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A Quantile Approach to the Interval Shannon Entropy

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Abstract. In this paper, we introduce and study quantile version of the Shannon entropy function via doubly truncated (interval) lifetime, which includes the residual and past lifetimes as special case. We aim to study the use of proposed measure in characterization of distribution functions. Further, we describe a stochastic order and a weighted distribution based on this entropy and show their properties. Finally, some results have been obtained for some distributions such as Uniform, Exponential, Pareto I, Power function and Govindarajulu. Also by analysing a real data the subject has been illustrated.

Keywords. Shannon entropy, quantile function, generalized failure rate, quantile doubly truncated Shannon entropy.

MSC 2010: 94A17; 62E10.

1 Introduction

Let X be a non-negative absolutely continuous random variable representing the lifetime of a component, then if the component is still operating at time t, the residual lifetime is represented by random variable, $X_t = X - t|X \geqslant t$, and if the component is found to be 'down' at time t, the time elapsed from its failure to time t is denoted by random variable $X_t^* = t - X|X \leqslant t$ and has called inactivity or past lifetime.

Also due to generalizing the two previous situations if we know further information about lower or upper bounds of lifetime of component then we have doubly truncated (interval) random variables of forms $X_{t_1,t_2} = X - t_1|t_1 \leqslant X \leqslant t_2$ and $X^*_{t_1,t_2} = t_2 - X|t_1 \leqslant X \leqslant t_2$ where $(t_1, t_2) \in D^* = \{(t_1, t_2) : F(t_1) < F(t_2)\}$. The random variable X_{t_1,t_2} is used when the component is still working at time t_1 and we know that it will not operating greater than time t_2 and in dual case for random variable $X^*_{t_1,t_2}$ it is used when the component is not working at time t_1 and we know that the failure has accured in interval (t_1, t_2). For various results on doubly truncated random variable in reliability and information theory, we refer to Sankaran and Sunoj (2004), Khorashadizadeh et al. (2013), Kayal and Moharana (2016), Kundu (2017) and Kumar et al. (2019).

As it is well known in information theory, the Shannon entropy (Shannon, 1948) and its dynamic versions includes residual entropy (Ebrahimi, 1996), past entropy (Di Crescenzo and Longobardi, 2002) and interval entropy (Misagh and Yari, 2010, 2011, 2012) are the main measures of the uncertainty contained in random variables X, residual lifetime, past lifetime and doubly truncated lifetime which are respectively defined by

$$\begin{aligned} H_X &= -\int_0^\infty f(x)\log f(x)dx, \\ RE_X(t) &= -\int_t^\infty \frac{f(x)}{\overline{F}(t)}\log\frac{f(x)}{\overline{F}(t)}dx, \\ PE_X(t) &= -\int_0^t \frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}dx, \\ H_X(t_1,t_2) &= -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)}\log\frac{f(x)}{F(t_2)-F(t_1)}dx, \end{aligned} \qquad (1)$$

where $(t_1, t_2) \in D^*$ and $H_X(0,\infty)$ is the Shannon entropy and $H_X(t,\infty)$ is the residual entropy $(RE_X(t))$ and also $H_X(0,t)$ is the past entropy $(PE_X(t))$. Recently, many researchers have pay attention to use of quantile function (QF) instead of distribution function, which is very usefull in data analysis in applied statistics and is defined for any random variables by,

$$Q(p) = F^{-1}(p) = \inf\{x|F(x) \geqslant p\}, \quad 0 \leqslant p \leqslant 1.$$

Quantile functions with respect to distribution functions are less influenced by extreme observation. On the other hand, some important lifetime distributions in reliability are known by their quantile functions like Lambda distributions, Power-Pareto distribution and Govindarajulu distribution, so using quantile version of reliability and information measures is the best way to analysis the data for such distributions.

For more properties and usefulness of QF in reliability and information theories we refer readers to the books by Gilchrist (2000) and Nair et al. (2013) and the works by Midhu et al. (2013), Nanda et al. (2014), Sankaran et al. (2016), Belzunce and Martinez-Riquelme (2017), Kundu and Patra (2018), Sankaran and Sunoj (2017), Baratpour and Khammar (2018), Kumar (2018) and Kumar et al. (2019).

From now on, assume that F is continuous, so, $F(Q(u)) = FQ(u) = u$. Parzen (1979) have defined the density quantile function with $fQ(u) = f(Q(u))$ and quantile density function with $q(u) = \frac{\partial Q(u)}{\partial u}$, hence $q(u)fQ(u) = 1$.

From the perspective of quantile function, Sunoj and Sankaran (2012) introduced a quantile based Shannon entropy and residual entropy $RE_q(u)$ respectively by,

$$H_q = \int_0^1 \log q(u)du,$$

and

$$RE_q(u) = \log(1-u) + (1-u)^{-1} \int_u^1 \log q(p)dp. \tag{2}$$

They showed that, contrary to the residual entropy of form (1), $RE_q(u)$ can uniquely determine the quantile function. Also, Qiu (2018) have obtained more results of (2). Similarly, Sunoj et al. (2013) have studied the past entropy in terms of QF by,

$$PE_q(u) = \log(u) + u^{-1} \int_0^u \log q(p)dp. \tag{3}$$

In this paper we define the quantile version of the doubly truncated (interval) Shannon entropy and study some of it properties including characterization, stochastic order, weighted cases and real data analysis.

2 Interval Quantile based Shannon Entropy

In doubly truncation, Navarro and Ruiz (1996) have defined the generalized failure rate (GFR) by,

$$\begin{cases} h_1(t_1,t_2) = \frac{f(t_1)}{\overline{F}(t_1)-\overline{F}(t_2)}, \\ h_2(t_1,t_2) = \frac{f(t_2)}{F(t_2)-F(t_1)}, \end{cases}$$

where $(t_1,t_2) \in D^*$. So, in this section, in order to generalize the measures (2) and (3), we first define the quantile GFR by,

$$\begin{cases} \Lambda_1(u_1,u_2) = \frac{1}{(u_2-u_1)q(u_1)}, \\ \Lambda_2(u_1,u_2) = \frac{1}{(u_2-u_1)q(u_2)}, \end{cases}$$

where $(u_1,u_2) \in D = \{(u_1,u_2); Q(u_1) < Q(u_2)\}$.

Although given both $h_1(t_1,t_2)$ and $h_2(t_1,t_2)$ the distribution function can be characterized (Navarro and Ruiz, 1996), but given one of $\Lambda_1(u_1,u_2)$ or $\Lambda_2(u_1,u_2)$ the quantile density function is uniquely determined for fixed $u \in (0,1)$ by $q(p) = [(u-p)\Lambda_1(p,u)]^{-1}$, for $(p,u) \in D$.

In similar way of previous measures, the doubly truncated (interval) quantile Shannon entropy function is defined by,

$$\begin{aligned} H_q(u_1,u_2) &= -\int_{u_1}^{u_2} \frac{f(Q(p))}{u_2-u_1} \log \frac{f(Q(p))}{u_2-u_1} dQ(p) \\ &= \log(u_2-u_1) + \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log(q(p))dp \qquad (4) \\ &= 1 - \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log(\Lambda_1(p,u_2)dp \\ &= 1 - \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log(\Lambda_2(u_1,p)dp, \end{aligned}$$

where $(u_1,u_2) \in D$.

Definition 1. The random variable X is said to have decreasing (increasing) interval quantile entropy DIQE (IIQE) property in terms of u_i if and only if for any fixed u_j, $H_q(u_1,u_2)$ is decreasing (increasing) with respect to u_i, where $i,j = 1,2$, $i \neq j$ and $(u_1,u_2) \in D$.

Now, differentiating Eq.(4) with respect u_1, we get

$$\begin{aligned}
\frac{\partial H_q(u_1,u_2)}{\partial u_1} &= \frac{-1}{u_2-u_1}+\frac{1}{(u_2-u_1)^2}\int_{u_1}^{u_2}\log q(p)dp-\frac{1}{u_2-u_1}\log q(u_1)\\
&= \frac{1}{u_2-u_1}\left\{H_q(u_1,u_2)-\log(u_2-u_1)-\log q(u_1)-1\right\} \quad (5)\\
&= \frac{1}{u_2-u_1}\left\{H_q(u_1,u_2)+\log\Lambda_1(u_1,u_2)-1\right\}.
\end{aligned}$$

So, $H_q(u_1,u_2)$ is increasing with respect to u_1 if and only if, for all $(u_1,u_2)\in D$,

$$H_q(u_1,u_2)\geqslant \log(u_2-u_1)+\log q(u_1)+1,$$

or equivalently

$$H_q(u_1,u_2)\geqslant -\log\Lambda_1(u_1,u_2)+1.$$

Similarly differentiating with respect to u_2 we have,

$$\begin{aligned}
\frac{\partial H_q(u_1,u_2)}{\partial u_2} &= \frac{1}{u_2-u_1}-\frac{1}{(u_2-u_1)^2}\int_{u_1}^{u_2}\log q(p)dp+\frac{1}{u_2-u_1}\log q(u_2)\\
&= \frac{1}{u_2-u_1}\left\{-H_q(u_1,u_2)+\log(u_2-u_1)+\log q(u_2)+1\right\}\\
&= \frac{1}{u_2-u_1}\left\{-H_q(u_1,u_2)-\log\Lambda_2(u_1,u_2)+1\right\}.
\end{aligned}$$

Hence, $H_q(u_1,u_2)$ is increasing with respect to u_2 if and only if, for all $(u_1,u_2)\in D$,

$$H_q(u_1,u_2)\leqslant \log(u_2-u_1)+\log q(u_2)+1,$$

or equivalently

$$H_q(u_1,u_2)\leqslant -\log\Lambda_2(u_1,u_2)+1.$$

3 Examples

In this section the interval quantile Shannon entropy for some distributions are obtained.

- Exponential distribution
  If X have the Exponential distribution with parameter λ. Then $Q(p) = \frac{1}{\lambda}(-\log(1-p))$, $q(p) = \frac{1}{\lambda(1-p)}$ and the interval quantile Shannon entropy is,

$$\begin{aligned} H_q(u_1, u_2) &= \log(u_2 - u_1) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} \log(\lambda(1-p))dp \\ &= \log(u_2 - u_1) - \log\lambda - \frac{(1-u_1)}{(u_2-u_1)} \log(1-u_1) \\ &\quad -\frac{(u_2 - 1)}{(u_2 - u_1)} \log(1-u_2) + 1. \end{aligned}$$

  In special case of $u_1 = u$ and $u_2 = 1$ we have quantile residual entropy,

$$RE_q(u) = 1 - \log\lambda,$$

  and when $u_1 = 0$ and $u_2 = u$ we have quantile past entropy,

$$PE_q(u) = 1 - \log\lambda + \log u + (\frac{1-u}{u}) \log(1-u).$$

- Pareto I distribution
  Let X have the Pareto I distribution with $Q(p) = \sigma(1-p)^{-\frac{1}{\alpha}}$ and $q(p) = \frac{\sigma}{\alpha}(1-p)^{-1-\frac{1}{\alpha}}$. Also, we have

$$\begin{aligned} H_q(u_1, u_2) &= \log(u_2 - u_1) + \frac{1}{u_2 - u_1}\{\log(\frac{\sigma}{\alpha}(u_2 - u_1)) \\ &\quad + \int_{u_1}^{u_2} (-1 - \frac{1}{\alpha}) \log(1-p)dp \\ &= K + \log(u_2 - u_1) - \frac{(\alpha+1)}{\alpha(u_2 - u_1)} \log\frac{(1-u_1)^{1-u_1}}{(1-u_2)^{1-u_2}}, \end{aligned}$$

$$RE_q(u) = K - \log(1-u)^{\frac{1}{\alpha}},$$
$$PE_q(u) = K + \log u(1-u)^{(1+\frac{1}{\alpha})(\frac{1}{u}-1)},$$

where $K = \log\frac{\sigma}{\alpha} + \frac{\alpha+1}{\alpha}$.

- Uniform distribution
  Let X have the Uniform distribution with parameters (a,b) and $Q(p) = a+(b-a)p$ and $q(p) = b-a$. Then quantile doubly truncated Shannon entropy for this distribution is,

$$H_q(u_1,u_2) \quad = \quad \log(u_2-u_1) + \log(b-a).$$

In special case of $u_1 = u$ and $u_2 = 1$ we have quantile residual entropy,

$$RE_q(u) = \log(1-u) + \log(b-a),$$

and when $u_1 = 0$ and $u_2 = u$ we have quantile past entropy,

$$PE_q(u) = \log u + \log(b-a).$$

- Power function distribution
  Let X have the Power function distribution with parameter (α,β) and $Q(p) = \alpha\, p^{\frac{1}{\beta}}$ and $q(p) = \frac{\alpha}{\beta}\, p^{\frac{1}{\beta}-1}$. Then quantile doubly truncated Shannon entropy for this distribution is given by,

$$\begin{aligned} H_q(u_1,u_2) &= \log(u_2-u_1) + \frac{1}{u_2-u_1}\{\log(\frac{\alpha}{\beta})(u_2-u_1) \\ &\quad + \int_{u_1}^{u_2}(\frac{1}{\beta}-1)\log(p)dp\} \\ &= K + \log(u_2-u_1) - \frac{(\frac{1}{\beta}-1)(u_1)}{(u_2-u_1)}\log(u_1) \\ &\quad + \frac{(\frac{1}{\beta}-1)(u_2)}{(u_2-u_1)}\log(u_2), \end{aligned}$$

where $K = \log\frac{\alpha}{\beta} - \frac{1}{\beta} + 1$. In special case of $u_1 = u$ and $u_2 = 1$ we have quantile residual entropy,

$$RE_q(u) = K + \log(1-u) + (\frac{\beta-1}{\beta})\frac{u\log u}{(1-u)},$$

and when $u_1 = 0$ and $u_2 = u$ we have quantile past entropy,

$$PE_q(u) = K + \frac{1}{\beta}\log u.$$

- Govindarajulu distribution
  This distribution have introduced by Govindarajulu (1977) with quantile function of the form

$$Q(u) = \theta + \alpha\{(\beta+1)u^{\beta} - \beta u^{\beta+1}\}; \quad 0 \leqslant u \leqslant 1,\ \alpha,\beta > 0, \theta \in R, \quad (6)$$

which have not closed analytic form of distribution function. In this case we have,

$$\begin{aligned} H_q(u_1,u_2) &= K + \ln\left((u_2-u_1)\left(\frac{u_2^{u_2(\beta-1)}(1-u_2)^{u_2-1}}{u_1^{u_1(\beta-1)}(1-u_1)^{u_1-1}}\right)^{\frac{1}{u_2-u_1}}\right), \\ RE_q(u) &= K + 2\ln(1-u) - \frac{u(\beta-1)}{1-u}, \\ PE_q(u) &= K + \ln(u^{\beta}(1-u)^{1-\frac{1}{u}}), \end{aligned}$$

where $K = \ln(\alpha\beta(\beta+1)) - \beta$.

4 Characterization

The next theorem shows that $H_q(u_1,u_2)$ can determine the distribution uniquely.

***Theorem* 1.** *Let X be a non-negative absolutely continuous random variable then the quantile density function is uniquely determined for fixed $u \in (0,1)$ by interval quantile entropy via relation,*

$$q(p) = \exp\{1 - \log(u-p) + H_q(p,u) - (u-p)H'_q(p,u)\}; \quad (p,u) \in D.$$

Proof. The proof follows from (5).

Theorem 2. *If X is IIQE (DIQE) in terms of u_i, $i = 1, 2$ and if $\phi(.)$ is any non-negative, increasing and convex (concave) function, then $\phi(X)$ is also IIQE (DIQE) in terms of u_i, $i = 1, 2$.*

Proof. Let $g(t)$ be the probability density function of $T = \phi(X)$, so $g(t) = \frac{f(\phi^{-1}(t))}{\phi'(\phi^{-1}(t))}$ and hence, $g(Q_T(u)) = \frac{1}{q_T(u)} = \frac{fQ(u)}{\phi' Q(u)} = \frac{1}{q_X(u)\phi' Q(u)}$, then the interval quantile Shannon entropy of T, $H_q^T(u_1, u_2)$ is as follows,

$$\begin{aligned} H_q^T(u_1,u_2) &= \log(u_2-u_1) + \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log(q_T(p))dp \\ &= \log(u_2-u_1) + \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log(q_X(p)\phi'(Q(p)))dp \\ &= \log(u_2-u_1) + \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log(q_X(p))dp + \frac{1}{u_2-u_1}\int_{u_1}^{u_2} \log(\phi'(Q(p))dp \\ &= H_q^X(u_1,u_2) + E(\log(\phi'(X))|\, \phi^{-1}(u_1) < X < \phi^{-1}(u_2)), \end{aligned}$$

where $(u_1, u_2) \in D$ and $H_q^X(u_1, u_2)$ and $H_q^T(u_1, u_2)$ are the interval quantile Shannon entropy of X and T respectively. Now if X is IIQE and $\phi(x)$ is non-negative, increasing and convex then $\phi(x)$ is also IIQE. So the proof is completed.

It should be noted that all characterization problems of distribution functions discussed in literature are hold for quantile version of measures with transformation $x = Q(u)$. But, new useful characterizations can be obtianed in the quantile case. Nair et al. (2013) have obtained characterization results based on linear hazard quantile function and mean residual quantile function.

The following theorems characterize different distributions using $H_q(u_1, u_2)$, which their proofs are direct.

Theorem 3. *The non-negative random variable X have interval quantile Shannon entropy of the form,*

$$H_q(u_1,u_2) = \frac{1}{a(u_2-u_1)} \ln\left(\frac{b^b}{(a(u_2-u_1)+b)^{a(u_2-u_1)+b}}\right), \tag{7}$$

if and only if

$$\Lambda_i(u_1,u_2)=a(u_2-u_1)+b;\quad i=1,2, \tag{8}$$

where $(u_1,u_2)\in D, a,b>0.$

It can be seen that the Equations (7) and (8) are hold if and only if for fixed $u\in(0,1)$,

$$\begin{aligned} q(p) &= \frac{1}{a(u-p)^2+b(u-p)},\\ Q(p) &= \frac{1}{b}\ln\left(\frac{u(a(u-p)+b)}{(u-p)(au+b)}\right). \end{aligned}$$

***Corollary* 1.** *In special case of* $b=0$ *in Theorem 3, the lifetime random variable* X *have interval quantile Shannon entropy of the form* $H_q(u_1,u_2)=\ln\frac{1}{a(u_2-u_1)}$ *if and only if* $\Lambda_i(u_1,u_2)=a(u_2-u_1); i=1,2$ *and also,* $q(p)=\frac{1}{a(u-p)^2}$ *and* $Q(p)=\frac{p}{au(u-p)}$.

***Theorem* 4.** *The non-negative random variable* X *have interval quantile Shannon entropy of the form,*

$$H_q(p,u)=a\ln(u-p)+b,$$

if and only if for any fixed $u\in(0,1)$

$$Q(p)=K\left(u^a-(u-p)^a\right),$$

where $(p,u)\in D$ *and* $K=\frac{e^{a+b+1}}{a}$ *is constant.*

***Theorem* 5.** *A random variable* X *has quantile density function of the form*

$$q(p)=Kp^{\alpha}(u-p)^{-(\gamma+\alpha)}, \tag{9}$$

for any fixed $u\in(0,1)$ *and all* $(p,u)\in D$ *if and only if it satisfies the relationship*

$$H_q(p,u)=\log\left(Ku^{\frac{au}{u-p}}(u-p)^{-(\gamma+\alpha)+1}\right),$$

where α *and* γ *are real constants.*

Remark 1. The family of distributions (9) in special case of $u = 1$ contains several distributions such as Exponential ($\alpha = 0, \gamma = 1$), rescaled Beta ($\alpha = 0, \gamma > 1$), Pareto ($\alpha = 0, \gamma < 1$), Loglogistic ($\alpha = \lambda - 1, \gamma = 2$) and Govindarajulu ($\alpha = \beta - 1, \gamma = -\beta$).

5 Stochastic Order

In this section some stochastic order properties of interval quantile Shannon entropy are obtained.

Definition 2. The random variable X is said to have less IQE than Y if $H_q^Y(u_1, u_2) \geqslant H_q^X(u_1, u_2)$ for all $(u_1, u_2) \in D$, and we write $X \leqslant_{IQE} Y$.

For example if X and Y have Uniform distribution with parameters $(0, a)$ and $(0, b)$ respectively, and if $a < b$, then

$$H_q^X(u_1, u_2) = \log(u_2 - u_1) + \log(a) \leqslant \log(u_2 - u_1) + \log(b) = H_q^Y(u_1, u_2),$$

so, $X \leqslant_{IQE} Y$.

***Theorem* 6.** *Let $Z_1 = a_1 X + b_1$ and $Z_2 = a_2 Y + b_2$, where $a_1 > 0, a_2 > 0$ and $b_1 > 0, b_2 > 0$. If $X \leqslant_{IQE} Y$ and $a_1 \leqslant a_2$, then $Z_1 \leqslant_{IQE} Z_2$.*

Proof. Using (4), the quantile density function of the variable $Z = aX + b$, where $a > 0$ and $b > 0$, is of the form,

$$H_q^Z(u_1, u_2) = H_q^X(u_1, u_2) + \log(a),$$

which proved the theorem.

6 Weighted Case

Fisher (1934) has introduced the concept of weighted distribution, which has many applications in many areas of applied statistics. This concept is related to the case of when our observations are recorded with some weight function like $w(x) > 0$. For more properties of weighted distributions and its applications one can see the works by Navarro et al. (2006), Bartoszewicz (2009) and Gupta and Arnold (2016) and references therein. Let $w(x)$ be a non-negative function of x such that $E(w(X)) = \int_0^\infty w(x)f(x)dx$ is finite. Then the corresponding probability density and distribution functions weighted random variable X^w are given by

$$\begin{aligned} f^w(x) &= \frac{w(x)f(x)}{E(w(X))}, \\ F^w(x) &= \frac{E(w(X)|X \leqslant x)}{E(w(X))}F(x). \end{aligned}$$

When the weight function depends only on the length of the unit of interest (i.e. $w(x) = x$), the $X^w \equiv X^*$ is called length-biased or a size-biased random variable with probability density and distribution functions as follow,

$$\begin{aligned} f^*(x) &= \frac{xf(x)}{E(X)} \\ F^*(x) &= E(X)^{-1}\int_0^x tf(t)dt. \end{aligned}$$

The corresponding weighted density and distribution function based on quantile functions can be expressed in the following forms,

$$\begin{aligned} f^w(Q(u)) &= \frac{w(Q(u))f(Q(u))}{\mu^w}, \\ F^w(Q(u)) &= \frac{E\left(w(Q(U))|Q(U) \leqslant Q(u)\right)}{\mu^w}u, \end{aligned}$$

where

$$\mu^w = \int_0^1 w(Q(p))f(Q(p))dQ(p) = \int_0^1 w(p)dp < \infty.$$

Hence, it is easy to see that the weighted quantile density function is $q^w(u) = \frac{\mu^w\, q(u)}{w(Q(u))}$. So, the weighted interval quantile Shannon entropy can be defined by

$$\begin{aligned} H_q^w(u_1,u_2) &= \log(u_2-u_1)+\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log(q^w(p))dp \\ &= H_q(u_1,u_2)+\log\mu^w-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log w(Q(p))dp, \end{aligned}$$

and in speacial case of length-biased ($w(Q(p)) = Q(p)$), we have,

$$H_q^w(u_1,u_2)=H_q(u_1,u_2)+\log E_q(X)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\log Q(p)dp.$$

Theorem 7. *If X is IIQE (DIQE) in terms of u_i, $i=1,2$ and if $w(x)$ is any non-negative, increasing and convex (concave) function, then X^w is also IIQE (DIQE) in terms of u_i, $i=1,2$.*

Proof. The proof is similar to that of Theorem 2.

Corollary 2. *In case of length-biased ($w(x)=x$), the IIQE (DIQE) properties of X, implies the IIQE (DIQE) of X^*.*

7 Data analysis

The intended data include the production of gas oil (in cubic meters per day) in Iranian refineries from 1998 to 2014 (extracted from National Statistics Center database) as follows:

$$64731, 69545, 69945, 70879, 71923, 73154, 77037, 79215, 80473, 81549,$$
$$84957, 88702, 90951, 94677, 93595, 97689, 96016$$

Shokrani and Khorashadizadeh (2019) have shown that these data have Govindarajulu distribution of the form (6) with parameters $\theta = 66890.09, \alpha = 31052.52$ and $\beta = 2.26$. For this data, we have computed the quantile interval Shannon entropy as follows,

$$H_q(u_1,u_2)=7.79+\ln\left((u_2-u_1)\left(\frac{{u_2}^{1.26\,u_2}\,(1-u_2)^{u_2-1}}{{u_1}^{1.26\,u_1}\,(1-u_1)^{u_1-1}}\right)^{(u_2-u_1)^{-1}}\right).$$

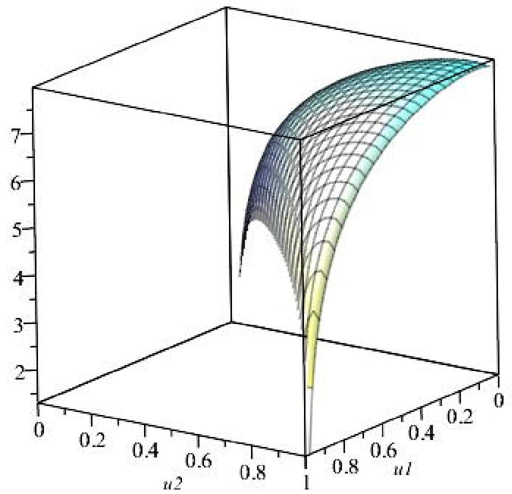

Figure 1. The quantile interval Shannon entropy of the production of gas oil data

The Figure 1 shows the values of $H_q(u_1, u_2)$ for different values of (u_1, u_2), which shows that the maxiumum uncertainty of production of gas oil data is 7.79.

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