J. Statist. Res. Iran **16** (2019): 101–120
DOI: ???

New Results on Stochastic Comparison of Series and Parallel Systems Comprising Heterogeneous Generalized Modified Weibull Components

Esmaeil Bashkar

Velayat University

Received: 3/25/2020 Approved: 10/30/2020

Abstract. In this work, we study various stochastic orderings of the smallest and largest order statistics arising from independent heterogeneous generalized modified Weibull (GMW) random variables. We also conduct stochastic comparison on the extreme order statistics from GMW samples with Archimedean copulas. The results established in this paper strengthen and generalize those known in the Balakrishnan et al. (2018).

Keywords. Majorization; order statistics; series and parallel systems; stochastic orders; generalized modified Weibull distribution.

MSC 2010: 60E15, 60E10.

1 Introduction

Series and parallel systems are two basic systems which play important roles in various applications in reliability engineering. An n-component system with series (parallel) structure fails (works) if at least one of the components of the system fails (works). Let $X_1, X_2, ..., X_n$ denote the lifetimes of n components that can be used to built up an n component system. If $X_{1:n} \leq \ldots \leq X_{n:n}$ denote the ordered lifetimes of the components then it is known that $X_{1:n}$ and $X_{n:n}$ correspond to the lifetimes of series and parallel systems, respectively. Reliability and stochastic properties of series and parallel systems have been considered by various researchers under different scenarios. For example, stochastic comparisons of the lifetimes of series and parallel systems, in the case of heterogeneous component lifetimes with Weibull distributions, are considered in Khaledi and Kochar (2006), Fang and Zhang (2013), Torrado (2015), Li and Li (2015), Torrado and Kochar (2015) and Fang and Balakrishnan (2016). Attempts have also been made by Fang and Zhang (2015) and Kundu and Chowdhury (2016) in the case of heterogeneous components with exponentiated Weibull (EW) distributions, and by Balakrishnan et al. (2014) in the case of heterogeneous components with generalized exponential (GE) distributions. For a recent review on the topic one can refer to Balakrishnan and Zhao (2013).

The aim of the present note is to compare the lifetimes of series and parallel systems with heterogeneous components where the component lifetimes are distributed as a four-parameter generalized modified Weibull (GMW) distribution with cumulative distribution function (cdf)

$$F(x) = (1 - \exp(-\alpha x^{\gamma} \exp(\lambda x))^{\beta} \quad (1)$$

where $\alpha > 0$ is a scale parameter, and $\gamma > 0$ and $\beta > 0$ are shape parameters. The parameter λ works as a factor of fragility in the survival of an individual as time increases. This model which has recently been studied by Carrasco et al. (2008), has both monotone and nonmonotone hazard rates. If a random variable X has GMW distribution in (1), then we write $X \sim \text{GMW}(\alpha, \gamma, \lambda, \beta)$. For $\lambda = 0$ and $\beta = 1$, we obtain the Weibull distribution and we have modified Weibull distributions when $\beta = 1$. For $\lambda = 0$ the model (1) reduces to the EW distribution constructed by Mudholkar and Srivastava (1993). Our established results extend some known results dealing with Weibull and EW distributions. Furthermore, the results obtained here strengthen and generalize those known in the Balakrishnan et al. (2018).

The rest of the paper is organized as follows: In Section 2, we give the required definitions and some useful lemmas which are used throughout the paper. In Section 3, we discuss stochastic comparisons of the lifetimes of series and parallel systems with independent heterogeneous GMW components in terms of the usual stochastic order, hazard rate order and the reversed hazard rate order. Section 4 discusses some results on series systems with dependent components where the structure of dependency is expressed according to the Archimedean copula. Throughout this paper, we use the notations $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_{++} = (0, +\infty)$, $\mathbb{D}_+ = \{(x_1, x_2, \ldots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n > 0\}$, $\mathbb{E}_+ = \{(x_1, x_2, \ldots, x_n) : 0 < x_1 \leq x_2 \leq \cdots \leq x_n\}$ and use the term 'increasing' for nondecreasing and 'decreasing' for nonincreasing.

2 Preliminaries

In this section, we review some definitions and well-known notions of stochastic orders and majorization concepts and lemmas that are useful for proving our main results. Let X and Y be two non-negative continuous random variables with distribution functions F and G, density functions f and g, and the survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. Denote also the reverse hazard rate functions $r_F = \frac{f}{F}$ and $r_F = \frac{f}{F}$, hazard rate functions of X and Y by $h_F = \frac{f}{\bar{F}}$ and $h_G = \frac{g}{\bar{G}}$, respectively. The following definition introduces some well-known orders that compare the magnitude of two random variables.

Definition 1. X is said to be smaller than Y in the

(i) stochastic order, denoted by $X \leq_{\text{st}} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x;

(ii) hazard rate order, denoted by $X \leq_{\text{hr}} Y$, if $h_F(x) \geq h_G(x)$ for all x.

(iii) reversed hazard rate order, denoted by $X \leq_{rh} Y$, if $r_F(x) \leq r_G(x)$ for all x.

Note that the hazard rate order, and the reversed hazard rate order implies the usual stochastic order. For more on these stochastic orders one may refer to Shaked and Shanthikumar (2007) and Li and Li (2013).

A real function ϕ is said to be n-monotone on $(a, b) \subseteq (-\infty, +\infty)$ if $(-1)^{n-2}\phi^{(n-2)}$ is decreasing and convex in (a, b) and $(-1)^k \phi^{(k)}(x) \geq 0$ for all $x \in (a, b), k = 0, 1, \ldots, n-2$, where $\phi^{(i)}(.)$ is the ith derivative of $\phi(.)$. For a n-monotone ($n \geq 2$) function $\phi : [0, +\infty) \longrightarrow [0, 1]$ with $\phi(0) = 1$ and $\lim_{x \to +\infty} \phi(x) = 0$, let $\psi = \phi^{-1}$ be the pseudo-inverse, then

$$C_\phi(u_1, \ldots, u_n) = \phi(\psi(u_1) + \ldots + \psi(u_n)), \quad \text{for all}\, u_i \in [0, 1], i = 1, \ldots, n,$$

is called an Archimedean copula with the generator ϕ. Archimedean copulas cover a wide range of dependence structures including the independence copula with the generator $\phi(t) = e^{-t}$. For more detail on Archimedean copula, see, McNeil and Neslehova (2009) and Nelsen (2006).

The notion of majorization is essential for the understanding of the stochastic inequalities for comparing order statistics. In the sequel, we use the notation $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ to denote the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, \dots, x_n)$.

Definition 2. The vector $\boldsymbol{x}$ is said to be

(i) weakly submajorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \preceq_{\mathrm{w}} \boldsymbol{y}$) if $\sum_{i=j}^{n} x_{(i)} \leq \sum_{i=j}^{n} y_{(i)}$ for all $j = 1, \dots, n$,

(ii) weakly supermajorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{\mathrm{w}}{\preceq} \boldsymbol{y}$) if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \dots, n$,

(iii) majorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y}$) if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \dots, n-1$.

Bon and Păltănea (1999) introduced the order of p-larger.

Definition 3. A vector $\boldsymbol{x}$ in $\mathbb{R}_+^n$ is said to be p-larger than another vector $\boldsymbol{y}$ in $\mathbb{R}_+^n$ (denoted by $\boldsymbol{x} \overset{\mathrm{p}}{\succeq} \boldsymbol{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, \qquad \text{for } j = 1, \dots, n.$$

Definition 4. A real valued function φ defined on a set $A \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on A if

$$\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y} \quad \text{on} \quad A \Longrightarrow \varphi(\boldsymbol{x}) \leq (\geq) \varphi(\boldsymbol{y}).$$

For more details on the above partial orders of real vectors, Schur-convexity and Schur-concavity we refer readers to Marshall et al. (2011). Let us introduce the following lemmas which will be used in the next sections to prove the results.

***Lemma* 1** (Marshall et al. (2011), Theorem 3.A.4)**.** *Suppose $\mathbb{I} \subset \mathbb{R}$ is an open interval and $\Phi : \mathbb{I}^n \longrightarrow \mathbb{R}_+$ is continuously differentiable. Necessary and sufficient conditions for Φ to be Schur-convex (Schur-concave) on $\mathbb{I}^n$ are*

(i) Φ is symmetric on $\mathbb{I}^n$,

(ii) for $i \neq j$ *and all* $z \in \mathbb{I}^n$,

$$(z_i - z_j)\left(\frac{\partial \Phi(z)}{\partial z_i} - \frac{\partial \Phi(z)}{\partial z_j}\right) \geq (\leq) 0,$$

where $\frac{\partial \Phi(z)}{\partial z_i}$ *denotes the partial derivative of* Φ *with respect to its* i*-th argument.*

Lemma* 2** (Marshall et al. (2011), Theorem 3.A.8).*** *For a function* l *on* $A \in \mathbb{R}^n$, $\boldsymbol{x} \preceq_{\mathrm{w}} (\overset{\mathrm{w}}{\preceq})\boldsymbol{y}$ *implies* $l(\boldsymbol{x}) \leq l(\boldsymbol{y})$ *if and only if it is increasing (decreasing) and Schur-convex on* A.

Lemma* 3*. *The function* $g : (0, \infty) \times (0, 1) \longrightarrow (0, \infty)$ *defined as*

$$g(\beta, x) = \frac{\beta x^{\beta - 1}}{1 - x^{\beta}}$$

is a decreasing function of β.

Proof. It is easy to see that $g(\beta, x) = \dfrac{1}{x} \frac{\beta x^{\beta}}{1 - x^{\beta}}$ and

$$\frac{\partial g(\beta, x)}{\partial \beta} \overset{\text{sgn}}{=} (x^{\beta} + x^{\beta} \log(x)\beta) \times (1 - x^{\beta}) + \beta x^{2\beta} \log(x) =$$

$$x^{\beta}(1 + \beta \log(x) - x^{\beta}) =: f(\beta).$$

Then, it follows that

$$\frac{\partial f(\beta)}{\partial \beta} = \log(x)(1 - x^{\beta}) < 0.$$

Therefore, $f(\beta) < f(0) = 0$, and so $\dfrac{\partial g(\beta, x)}{\partial \beta} < 0$. The assertion then follows. □

3 Main Results

3.1 Independent Random Variables

In this section, we provide some new comparison results on the lifetimes of series and parallel systems arising from independent heterogeneous GMW random variables.

The following results discuss the variability ordering of lifetimes of parallel systems with independent heterogeneous GMW components in terms of the reversed hazard rate order.

Theorem 1. *Let $X_1, \ldots, X_n$ be a set of independent random variables with $X_i \sim \mathrm{GMW}(\alpha_i, \gamma, \lambda, \beta)$, $i = 1, \ldots, n$. Further, let $X_1^*, \ldots, X_n^*$ be another set of independent random variables with $X_i^* \sim \mathrm{GMW}(\alpha_i^*, \gamma, \lambda, \beta)$, $i = 1, \ldots, n$. If $(\alpha_1, \ldots, \alpha_n) \overset{\mathrm{w}}{\preceq} (\alpha_1^*, \ldots, \alpha_n^*)$, then $X_{n:n} \leq_{\mathrm{rh}} X_{n:n}^*$.*

Proof. Let us consider a fixed $x > 0$, then the reversed hazard rate of $X_{n:n}$ is given by

$$\tilde{r}_{X_{n:n}}(x) = \frac{\beta(\lambda x + \gamma)}{x} \sum_{i=1}^{n} \frac{u_i}{e^{u_i} - 1}$$

where $u_i = \alpha_i x^{\gamma} e^{\lambda x}$. From Lemma 2, the proof follows if we prove that, for each $x > 0$, $\tilde{r}_{X_{n:n}}(x)$ is Schur-convex and decreasing in α_i's. For the proof of the first part, see the work of Balakrishnan et al. (2018). The proof of the second part is easy. Thus, it is omitted for the sake of conciseness. □

Note that $(\alpha_1, \ldots, \alpha_n) \overset{\mathrm{m}}{\preceq} (\alpha_1^*, \ldots, \alpha_n^*)$ implies $(\alpha_1, \ldots, \alpha_n) \overset{\mathrm{w}}{\preceq} (\alpha_1^*, \ldots, \alpha_n^*)$, Theorem 1 substantially improves Theorem 1 of Balakrishnan et al. (2018).

Example 1. Let (X_1, X_2, X_3) $((X_1^*, X_2^*, X_3^*))$ be a vector of independent heterogeneous GMW random variables. Set $\gamma = 2, \lambda = 1.2, \beta = 0.5$, $(\alpha_1, \alpha_2, \alpha_3) = (0.1, 4, 6)$ and $(\alpha_1^*, \alpha_2^*, \alpha_3^*) = (0.1, 1, 8)$. Obviously $(\alpha_1, \alpha_2, \alpha_3) \overset{\mathrm{w}}{\preceq} (\alpha_1^*, \alpha_2^*, \alpha_3^*)$, and $X_{3:3}$ and $X_{3:3}^*$ are ordered in the reversed hazard rate order, as can be seen in Fig. 1.

We now generalize Theorem 2 to a wider range of the shape parameters as follows.

Theorem 2. *Let the conditions of Theorem 1 be met. Then, for any $\gamma, \lambda, \beta > 0$, if $\alpha_i \leq \alpha_i^*$, $i = 1, \ldots, n$, we have $X_{n:n} \geq_{\mathrm{rh}} X_{n:n}^*$.*

Proof. Using the definition of the reversed hazard rate order and the fact that $\tilde{r}_{X_{n:n}}(x)$ is decreasing in each α_i, the required result follows readily. □

Theorem 3. *For $i = 1, \ldots, n$, let X_i and X_i^* be two sets of mutually independent random variables with $X_i \sim \mathrm{GMW}(\alpha_i, \gamma, \lambda, \beta_i)$ and $X_i^* \sim \mathrm{GMW}(\alpha_i^*, \gamma, \lambda, \beta_i)$. Further, suppose that $\boldsymbol{\beta} \in \mathbb{E}_+$ and $\boldsymbol{\alpha}, \boldsymbol{\alpha}^* \in \mathbb{D}_+$. Then for all $\gamma, \lambda > 0$, $(\alpha_1, \ldots, \alpha_n) \overset{\mathrm{w}}{\preceq} (\alpha_1^*, \ldots, \alpha_n^*)$ implies $X_{n:n} \leq_{\mathrm{rh}} X_{n:n}^*$*

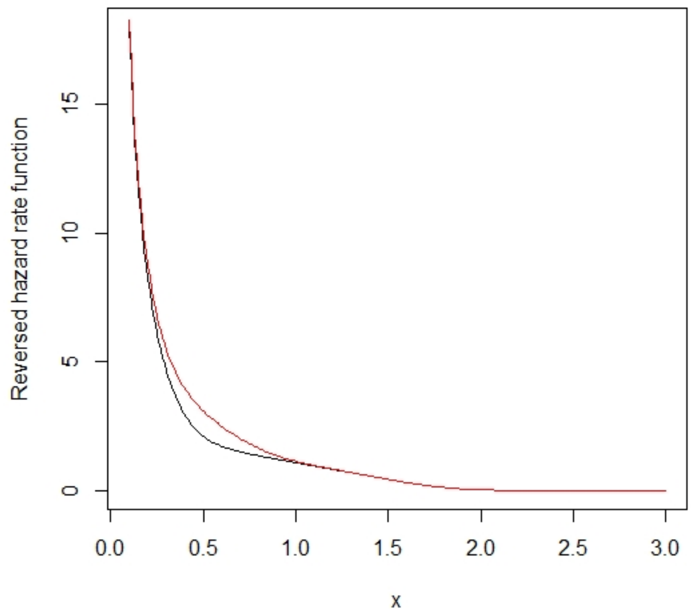

Figure 1. Plot of the reversed hazard rate functions of $X_{3:3}$ (black line) and $X^*_{3:3}$ (red line)

Proof. The reversed hazard rate functions of $X_{n:n}$ is given by

$$\tilde{r}_{X_{n:n}}(x) = \frac{(\lambda x + \gamma)}{x} \sum_{i=1}^{n} \frac{\beta_i \alpha_i x^\gamma e^{\lambda x}}{e^{\alpha_i x^\gamma e^{\lambda x}} - 1}.$$

Let $g_i(y) = \dfrac{\beta_i y x^\gamma e^{\lambda x}}{e^{y x^\gamma e^{\lambda x}} - 1} = \beta_i h(y x^\gamma e^{\lambda x})$, where, by Lemma 8 of Balakrishnan et al. (2018), $h(y) = \dfrac{y}{e^y - 1}$ is convex and decreasing in $y > 0$, which implies that for any two real numbers $a \geq b$, $g'_i(a) \geq g'_i(b)$. So, by Proposition H.2 of Marshall et al. (2011) , $\tilde{r}_{X_{n:n}}(x)$ is Schur convex. Thus, the result follows from Lemma 8 of Balakrishnan et al. (2018) and lemma 2. □

Theorem 4. *For $i = 1, \ldots, n$, let X_i and X^*_i be two sets of mutually independent random variables with $X_i \sim \mathrm{GMW}(\alpha_i, \gamma, \lambda, \beta_i)$ and $X^*_i \sim \mathrm{GMW}(\alpha^*_i, \gamma, \lambda, \beta^*_i)$.*

Then, for any $\gamma, \lambda > 0$, if $\alpha_i \geq \alpha_i^*$ and $\beta_i \leq \beta_i^*$, $i = 1, \ldots, n$, we have $X_{n:n} \leq_{\mathrm{rh}} X_{n:n}^*$.

Proof. Using the definition of the reversed hazard rate order and the fact that $\tilde{r}_{X_{n:n}}(x)$ is decreasing in each α_i and increasing in each β_i, the required result follows readily. □

Theorem 5. *For $i = 1, \ldots, n$, let X_i and X_i^* be two sets of mutually independent random variables with $X_i \sim \mathrm{GMW}(\alpha_i, \gamma, \lambda, \beta_i)$ and $X_i^* \sim \mathrm{GMW}(\alpha_i, \gamma, \lambda, \beta_i^*)$. If $(\beta_1, \ldots, \beta_n) \overset{\mathrm{m}}{\preceq} (\beta_1^*, \ldots, \beta_n^*)$ and (i) $\boldsymbol{\beta}, \boldsymbol{\beta^*} \in \mathbb{D}_+$ and $\boldsymbol{\alpha} \in \mathbb{E}_+$, then $X_{n:n} \leq_{\mathrm{rh}} X_{n:n}^*$; (ii) $\boldsymbol{\beta}, \boldsymbol{\beta^*}, \boldsymbol{\alpha} \in \mathbb{D}_+$, then $X_{n:n} \geq_{\mathrm{rh}} X_{n:n}^*$*

Proof. Assuming $g(x_i) = x_i$, $u_i = \dfrac{\alpha_i x^\gamma e^{\lambda x}}{e^{\alpha_i x^\gamma e^{\lambda x}} - 1}$ and

$$\varphi(\boldsymbol{\beta}) = \frac{(\lambda x + \gamma)}{x} \sum_{i=1}^{n} u_i g(\beta_i)$$

and by noting the fact that $u_i \in \mathbb{D}_+(u_i \in \mathbb{E}_+)$ whenever $\alpha_i \in \mathbb{E}_+(\alpha_i \in \mathbb{D}_+)$ (by Lemma 8 of Balakrishnan et al. (2018)) and using Theorem 3.1 and Theorem 3.2 of Kundu et al. (2016) it can be proved that $\tilde{r}_{X_{n:n}}(x)$ is Schur convex under condition (i), and Schur concave under condition (ii). This proves the result. □

The following theorem follows from Lemma 2 and Theorem 5.

Theorem 6. *For $i = 1, \ldots, n$, let X_i and X_i^* be two sets of mutually independent random variables with $X_i \sim \mathrm{GMW}(\alpha_i, \gamma, \lambda, \beta_i)$ and $X_i^* \sim \mathrm{GMW}(\alpha_i, \gamma, \lambda, \beta_i^*)$. (i) If $\boldsymbol{\beta}, \boldsymbol{\beta^*} \in \mathbb{D}_+$ and $\boldsymbol{\alpha} \in \mathbb{E}_+$, then $(\beta_1, \ldots, \beta_n) \preceq_w (\beta_1^*, \ldots, \beta_n^*)$ implies $X_{n:n} \leq_{\mathrm{rh}} X_{n:n}^*$; (ii) If $\boldsymbol{\beta}, \boldsymbol{\beta^*}, \boldsymbol{\alpha} \in \mathbb{D}_+$, then $(\beta_1, \ldots, \beta_n) \overset{\mathrm{w}}{\preceq} (\beta_1^*, \ldots, \beta_n^*)$ implies $X_{n:n} \geq_{\mathrm{rh}} X_{n:n}^*$*

The following result considers the comparison of the lifetimes of two series systems in terms of the usual stochastic order when the shape parameters (α) of the distribution of component lifetimes of a system is weakly majorized by the shape parameters (α^*) of the other system.

Theorem 7. *Let $X_1, \ldots, X_n$ be a set of independent random variables with $X_i \sim \mathrm{GMW}(\alpha_i, \gamma, \lambda, \beta)$, $i = 1, \ldots, n$. Further, let $X_1^*, \ldots, X_n^*$ be another set* *of independent random variables with* $X_i^* \sim \mathrm{GMW}(\alpha_i^*, \gamma, \lambda, \beta)$*,* $i = 1, \ldots, n$*. Then, (i) for* $(\alpha_1, \ldots, \alpha_n) \preceq_w (\alpha_1^*, \ldots, \alpha_n^*)$*, we have* $X_{1:n} \geq_{\mathrm{st}} X_{1:n}^*$ *if* $\beta \geq 1$*; (ii)* $(\alpha_1, \ldots, \alpha_n) \overset{\mathrm{w}}{\preceq} (\alpha_1^*, \ldots, \alpha_n^*)$*, we have* $X_{1:n} \leq_{st} X_{1:n}^*$ *if* $0 < \beta \leq 1$*.*

Proof. To prove the first part (second part), it is sufficient to show that, for $\beta \geq 1$ ($0 < \beta \leq 1$), the survival function of $X_{1:n}$ is Schur-concave (Schur-convex) in $(\alpha_1, \ldots, \alpha_n)$ and is decreasing in $\alpha_i, i = 1, \ldots, n$.. The survival function of $X_{1:n}$ is given by

$$\bar{F}_{X_{1:n}}(x) = \prod_{i=1}^{n}(1 - (1 - \exp(-\alpha_i x^\gamma \exp(\lambda x)))^\beta), x > 0. \tag{2}$$

Differentiating (2) partially with respect to $\alpha_i, i = 1, \ldots, n$, we get

$$\frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \alpha_i} = -x^\gamma \exp(\lambda x) \bar{F}_{X_{1:n}}(x) \frac{\beta \exp(-\alpha_i x^\gamma \exp(\lambda x))}{1 - (1 - \exp(-\alpha_i x^\gamma \exp(\lambda x)))} \leq 0. \tag{3}$$

Thus, the survival function of $X_{1:n}$ given by (2) is decreasing in $\alpha_i, i = 1, \ldots, n$. To prove its Schur-concavity (Schur-convexity) see Theorem 2 of Balakrishnan et al. (2018). □

Note that $(\alpha_1, \ldots, \alpha_n) \overset{\mathrm{m}}{\preceq} (\alpha_1^*, \ldots, \alpha_n^*)$ implies both $(\alpha_1, \ldots, \alpha_n) \preceq_{\mathrm{w}} (\alpha_1^*, \ldots, \alpha_n^*)$ and $(\alpha_1, \ldots, \alpha_n) \overset{\mathrm{w}}{\preceq} (\alpha_1^*, \ldots, \alpha_n^*)$, Theorem 7 substantially improves Theorem 2 of Balakrishnan et al. (2018).

Example 2. (i) Let (X_1, X_2, X_3) $((X_1^*, X_2^*, X_3^*))$ be a vector of independent heterogeneous GMW random variables. Set $\gamma = 1.3, \lambda = 1.6, \beta = 1.3$, $(\alpha_1, \alpha_2, \alpha_3) = (5, 2, 0.2)$ and $(\alpha_1^*, \alpha_2^*, \alpha_3^*) = (8, 3, 0.3)$. Obviously $(\alpha_1, \alpha_2, \alpha_3) \preceq_w (\alpha_1^*, \alpha_2^*, \alpha_3^*)$, and $X_{1:3} \geq_{st} X_{1:3}^*$, as can be seen in Fig. 2 (A). (ii)Let (X_1, X_2, X_3) $((X_1^*, X_2^*, X_3^*))$ be a vector of independent heterogeneous GMW random variables. Set $\gamma = 1.3, \lambda = 1.6, \beta = 0.3$, $(\alpha_1, \alpha_2, \alpha_3) = (0.2, 4, 9)$ and $(\alpha_1^*, \alpha_2^*, \alpha_3^*) = (0.1, 1, 6)$. Obviously $(\alpha_1, \alpha_2, \alpha_3) \overset{\mathrm{w}}{\preceq} (\alpha_1^*, \alpha_2^*, \alpha_3^*)$, and $X_{1:3} \leq_{st} X_{1:3}^*$, as can be seen in Fig. 2 (B).

Naturally, one may wonder whether the following statement is actually also true: For $0 < \beta \leq 1$, $(\alpha_1, \ldots, \alpha_n) \overset{\mathrm{p}}{\preceq} (\alpha_1^*, \ldots, \alpha_n^*)$ gives rise to the usual stochastic order between $X_{1:n}$ and $X_{1:n}^*$. The following example gives negative answer to these two conjecture.

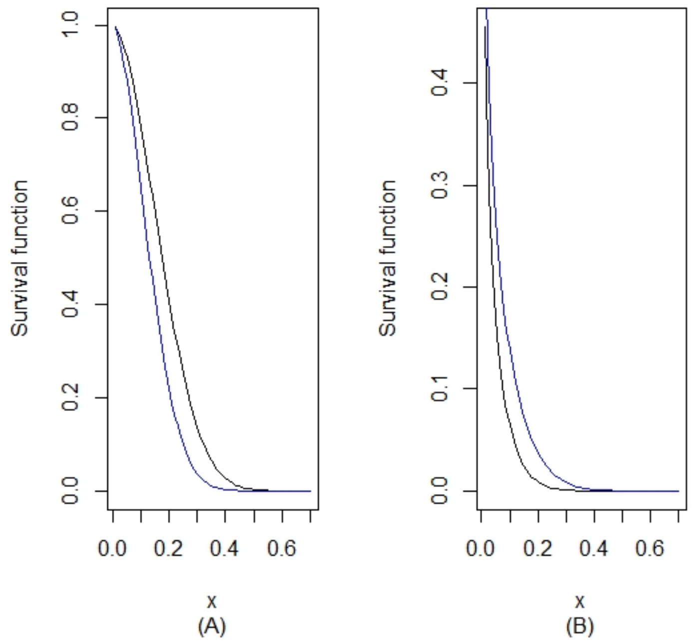

Figure 2. Plot of the survival functions of $X_{1:3}$ (black line) and $X^*_{1:3}$ (blue line)

Example 3. Let (X_1, X_2) $((X^*_1, X^*_2))$ be a vector of independent heterogeneous GMW random variables. Set $\beta = 0.5, \gamma = 2$ and $\lambda = 3$. For $(\alpha_1, \alpha_2) = (2, 3) \overset{\text{p}}{\preceq} (1, 5.5) = (\alpha^*_1, \alpha^*_2)$, $X_{1:2} \geq_{\text{st}} X^*_{1:2}$, as can be seen in Fig. 3 (A); however, for $(\alpha_1, \alpha_2) = (1.1, 6) \overset{\text{p}}{\preceq} (1, 2.25) = (\alpha^*_1, \alpha^*_2)$, $X_{1:2} \leq_{\text{st}} X^*_{1:2}$, as can be seen in Fig. 3 (B) So, $(\alpha_1, \alpha_2) \overset{\text{p}}{\preceq} (\alpha^*_1, \alpha^*_2)$ implies neither $X_{1:2} \leq_{\text{st}} X^*_{1:2}$ nor $X_{1:2} \geq_{\text{st}} X^*_{1:2}$ for $0 < \beta \leq 1$.

Now, we discuss stochastic comparison between the lifetimes of two series systems in the sense of the hazard rate order.

Theorem 8. *Let $X_1, \ldots, X_n$ be a set of independent random variables with $X_i \sim \text{GMW}(\alpha, \gamma, \lambda, \beta_i)$, $i = 1, \ldots, n$. Further, let $X^*_1, \ldots, X^*_n$ be another set of independent random variables with $X^*_i \sim \text{GMW}(\alpha, \gamma, \lambda, \beta^*_i)$, $i = 1, \ldots, n$. If $(\beta_1, \ldots, \beta_n) \overset{\text{w}}{\succeq} \beta^*_1, \ldots, \beta^*_n)$, then $X_{1:n} \leq_{\text{hr}} X^*_{1:n}$.*

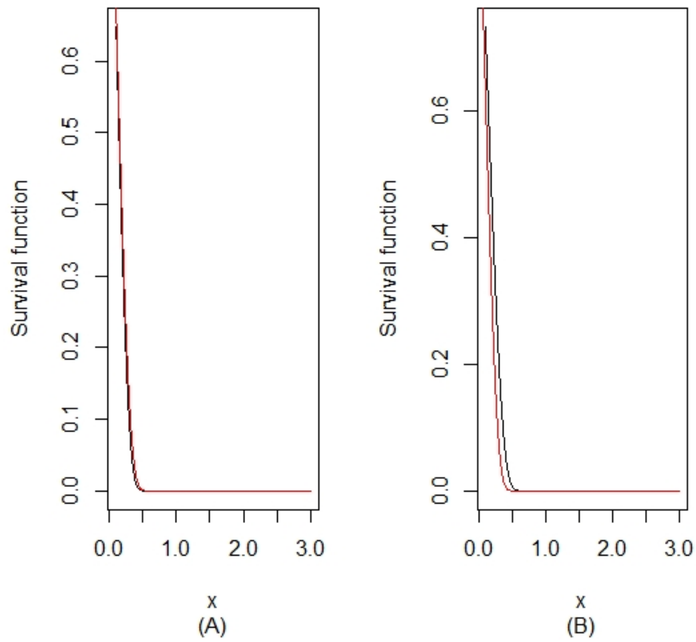

Figure 3. Plot of the survival functions of $X_{1:2}$ (red line) and $X^*_{1:3}$ (black line)

Proof. The hazard rate of $X_{1:n}$ can be obtained as

$$h_{X_{1:n}}(x) = (\alpha x^{\gamma} \exp(\lambda x))(\lambda + \frac{\gamma}{x})(\exp(-\alpha x^{\gamma} \exp(\lambda x)))\times$$

$$\sum_{i=1}^{n} g(\beta_i, 1 - \exp(-\alpha x^{\gamma} \exp(\lambda x))).$$

where g is given as in Lemma 3. Using Lemma 2, it is enough to show that the function $h_{X_{1:n}}(x)$ is Schur-convex and decreasing in β_i's. The partial derivatives of $h_{X_{1:n}}(x)$ with respect to β_i are

$$\frac{\partial h_{X_{1:n}}(x)}{\partial \beta_i} =$$

$$(\alpha x^\gamma \exp(\lambda x))(\lambda + \frac{\gamma}{x})(\exp(-\alpha x^\gamma \exp(\lambda x)))\frac{\partial g(\beta_i, 1-\exp(-\alpha x^\gamma \exp(\lambda x)))}{\partial \beta_i}. \tag{4}$$

Furthermore, from Lemma 3, $g(\beta, x)$ is a decreasing function of β. Using this, we obtain $\frac{\partial h_{X_{1:n}}(x)}{\partial \beta_i} \leq 0$. Thus, the hazard rate of $X_{1:n}$ is decreasing in $\beta_i, i = 1, \ldots, n$. To prove its Schur-convexity see Theorem 3 of Balakrishnan et al. (2018). □

The following result considers the comparison of the lifetimes of parallel systems in terms of the usual stochastic order with respect to the scale parameter λ.

***Theorem* 9.** *Let $X_1, \ldots, X_n$ be a set of independent random variables with $X_i \sim \mathrm{GMW}(\alpha, \gamma, \lambda_i, \beta)$, $i = 1, \ldots, n$. Further, let $X_1^*, \ldots, X_n^*$ be another set of independent random variables with $X_i^* \sim \mathrm{GMW}(\alpha, \gamma, \lambda_i^*, \beta)$, $i = 1, \ldots, n$. Then, for $(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{w}}{\preceq} (\lambda_1^*, \ldots, \lambda_n^*)$, we have $X_{n:n} \leq_{\mathrm{st}} X_{n:n}^*$.*

Proof. The cdf of $X_{n:n}$ is given by

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n}(1 - \exp(-\alpha x^\gamma \exp(\lambda_i x))^\beta), x > 0. \tag{5}$$

Using Lemma 2, it is enough to show that the function $F_{X_{n:n}}(x)$ is Schur-concave and increasing in λ_i's. Differentiating (5) partially with respect to $\lambda_i, i = 1, \ldots, n$, we get

$$\frac{\partial F_{X_{n:n}}(x)}{\partial \lambda_i} = x\beta F_{X_{n:n}}(x)\frac{\alpha x^\gamma \exp(\lambda_i x)}{\exp(\alpha x^\gamma \exp(\lambda_i x)) - 1} \geq 0. \tag{6}$$

Thus, the cdf of $X_{n:n}$ is increasing in $\lambda_i, i = 1, \ldots, n$. To prove its Schur-concavity see Theorem 4 of Balakrishnan et al. (2018). □

The next result considers the comparison of the lifetimes of series systems in terms of the usual stochastic order with respect to the scale parameter λ.

***Theorem* 10.** *Let $X_1, \ldots, X_n$ be a set of independent random variables with $X_i \sim \mathrm{GMW}(\alpha, \gamma, \lambda_i, \beta)$, $i = 1, \ldots, n$. Further, let $X_1^*, \ldots, X_n^*$ be another set of independent random variables with $X_i^* \sim \mathrm{GMW}(\alpha, \gamma, \lambda_i^*, \beta)$, $i = 1, \ldots, n$. Then, for $(\lambda_1, \ldots, \lambda_n) \preceq_{\mathrm{w}} (\lambda_1^*, \ldots, \lambda_n^*)$, we have $X_{1:n} \geq_{\mathrm{st}} X_{1:n}^*$ if $\beta \geq 1$.*

Proof. The survival function of $X_{1:n}$ is given by

$$\bar{F}_{X_{1:n}}(x) = \prod_{i=1}^{n}(1 - (1 - \exp(-\alpha x^{\gamma} \exp(\lambda_i x)))^{\beta}), x > 0. \quad (7)$$

Its easy to see that $\bar{F}_{X_{1:n}}(x)$ is decreasing in $\lambda_i, i = 1, \ldots, n$. To prove its Schur-concavity see Theorem 4 of Balakrishnan et al. (2018). □

3.2 Interdependent Variables with Archimedean Copulas

Recently, some authors paid their attention to comparing order statistics of dependent samples. For instance, see, Bashkar et al. (2017), Li and Li (2015), Li and Fang (2015) and Fang et al. (2015). In this section, we derive some new results on the usual stochastic order between the minimum of order statistics of two heterogeneous random vectors with the dependent components having Archimedean copula structure with GMW marginals. Before going into the details, let us recall one important general family of distributions. We say that random variable X belongs to the Exponentiated Scale (ES) family of distributions if $X \sim H(x) = [G(\lambda x)]^{\alpha}$, where $\alpha, \lambda > 0$ and G is called the baseline distribution function which we assume that is absolutely continuous. In the sequel, we denote this family by $\mathrm{ES}(\alpha, \lambda)$. Bashkar et al. (2017) proved the following general result.

***Theorem* 11.** *Suppose for* $i = 1, ..., n$, $X_i \sim \mathrm{ES}(\alpha_i, \lambda)$ *and* $X_i^* \sim \mathrm{ES}(\alpha_i^*, \lambda)$ *share a common Archimedean survival copula with generator* ϕ. *Then,* $X_{1:n} \leq_{\mathrm{st}} X_{1:n}^*$ *if* $(\alpha_1, \ldots, \alpha_n) \overset{\mathrm{w}}{\succeq} (\alpha_1^*, \ldots, \alpha_n^*)$.

The following result, for the GMW samples with Archimedean survival copulas, follows immediately from Theorem 11.

***Theorem* 12.** *Suppose for* $i = 1, ..., n$, $X_i \sim \mathrm{GMW}(\alpha, \gamma, \lambda, \beta_i)$ *and* $X_i^* \sim \mathrm{GMW}(\alpha, \gamma, \lambda, \beta_i^*)$ *share a common Archimedean survival copula with generator* ϕ. *Then,* $X_{1:n} \leq_{\mathrm{st}} X_{1:n}^*$ *if* $(\beta_1, \ldots, \beta_n) \overset{\mathrm{w}}{\succeq} (\beta_1^*, \ldots, \beta_n^*)$.

In the sequel, by $\boldsymbol{X} \sim \mathrm{GMW}(\boldsymbol{\beta}, \alpha, \gamma, \lambda, \phi)$ we mean that $\mathbf{X}$ has the Archimedean copula with generator ϕ and for $i = 1, ..., n$, $X_i \sim GMW(\beta_i, \alpha, \gamma, \lambda)$. The smallest order statistic $X_{1:n}$ of the sample $\boldsymbol{X} \sim \mathrm{GMW}(\boldsymbol{\beta}, \alpha, \gamma, \lambda, \phi)$ has the survival function

$$\bar{G}_{X_{1:n}}(x) = \phi\Big(\sum_{i=1}^{n} \psi(1 - (F(\alpha, \gamma, \lambda, x))^{\beta_i})\Big) \equiv J(\boldsymbol{\beta}, \alpha, \gamma, \lambda, x, \phi) \quad (8)$$

where, $F(\alpha,\gamma,\lambda,x) = 1-\exp(-\alpha x^{\gamma}\exp(\lambda x))$.

***Theorem* 13.** *For* $\boldsymbol{X} \sim \mathrm{GMW}(\boldsymbol{\beta},\alpha,\gamma,\lambda,\phi_1)$ *and* $\boldsymbol{X^*} \sim \mathrm{GMW}(\boldsymbol{\beta^*},\alpha,\gamma,\lambda,\phi_2)$, *if* $\psi_2\circ\phi_1$ *is super-additive, then* $(\beta_1,\ldots,\beta_n)\overset{\mathrm{w}}{\succeq}(\beta_1^*,\ldots,\beta_n^*)$ *implies* $X_{1:n} \leq_{\mathrm{st}} X^*_{1:n}$.

Proof. According to Equation (8), $X_{1:n}$ and $X^*_{1:n}$ have their respective survival functions $J(\boldsymbol{\beta},\alpha,\gamma,\lambda,x,\phi_1)$ and $J(\boldsymbol{\beta},\alpha,\gamma,\lambda,x,\phi_2)$ for $x\geq 0$. First we show that $J(\boldsymbol{\beta},\alpha,\gamma,\lambda,x,\phi_1)$ is increasing and Schur-concave function of $\beta_i, i=1,\ldots,n$. Since ϕ_1 is decreasing, we have

$$\frac{\partial J(\boldsymbol{\beta},\alpha,\gamma,\lambda,x,\phi_1)}{\partial\alpha_i} = -$$

$$\frac{(F(\alpha,\gamma,\lambda,x))^{\beta_i}\log(F(\alpha,\gamma,\lambda,x))\phi_1'\big(\sum_{i=1}^n\psi(1-(F(\alpha,\gamma,\lambda,x))^{\beta_i})\big)}{\phi_1'(\psi(1-(F(\alpha,\gamma,\lambda,x))^{\beta_i}))} \geq 0,$$

$$\text{for all}\quad x>0,$$

That is, $J(\boldsymbol{\beta},\alpha,\gamma,\lambda,x,\phi_1)$ is increasing in β_i for $i=1,\ldots,n$.

To prove its Schur-concavety, from lemma 1, we need to show that for $i\neq j$,

$$(\beta_i-\beta_j)\big(\frac{\partial J(\boldsymbol{\beta},\alpha,\gamma,\lambda,x,\phi_1)}{\partial\beta_i}-\frac{\partial J(\boldsymbol{\beta},\alpha,\gamma,\lambda,x,\phi_1)}{\partial\beta_j}\big)\leq 0,$$

that is, for $i\neq j$,

$$-\log(F(\alpha,\gamma,\lambda,x))\phi_1'\big(\sum_{i=1}^n\psi_1(1-(F(\alpha,\gamma,\lambda,x))^{\beta_i}))\big)(\beta_i-\beta_j)\times$$

$$\left(\frac{(F(\alpha,\gamma,\lambda,x))^{\beta_i}}{\phi_1'(\psi_1(1-(F(\alpha,\gamma,\lambda,x))^{\beta_i}))}-\frac{(F(\alpha,\gamma,\lambda,x))^{\beta_j}}{\phi_1'(\psi_1(1-(F(\alpha,\gamma,\lambda,x))^{\beta_j}))}\right)\leq 0. \quad (9)$$

Now, let us consider the function $g(\beta)=\dfrac{(F(\alpha,\gamma,\lambda,x))^{\beta_i}}{\phi_1'(\psi(1-(F(\alpha,\gamma,\lambda,x))^{\beta_i}))}$. $g(\beta)$ is increasing with respect to β, from which it follows that (9) holds. According to Lemma 2, $\boldsymbol{\beta}\overset{\mathrm{w}}{\succeq}\boldsymbol{\beta^*}$ implies $J(\boldsymbol{\beta},\alpha,\gamma,\lambda,x,\phi_1)\leq J(\boldsymbol{\beta^*},\alpha,\gamma,\lambda,x,\phi_1)$. On the other hand, since $\psi_2\circ\phi_1$ is super-additive by Lemma A.1. of Li and Fang

(2015), we have $J(\boldsymbol{\beta}^*, \alpha, \gamma, \lambda, x, \phi_1) \leq J(\boldsymbol{\beta}^*, \alpha, \gamma, \lambda, x, \phi_2)$. So, it holds that

$$J(\boldsymbol{\beta}, \alpha, \gamma, \lambda, x, \phi_1) \leq J(\boldsymbol{\beta}^*, \alpha, \gamma, \lambda, x, \phi_1) \leq J(\boldsymbol{\beta}^*, \alpha, \gamma, \lambda, x, \phi_2).$$

That is, $X_{1:n} \leq_{\text{st}} X^*_{1:n}$. □

Theorem 13 generalizes the result of the Theorem 12 to GMW samples with not necessarily a common dependence structure. Theorem 14 generalizes the result of Bashkar et al. (2017) to ES samples with not necessarily a common dependence structure. The smallest order statistic $X_{1:n}$ of the sample $\boldsymbol{X} \sim \text{ES}(\boldsymbol{\alpha}, \lambda, \phi)$ gets survival function

$$\bar{G}_{X_{1:n}}(x) = \phi\big(\sum_{i=1}^{n} \psi(1 - G^{\alpha_i}(\lambda x))\big) = J_1(\boldsymbol{\alpha}, \lambda, x, \phi) \tag{10}$$

Theorem 14. *For $\boldsymbol{X} \sim \text{ES}(\boldsymbol{\alpha}, \lambda, \phi_1)$ and $\boldsymbol{X}^* \sim \text{ES}(\boldsymbol{\alpha}^*, \lambda, \phi_2)$, if $\psi_2 \circ \phi_1$ is super-additive, then $\boldsymbol{\alpha} \overset{\text{w}}{\succeq} \boldsymbol{\alpha}^*$ implies $X_{1:n} \leq_{\text{st}} X^*_{1:n}$.*

Proof. According to Equation (10), $X_{1:n}$ and $X^*_{1:n}$ have their respective survival functions $J_1(\boldsymbol{\alpha}, \lambda, x, \phi_1)$ and $J_1(\boldsymbol{\alpha}^*, \lambda, x, \phi_2)$ for $x \geq 0$.

First we show that $J_1(\boldsymbol{\alpha}, \lambda, x, \phi_1)$ is increasing and Schur-concave function of $\alpha_i, i = 1, \ldots, n$. Since ϕ_1 is decreasing, we have

$$\frac{\partial J_1(\boldsymbol{\alpha}, \lambda, x, \phi_1)}{\partial \alpha_i} = -\frac{G^{\alpha_i}(\lambda x) \log(G(\lambda x)) \phi_1'\big(\sum_{i=1}^{n} \psi(1 - G^{\alpha_i}(\lambda x))\big)}{\phi_1'(\psi(1 - G^{\alpha_i}(\lambda x)))} \geq 0,$$

$$\text{for all} \quad x > 0,$$

That is, $J_1(\boldsymbol{\alpha}, \lambda, x, \phi_1)$ is increasing in α_i for $i = 1, \ldots, n$.

To prove its Schur-concavety, it follows from lemma 1 that we have to show that for $i \neq j$,

$$(\alpha_i - \alpha_j)\big(\frac{\partial J_1(\boldsymbol{\alpha}, \lambda, x, \phi_1)}{\partial \alpha_i} - \frac{\partial J_1(\boldsymbol{\alpha}, \lambda, x, \phi_1)}{\partial \alpha_j}\big) \leq 0,$$

that is, for $i \neq j$,

$$-\log(G(\lambda x)) \phi_1'\big(\sum_{i=1}^{n} \psi_1(1 - G^{\alpha_i}(\lambda x)))\big)(\alpha_i - \alpha_j)$$

$$\left(\frac{G^{\alpha_i}(\lambda x)}{\phi_1'(\psi_1(1-G^{\alpha_i}(\lambda x)))}-\frac{G^{\alpha_j}(\lambda x)}{\phi_1'(\psi_1(1-G^{\alpha_j}(\lambda x)))}\right)\leq 0. \tag{11}$$

Now, let us consider the function $g(\alpha)=\dfrac{G^{\alpha}(\lambda x)}{\phi'(\psi(1-G^{\alpha}(\lambda x)))}$. Taking derivative with respect to α, we get

$$g'(\alpha)\overset{\text{sgn}}{=}G^{\alpha}(\lambda x)\log(G(\lambda x))\phi'(\psi(1-G^{\alpha}(\lambda x)))$$
$$+\frac{G^{2\alpha}(\lambda x)\log(GF(\lambda x))}{\phi'(\psi(1-G^{\alpha}(\lambda x)))}\phi''(\psi(1-G^{\alpha}(\lambda x)))\geq 0.$$

Thus, $g(\alpha)$ is increasing with respect to α, from which it follows that (11) holds. According to Lemma 2 $\boldsymbol{\alpha}\overset{\text{w}}{\succeq}\boldsymbol{\alpha}^*$ implies $J_1(\boldsymbol{\alpha},\lambda,x,\phi_1)\leq J_1(\boldsymbol{\alpha}^*,\lambda,x,\phi_1)$. On the other hand, since $\psi_2\circ\phi_1$ is super-additive by Lemma A.1. of Li and Fang (2015), we have $J_1(\boldsymbol{\alpha}^*,\lambda,x,\phi_1)\leq J_1(\boldsymbol{\alpha}^*,\lambda,x,\phi_2)$. So, it holds that

$$J_1(\boldsymbol{\alpha},\lambda,x,\phi_1)\leq J_1(\boldsymbol{\alpha}^*,\lambda,x,\phi_1)\leq J_1(\boldsymbol{\alpha}^*,\lambda,x,\phi_2).$$

That is, $X_{1:n}\leq_{\text{st}}X^*_{1:n}$. □

Note that if in Theorem 14, we take $\lambda=1$, then we get the following result for the proportional reversed hazards (PRH) model.

***Corollary* 1.** *Suppose $\boldsymbol{X}\sim PRH(\boldsymbol{\alpha},\phi_1)$ and $\boldsymbol{X}^*\sim PRH(\boldsymbol{\alpha}^*,\phi_2)$ and $\phi_2\circ\psi_1$ is super-additive. Then $\boldsymbol{\alpha}\overset{\text{w}}{\succeq}\boldsymbol{\alpha}^*$ implies $X_{1:n}\leq_{\text{st}}X^*_{1:n}$.*

In the ES family, Bashkar et al. (2018) obtained some sufficient conditions for the comparison of parallel systems under some well-known stochastic orderings. Their results are stated in the following theorem.

***Theorem* 15.** *For $\boldsymbol{X}\sim\text{ES}(\boldsymbol{\alpha},\lambda,\phi_1)$ and $\boldsymbol{X}^*\sim\text{ES}(\boldsymbol{\alpha}^*,\lambda,\phi_2)$,*

(i) if ϕ_1 or ϕ_2 is log-convex, and $\psi_2\circ\phi_1$ is super-additive, then $(\alpha_1,\ldots,\alpha_n)\succeq_{\text{w}}(\alpha_1^*,\ldots,\alpha_n^*)$ implies $X_{n:n}\geq_{\text{st}}X^*_{n:n}$;

(ii) if ϕ_1 or ϕ_2 is log-concave, and $\psi_1\circ\phi_2$ is super-additive, then $(\alpha_1,\ldots,\alpha_n)\overset{\text{w}}{\succeq}(\alpha_1^*,\ldots,\alpha_n^*)$ implies $X_{n:n}\leq_{\text{st}}X^*_{n:n}$.

The following result follows immediately from Theorem 15.

***Theorem* 16.** *For $\boldsymbol{X}\sim\text{GMW}(\boldsymbol{\beta},\alpha,\gamma,\lambda,\phi_1)$ and $\boldsymbol{X}^*\sim\text{GMW}(\boldsymbol{\beta}^*,\alpha,\gamma,\lambda,\phi_2)$,* *(i)* *if* ϕ_1 *or* ϕ_2 *is log-convex, and* $\psi_2 \circ \phi_1$ *is super-additive, then* $(\beta_1, \ldots, \beta_n) \succeq_{\mathrm{w}} (\beta_1^*, \ldots, \beta_n^*)$ *implies* $X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*$;

(ii) *if* ϕ_1 *or* ϕ_2 *is log-concave, and* $\psi_1 \circ \phi_2$ *is super-additive, then* $(\beta_1, \ldots, \beta_n) \stackrel{\mathrm{w}}{\succeq} (\beta_1^*, \ldots, \beta_n^*)$ *implies* $X_{n:n} \leq_{\mathrm{st}} X_{n:n}^*$.

The following theorem provides some sufficient conditions under which weakly supermajorized order between the scale parameter vectors implies the usual stochastic order between the largest order statistics.

Theorem 17. *For* $\boldsymbol{X} \sim \mathrm{GMW}(\beta, \boldsymbol{\alpha}, \gamma, \lambda, \phi_1)$ *and* $\boldsymbol{X}^* \sim \mathrm{GMW}(\beta, \boldsymbol{\alpha}^*, \gamma, \lambda, \phi_2)$, *if* ϕ_1 *or* ϕ_2 *is log-convex, and* $\psi_2 \circ \phi_1$ *is super-additive, then* $(\alpha_1, \ldots, \alpha_n) \stackrel{\mathrm{w}}{\succeq} (\alpha_1^*, \ldots, \alpha_n^*)$ *implies* $X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*$.

Proof. The largest order statistic $X_{n:n}$ of the sample $\boldsymbol{X} \sim \mathrm{GMW}(\beta, \boldsymbol{\alpha}, \gamma, \lambda, \phi_1)$ gets distribution function

$$F_{X_{n:n}}(x) = \phi_1\big(\sum_{i=1}^n \psi_1((1 - \exp(-\alpha_i x^\gamma \exp(\lambda x)))^\beta)\big) = J_2(\beta, \boldsymbol{\alpha}, \gamma, \lambda, x, \phi_1) \tag{12}$$

We only prove the case that ϕ_1 is log-convex, and the other case can be finished similarly. First we show that $J_2(\beta, \boldsymbol{\alpha}, \gamma, \lambda, x, \phi_1)$ is increasing and Schur-concave function of $\alpha_i, i = 1, \ldots, n$. Since ϕ_1 is decreasing, we have

$$\frac{\partial J_2(\beta, \boldsymbol{\alpha}, \gamma, \lambda, x, \phi_1)}{\partial \alpha_i} = \beta x^\gamma e^{\lambda x} e^{-\alpha_i x^\gamma \exp(\lambda x)} (1 - \exp(-\alpha_i x^\gamma \exp(\lambda x)))^{\beta - 1}$$

$$\times \frac{\phi_1'\big(\sum_{i=1}^n \psi_1((1 - \exp(-\alpha_i x^\gamma \exp(\lambda x)))^\beta)\big)}{\phi_1'\big(\psi_1((1 - \exp(-\alpha_i x^\gamma \exp(\lambda x)))^\beta)\big)} \geq 0,$$

$$\text{for all} \quad x > 0,$$

That is, $J_2(\beta, \boldsymbol{\alpha}, \gamma, \lambda, x, \phi_1)$ is increasing in α_i for $i = 1, \ldots, n$. Furthermore, for $i \neq j$,

$$\frac{\partial J_2(\beta, \boldsymbol{\alpha}, \gamma, \lambda, x, \phi_1)}{\partial \alpha_i} - \frac{\partial J_2(\beta, \boldsymbol{\alpha}, \gamma, \lambda, x, \phi_1)}{\partial \alpha_j)} =$$

$$\beta x^\gamma e^{\lambda x} \phi_1'\big(\sum_{i=1}^n \psi_1((1 - \exp(-\alpha_i x^\gamma \exp(\lambda x)))^\beta)\big)$$

$$\times \left(\frac{e^{-\alpha_i x^\gamma \exp(\lambda x)}}{(1 - e^{-\alpha_i x^\gamma \exp(\lambda x)})} \times \frac{\phi_1(\psi_1(1 - e^{-\alpha_i x^\gamma \exp(\lambda x)}))}{\phi_1'\big(\psi_1(1 - e^{-\alpha_i x^\gamma \exp(\lambda x)})\big)} - \right.$$

$$\left.\frac{e^{-\alpha_j x^\gamma \exp(\lambda x)}}{(1-e^{-\alpha_j x^\gamma \exp(\lambda x)})} \times \frac{\phi_1(\psi_1(1-e^{-\alpha_j x^\gamma \exp(\lambda x)}))}{\phi_1'\big(\psi_1(1-e^{-\alpha_j x^\gamma \exp(\lambda x)})\big)}\right).$$

Note that the log-convexity of ϕ_1 implies the decreasing property of $\frac{\phi_1}{\phi_1'}$. Since $\psi_1(1-e^{-\alpha x^\gamma \exp(\lambda x)})$ is decreasing in $\alpha>0$, then $\frac{\phi_1(\psi_1(1-e^{-\alpha_i x^\gamma \exp(\lambda x)}))}{\phi_1'\big(\psi_1(1-e^{-\alpha_i x^\gamma \exp(\lambda x)})\big)}$ is increasing in $\alpha>0$. Also $\frac{e^{-\alpha x^\gamma \exp(\lambda x)}}{(1-e^{-\alpha x^\gamma \exp(\lambda x)})}$ is decreasing in $\alpha>0$, and thus $\frac{e^{-\alpha x^\gamma \exp(\lambda x)}}{(1-e^{-\alpha x^\gamma \exp(\lambda x)})} \times \frac{\phi_1(\psi_1(1-e^{-\alpha x^\gamma \exp(\lambda x)}))}{\phi_1'\big(\psi_1(1-e^{-\alpha x^\gamma \exp(\lambda x)})\big)}$ is increasing in $\alpha>0$. So, for $i \neq j$,

$$(\alpha_i-\alpha_j)\left(\frac{\partial J_2(\beta,\boldsymbol{\alpha},\gamma,\lambda,x,\phi_1)}{\partial\alpha_i} - \frac{\partial J_2(\beta,\boldsymbol{\alpha},\gamma,\lambda,x,\phi_1)}{\partial\alpha_j)}\right) \leq 0.$$

Then Schur-concavity of $J_2(\beta,\boldsymbol{\alpha},\gamma,\lambda,x,\phi_1)$ follows from lemma 1. According to Lemma 2 $(\alpha_1,\ldots,\alpha_n) \overset{\mathrm{w}}{\succeq} (\alpha_1^*,\ldots,\alpha_n^*)$ implies $J_2(\beta,\boldsymbol{\alpha},\gamma,\lambda,x,\phi_1) \leq J_2(\beta,\boldsymbol{\alpha}^*,\gamma,\lambda,x,\phi_1)$. On the other hand, since $\psi_2 \circ \phi_1$ is super-additive by Lemma A.1. of Li and Fang (2015), we have $J_2(\beta,\boldsymbol{\alpha}^*,\gamma,\lambda,x,\phi_1) \leq J_2(\beta,\boldsymbol{\alpha}^*,\gamma,\lambda,x,\phi_2)$. So, it holds that

$$J_2(\beta,\boldsymbol{\alpha},\gamma,\lambda,x,\phi_1) \leq J_2(\beta,\boldsymbol{\alpha}^*,\gamma,\lambda,x,\phi_1) \leq J_2(\beta,\boldsymbol{\alpha}^*,\gamma,\lambda,x,\phi_2).$$

That is, $X_{n:n} \geq_{\mathrm{st}} X_{n:n}^*$. □

4 Conclusions

In this paper, we have considered series and parallel systems with independent heterogeneous generalized modified Weibull components. These comparisons are made with respect to usual stochastic, hazard rate and reversed hazard rate orderings. We also conducted stochastic comparison on the extreme order statistics from GMW samples with Archimedean survival copulas. The results of the paper extend some known results in the Balakrishnan et al. (2018).

Acknowledgment

The author would like to thank the referees for their valuable comments and suggestions, which resulted in an improvement in the presentation of this

manuscript.

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Esmaeil Bashkar
Department of Statistics,
Velayat University,
Iranshahr, Iran.
email: *e.bashkar@velayat.ac.ir*