=1}^{2} \frac{\lambda_i e^{-\lambda_i t}}{1-e^{-\lambda_i}t} - \frac{2\delta(\lambda_1+\lambda_2)e^{-2(\lambda_1+\lambda_2)t}}{1+\delta e^{-2(\lambda_1+\lambda_2)t}} - \frac{\theta(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}}{1+\theta e^{-(\lambda_1+\lambda_2)t}}.
\end{aligned}$$

As we observe, $r_{T_{2:2}}(t)$ is decreasing but hazard rate doesn't have a monotone trend. The mean residual lifetime of T_2 is obtained as bellow:

$$\begin{aligned}
E(T_2 - t|T_2 > t) &= \int_0^\infty (x-t)\frac{f_{T_2}(t+x)}{\overline{F}_{T_2}(t)}dx \\
&= \frac{\lambda_1\lambda_2 e^{-(\lambda_1+\lambda_2)t}}{\overline{F}_{T_2}(t)} \int_t^\infty u \sum_{m=0}^{3}\sum_{n=0}^{3} a_{n,m}(\theta,\delta) \\
&\quad \times(1-e^{-\lambda_1(u+2t)})^m(1-e^{-\lambda_2(u+2t)})^n du \\
&= \frac{\lambda_1\lambda_2}{\overline{H}(t,t)} \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{i=0}^{m}\sum_{j=0}^{m} \frac{a_{n,m}(\theta,\delta)\binom{m}{i}\binom{n}{j}(-1)^{i+j}}{[i\lambda_1+j\lambda_2]^2} \\
&\quad \times e^{-2\lambda_1 t(i+1)-2\lambda_2 t(j+1)}.
\end{aligned}$$

4.1 Simulation Studies

In this section, we conduct a simulation study to highlight further properties of the new copula. Specifically, we concentrate on the estimation its parameters. In order to simulate a vector (X, Y) from the bivariate distribution H, it is sufficient to perform following steps:

Step 1: Generate two variables U and Z independently from uniform distributions on $[0, 1]$.

Step 2: Calculate $V = C_u^{-1}(Z)$, where

$$C_u(v) = C(V \leqslant v | U = u) = \frac{\partial C(u, v)}{\partial u}.$$

Step 3: By the probability integral transform, we have

$$U \overset{d}{=} F(X) \Leftrightarrow X \overset{d}{=} F^{-1}(U) \quad V \overset{d}{=} G(X) \Leftrightarrow Y \overset{d}{=} G^{-1}(V)$$

where $\overset{d}{=}$ denotes the equality of distributions. Then, the pair (X, Y) has the joint distribution H with marginal distributions F and G.
The software R was used for simulation and estimation. Now, we generate data from the bivariate distribution in (19) with four sets of parameters $(\theta, \delta, \lambda_1, \lambda_2) = (0.85, 0.5, 0.5, 0.3)$, $(\theta, \delta, \lambda_1, \lambda_2) = (0.45, 0.2, 2, 1)$, $(\theta, \delta, \lambda_1, \lambda_2) = (-0.55, 0.35, 4, 3)$ and $(\theta, \delta, \lambda_1, \lambda_2) = (-0.75, 0.85, 1.5, 2.5)$ with sample sizes 50, 150 and 200. All simulations consist of 1000 replications. In each case, we estimate parameters and compute the average estimates (AE), biases and the mean squared errors (MSE). Note that, to estimate the parameters in a bivariate distribution, the method of inference function for margins (IFM) (see Joe, 1997) is employed. Results are given in Table 7. It is seen that the performance of estimates are satisfactory. Moreover, as the sample size increases, MSE decreases for all parameters and the bias decreases in most cases.

4.2 Data Analysis

In this section, to examine the performance of our proposed distribution, we reanalyze the following two data sets. We consider several families of bivariate exponential distributions, such as ACVBE, Freund, Gumbel type II, Gumbel type III, Johnson-Kotz, Marshal-Olkin type A, Marshal-Olkin type B and Sarkar mentioned in Balakrishnan and Lai (2009). Also we consider the bivariate copula with exponential distributions in margins, such

Table 7. AE, bias, and MSE based on 1000 simulations of the bivariate distribution in (19) for n=50, 150 and 200.

		$(\theta,\delta,\lambda_1,\lambda_2)=(0.85,0.5,0.5,0.3)$			$(\theta,\delta,\lambda_1,\lambda_2)=(0.45,0.2,2,1)$		
n		**AE**	**bias**	**MSE**	**AE**	**bias**	**MSE**
50	$\widehat{\theta}$	0.7506	-0.0994	0.2205	0.3716	-0.0784	0.3696
	$\widehat{\delta}$	0.6086	0.1086	0.5042	0.4560	0.2560	1.0792
	$\widehat{\lambda}_1$	0.5021	0.0021	0.0710	1.9928	-0.0072	0.2819
	$\widehat{\lambda}_2$	0.3043	0.0043	0.0431	0.9914	-0.0085	0.1403
150	$\widehat{\theta}$	0.8129	-0.0371	0.0116	0.4110	-0.0390	0.1176
	$\widehat{\delta}$	0.5358	0.0358	0.0246	0.3833	0.1833	0.3356
	$\widehat{\lambda}_1$	0.4991	-0.0009	0.0407	2.0062	0.0062	0.1638
	$\widehat{\lambda}_2$	0.2987	-0.0013	0.0244	1.0032	0.0032	0.0819
200	$\widehat{\theta}$	0.8281	-0.0219	0.0032	0.4228	-0.0272	0.0767
	$\widehat{\delta}$	0.5112	0.0112	0.0061	0.3412	0.1412	0.2141
	$\widehat{\lambda}_1$	0.4994	-0.0006	0.0353	2.0014	0.0014	0.1415
	$\widehat{\lambda}_2$	0.3002	0.0002	0.0212	1.0007	0.0007	0.0707
		$(\theta,\delta,\lambda_1,\lambda_2)=(-0.55,0.35,4,3)$			$(\theta,\delta,\lambda_1,\lambda_2)=(-0.75,0.85,1.5,2.5)$		
n		**AE**	**bias**	**MSE**	**AE**	**bias**	**MSE**
50	$\widehat{\theta}$	-0.4492	0.1008	0.4666	-0.5726	0.1774	0.4238
	$\widehat{\delta}$	0.5289	0.1749	1.7340	0.5615	-0.2885	1.6080
	$\widehat{\lambda}_1$	3.9716	-0.0284	0.5625	1.5089	0.0089	0.2135
	$\widehat{\lambda}_2$	3.0171	0.0171	0.4270	2.5168	0.0168	0.3562
150	$\widehat{\theta}$	-0.5586	-0.0086	0.1475	-0.6748	0.0752	0.1152
	$\widehat{\delta}$	0.5021	0.1521	0.5619	0.5661	-0.2839	0.5058
	$\widehat{\lambda}_1$	3.9862	-0.0138	0.3257	1.5067	0.0067	0.1231
	$\widehat{\lambda}_2$	3.0015	0.0015	0.2451	2.4929	-0.0071	0.2036
200	$\widehat{\theta}$	-0.5578	-0.0078	0.1048	-0.6773	0.0727	0.0765
	$\widehat{\delta}$	0.4716	0.1216	0.3909	0.6073	-0.2427	0.3344
	$\widehat{\lambda}_1$	4.0017	0.0017	0.2830	1.5004	0.0004	0.1061
	$\widehat{\lambda}_2$	3.0082	0.0082	0.2128	2.4964	-0.0036	0.1765

Table 8. Parameter Estimates

Model	Param.	Est. (Std. Err.)	Model	Param.	Est. (Std. Err.)
Our Model	θ	-0.7617(0.1886)	Freund	α_1	0.0006 (0.0003)
	δ	0.8885(0.9964)		α_2	0.0247(0.0021)
	λ_1	0.0208(0.0018)		λ_2	0.0013(0.0001)
	λ_2	0.0014(0.0001)		λ_3	0.0023(0.0013)
Gumbel type II	θ	-0.6703(0.1978)	FGM	θ	-0.6703 (0.1978)
	λ_1	0.0014(0.0001)		λ_1	0.0014 (0.0001)
	λ_2	0.0208(0.0017)		λ_2	0.0209 (0.0017)
Marshal-Olkin A	λ_3	-0.0185(0.0026)	ACBVE	λ_1	0.0196(0.0018)
	λ_1	0.0197 (0.0026)		λ_2	0.0005(0.0003)
	λ_2	0.0264 (0.0022)		λ_3	0.0008 (0.0003)
Plakett	θ	0.4843 (0.1214)	Frank	θ	-1.5444(0.5357)
	λ_1	0.0014 (0.0001)		λ_1	0.0014 (0.0001)
	λ_2	0.0208 (0.0017)		λ_2	0.0208 (0.0017)

as AMH, Clayton, FGM, Frank, Galambos, Gumbel-Hugaard and Plakett. To select the best fitted model, we use $AIC = -2\log L + 2k$ (Akaikk, 1974), $BIC = -2\log L+k\log n$ (Schwarz, 1978), $AICC = -2\log L+2kn/(n-d-1)$ (Hurvich and Tsai, 1989), $HQIC = -2\log L + +2k\log\log n$ (Hannan and Quinn, 1979) and $CAIC = -2\log L+k(\log n+1)$ (Genest et al., 2008) where n is the number of observations, k is number of parameters in the model and L is the maximum value of the likelihood function.

The Bone Marrow Transplantation (BMT) Study

The BMT data set is available in Klein and Moeschberger (2003). 'T_2' represents disease free survival time (time to relapse, death or end of study) and 'T_P' represents time to return of platelets to normal levels. The spearman and kendall correlation coefficient of data are -0.2544 and -0.1806 with p-values 0.0027 and 0.0020, which implies that the dependence between 'T_2' and 'T_P' is statistically significant. The estimated parameters, obtained by the maximum likelihood estimation (MLE) method, with their standard errors are given in Table 10. Estimation results are given in Table 8 with standard errors in parentheses. Moreover, Table 9 reports AIC, AICC, BIC, HQIC, and CAIC values of several fitted models. Also, we use the procedure in Appendix A of Genest et al. (2008) to test that the our model is a good fit to the data or not. P-values are repoterd in Table 9. Results indicate that our proposed model fits better to the data compared to rival models.

Table 9. Comparison of some bivariate exponential distributions and bivariate copulas with exponential margins

Model	AIC	AICC	BIC	HQIC	CAIC	p-value
Our Model	3351.2	3351.5	3361.9	3355.95	3366.88	0.6931
Gumbel type II	3438.2	3438.4	3446.9	3440.77	3447.72	0.5100
Freund	3406.4	3406.7	3418.0	3409.83	3419.10	0.4432
Marshal-olkin A	3353.8	3354.0	3362.6	3356.37	3363.32	0.3812
ACBVE	3443.2	3443.4	3451.9	3445.77	3452.72	0.3861
FGM	3438.2	3438.4	3446.9	3440.77	3447.72	0.5114
Frank	3437.8	3438.0	3446.6	3440.37	3447.32	0.4100
Plakett	3438.2	3438.4	3447.0	3440.77	357.723	0.3977

So, an adequate bivariate distribution function for the data is given by

$$\begin{aligned} H(x,y) &= (1-e^{-0.0208x})(1-e^{-0.0014y})(1+0.8885e^{-0.0416x}e^{-0.0028y}) \\ &\quad \times(1-0.7617e^{-0.0208x}e^{-0.0014y}); \quad x>0, y>0 \end{aligned}$$

Note that the the spearman ans kendall correlations of the fitted model are -0.2002 and -0.1339 which are close to the sample correlations.

Myeloma Data

The data is taken from Krall, Uthoff and Harley (1975) in which 65 patients alkylation agents were treated. In the Myeloma data set, the variable "Time" represents the survival time in months from diagnosis and 'Age' represents age at diagnosis in years. As before, we estimate the parameters by maximum likelihood method that are given in Table 10. As seen in Table 11, results of our model has the lowest AIC, AICC, BIC, HQIC and CAIC, implying that the proposed model is the best one to fit the data. Note that we also checked Clayton, Galambos, Gumbel type III, Gumbel-Hugaard, Johnson-Kotz, Marshal-Olkin type B and Sarkar which did not fit the data well. Thus, to save the space, we have not reported them in Tables 10 and 11.

Concluding Remarks

In this paper, we proposed a class of copulas which was originated from our earlier work (Sharifonnasabi et al., 2018). We constructed a new bivariate distribution and studied its marginals together with the joint distribution

Table 10. Estimation results of various fitted models

Model	Param.	Est. (Std. Err.)	Model	Param.	Est. (Std. Err.)
Our Model	θ	-0.1381(1.1931)	Freund	α_1	0.0407(0.0054)
	δ	0.7133(1.7726)		α_2	0.0057 (0.0020)
	λ_1	0.0428(0.0103)		λ_1	0.04969(0.0176)
	λ_2	0.0167(0.0021)		λ_2	0.0227(0.0030)
Gumbel type II	θ	-0.3674(1.4152)	Plakett	θ	0.3941(0.3279)
	λ_1	0.0397(0.0091)		λ_1	0.0365(0.0068)
	λ_2	0.0166(0.0020)		λ_2	0.0166(0.0020)
Marshal-Olkin A	λ_1	0.0474(0.0059)	Frank	θ	-1.5915(2.5937)
	λ_2	0.0405(0.0084)		λ_1	0.0373(0.0086)
	λ_3	-0.025(0.0083)		λ_2	0.0166(0.0020)
ACVBE	λ_1	0.02435(0.0063)	AMH	θ	0.5322(1.3800)
	λ_2	0.0034(0.0016)		λ_1	0.0394(0.0068)
	λ_3	0.0194(0.0037)		λ_2	0.0166(0.0021)
FGM	θ	-0.3676(1.4153)			
	λ_1	0.0397(0.0091)			
	λ_2	0.0166(0.0021)			

Table 11. Comparison of some bivariate exponential distributions and bivariate copulas with exponential margins

Model	AIC	AICC	BIC	HQIC	CAIC
Our Model	1189.7	1190.37	1198.4	1193.13	1202.4
Gumbel type II	1211.7	1212.1	1218.2	1214.27	1221.22
Marshal-olkin A	1201.8	1202.2	1208.3	1204.37	1211.32
Freund	1195.1	1195.7	1203.8	1198.53	1207.80
ACBVE	1193.2	1193.6	1199.7	1195.77	1202.72
AMH	1211.7	1212.1	1218.2	1214.27	1221.22
FGM	1211.7	1212.1	1218.2	1214.27	1221.22
Frank	1211.6	1212.0	1218.1	1214.17	1221.12
Plakett	1211.4	1211.8	1217.9	1213.97	1220.92

of concomitants of its generalized ordered statistics. Moreover, we derived several explicit expressions for their moments and mgfs. It is shown that these expressions can be written as functions of moments and mgfs of the ordinary ordered statistics. We utilized these moments to obtain the minimum variance linear unbiased estimate of the location and scale parameters of the concomitants of ordered statistics from Burr and logistic distributions. In addition, in particular, we presented a bivariate exponential distribution whose univariate marginal distributions are exponential and investigated its different properties. One of the most important properties of this bivariate exponential distribution includes its hazard rate for parallel and series systems that are not monotone and thus this flexibility allows one to use this new bivariate model for different data sets. Results from the simulation studies show that as sample size increases, MSE decreases for all parameters and the bias decreases in most cases. Finally, we applied our model to two real-life data sets to show the potential of our new model in practical applications.

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