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A Repetitive Sampling-based Control Chart for Multivariate Weighed Poisson Distribution with Two Different Indexes

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Abstract. Control charts using repetitive group sampling have been an attractive topic in the last few years. This paper presents a control chart for multivariate weighed Poisson distribution by repetitive sampling with two different indexes. The effect of these two indexes on the performance of the control chart will be investigated based on the average sequence length criterion. Unlike almost all the research studies on this topic, this paper considers those cases in which the process parameters in the out-of-control situation are not necessarily a constant proportion of the process parameters in the control situation. In this paper, we will show that choosing appropriate statistics can be useful in the performance of the control chart and increasing its efficiency.

Keywords. Multi-attribute process control; average run length; multivariate Poisson distribution; repetitive sampling.
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1 Introduction

Control charts are important tools in statistical quality control for monitoring and improvement purposes. Control charts are dividable into attributes and variable control charts, depending on whether the quality characteristic is attributable or measurable. When process quality characteristics are classified on the nominal or ordinal scale, the corresponding monitoring process is regarded as a multivariate attribute process control Lu et al. (1998). It is noteworthy to mention that little work has been carried out on statistical procedures to monitor multivariate attribute processes and many problems in developing multi-attribute monitoring methods are still open Niaki and Akbari-Nasaji (2011); Niaki and Khedmati (2013). To study some reviews on the multi-attribute statistical process, control the readers may refer to Bersimis et al. (2007); Mason and Young (2002); Saghir and Lin (2015); Topalidou and Psarakis (2009). Many research studies focused on control charts for multivariate Poisson distribution. For more discussion, one may refer to Ali Raza and Aslam (2018); Aparisi et al. (2014); Epprecht et al. (2013); Garcia-Bustos et al. (2016); He et al. (2014); Ho and Costa (2009); Holgate (1964); Kuo and Chiu (2008); Laungrungrong et al. (2014); Niaki and Khedmati (2013). Ahmad et al. (2013) and Aslam et al. (2014) offered repetitive sampling in the field of control charts. Recently, Aslam et al. (2017) introduced a control chart for MP count data under repetitive sampling (RS), and showed this chart is more effective in identifying the process shift. Also, Cozzuzoli et al. (2018) proposed a control chart for multivariate weighted Poisson distribution, with a new index. In the present paper, a control chart for multivariate weighted Poisson distribution is proposed with two different indexes, in Section 2. while section 3 depicts a performance comparison of the proposed control charts in terms of ARL. The average run length (ARL) is used to measure the efficiency of a control chart for the detection of the out-of-control manufacturing process Montgomery (2013). Finally, some conclusions are brought in Section 4. Please note that unlike almost all the research studies on this topic, this paper has considered those cases in which the process parameters in the out-of-control situation are not necessarily a constant proportion of the process parameters in the control situation.

2 A Control Chart for Weighted Multivariate Poisson Distribution Using Repetitive Sampling

For evaluating the production process, a sampling inspection unit is collected from the process. Each identified defect have to be classified in one of the P category. Let $D = (D_1, \cdots, D_j, \cdots, D_P)$ be a vector of the p defect categories. The component D_j indicates the j-th defect category; D_1 is the least serious defect category, and D_P is the most severe defect category.

Let $X_j, j = 1, 2, \cdots, P$ be a number of non-conformities or flaws of the quality characteristics which follows jointly P-variate Poisson distribution with mean $\lambda_j, j = 1, 2, \cdots, p$. Similar to Karlis (2003); Krishnamoorthy (1951), we use the notation $X \sim MVP_P(\Lambda)$, where $\Lambda = (\lambda_1, \cdots, \lambda_P)$. Also, we assume that $Cov(X_i, X_j) = 0$ for $i \neq j$.

Let $d = (d_1, \cdots, d_j, \cdots, d_p)$ be a vector of increasing weights associated to the vector D vector, i.e ., $d_j < d_{j+1},\ j = 1, 2, \cdots, P - 1$. In general, d_j indicates the degree of quality loss that the j-th defect class introduces into the system. Jones et al. (1999) defined a index as a linear weighted combination of several Poisson random variables. In the other words, for the fixed vectors d and Λ, the index is

$$\xi = \sum_{j=1}^{P} d_j \lambda_j. \tag{1}$$

The following sampling statistics can be used to monitor the overall defectiveness parameter ξ

$$\hat{\xi} = \sum_{j=1}^{P} d_j \hat{\lambda}_j = \sum_{j=1}^{P} d_j X_j, \tag{2}$$

where X_j and $\hat{\xi}$ represent the unbiased estimator of the parameter λ_j and of the index ξ, respectively. Cozzucoli and Marozzi (2017) defined a new index of the overall defectiveness based on a modification of (1) and a two-sided Shewhart-type multivariate control chart with asymptotic probabilistic limits to monitor the defectiveness or demerit of the process.

Let $\Lambda_0 = (\lambda_{10}, \cdots, \lambda_{p0})$ be the parameter vector when process is in control. Cozzucoli and Marozzi (2017) introduced a statistic δ as follows

$$\delta = \sum_{j=1}^{p} d_j p_j, \tag{3}$$

where $p_j = \frac{\lambda_j}{\lambda_0}$ is a measure of the relative weight of the j-th defect class, and $\lambda_0 = \sum_{j=1}^{p} \lambda_{j0}$. They used the geometric progressive sequence for the weights, which is $d_j = rd_{j-1}$, where r is the common ratio of the geometric progression. The following sampling statistic may be used

$$\hat{\delta} = \sum_{j=1}^{p} d_j \hat{p}_j, \tag{4}$$

where $\hat{p}_j = \frac{X_j}{\sum_{j=1}^{p} X_j}$.

Case 1: proposed index by Jones

We use (1) as sampling statistics

$$\hat{\xi} = \sum_{j=1}^{P} d_j X_j,$$

Where d_j and X_j has been defined in above. The outer control limits for the proposed control chart are given by

$$\begin{aligned} LCL_1 &= \mu_{\hat{\xi}} - k_1\sigma_{\hat{\xi}} = \sum_{j=1}^{p} d_j\lambda_{j0} - k_1\big(\sum_{j=1}^{p} d_j^2\lambda_{j0}\big)^{\frac{1}{2}}, \\ UCL_1 &= \mu_{\hat{\xi}} + k_1\sigma_{\hat{\xi}} = \sum_{j=1}^{p} d_j\lambda_{j0} + k_1\big(\sum_{j=1}^{p} d_j^2\lambda_{j0}\big)^{\frac{1}{2}}. \end{aligned}$$

Also, the inner control limits for the proposed control chart are designed by

$$\begin{aligned} LCL_2 &= \mu_{\hat{\xi}} - k_2\sigma_{\hat{\xi}} = \sum_{j=1}^{p} d_j\lambda_{j0} - k_2\big(\sum_{j=1}^{p} d_j^2\lambda_{j0}\big)^{\frac{1}{2}}, \\ UCL_2 &= \mu_{\hat{\xi}} + k_2\sigma_{\hat{\xi}} = \sum_{j=1}^{p} d_j\lambda_{j0} + k_2\big(\sum_{j=1}^{p} d_j^2\lambda_{j0}\big)^{\frac{1}{2}}. \end{aligned}$$

• Inner control limits

$$LCL_2 = \max\Big[0, \sum_{j=1}^{p} d_j\lambda_{j0} - k_2\big(\sum_{j=1}^{p} d_j^2\lambda_{j0}\big)^{\frac{1}{2}}\Big], \quad UCL_2 = \sum_{j=1}^{p} d_j\lambda_{j0} + k_2\big(\sum_{j=1}^{p} d_j^2\lambda_{j0}\big)^{\frac{1}{2}}.$$

where k_1 and k_2 are the control limit coefficients and selected in such a way that a specific in-control ARL is attained.

Under normal approximation, The out-of-control probability ($P^0_{out.1}$) based on a single sample when the process is in-control is given by

$$P^0_{out.1} = P(\hat{\xi} \le LCL_1|\lambda_j = \lambda_{j0}) + P(\hat{\xi} \ge UCL_1|\lambda_j = \lambda_{j0}) = P^0_{out.1} = 2[1-\Phi(k_1)],$$

where $\Phi(.)$ denotes the cumulative distribution function of the standard normal distribution. The probability of repetition (P^0_{rep}) for the introduced control chart is given as the below

$$\begin{aligned} P^0_{rep} &= P(UCL_2 \le \hat{\xi} \le UCL_1|\lambda_j = \lambda_{j0}) + P(LCL_1 \le \hat{\xi} \le LCL_2|\lambda_j = \lambda_{j0}) \\ &= P^0_{rep} = 2[\Phi(k_1) - \Phi(k_2)]. \end{aligned}$$

Therefore, the probability of the process being declared to be out-of-control when the process is in control is given as

$$P^0_{out} = \frac{P^0_{out,1}}{1 - P^0_{rep}} = \frac{2[1 - \Phi(k_1)]}{1 - 2[\Phi(K_1) - \Phi(K_2)]}, \qquad ARL_0 = \frac{1}{P^0_{out}}.$$

Suppose now that the process parameter $\lambda_j = \lambda_{j0}$ is shifted to $\lambda_j = \lambda_{j1}$. Then, the probability of the process being declared to be out-of-control based on the signal sample when the process is shifted is given by

$$P^1_{out,1} = P[\hat{\xi} \le LCL_1|\lambda_j = \lambda_{j1}] + P[\hat{\xi} \ge UCL_1\lambda_j = \lambda_{j1}].$$

Similarly, the probability of repetition (P^1_{rep}) when the process is shifted is obtained

$$P^1_{rep} = P[UCL_2 \le \hat{\xi} \le UCL_1|\lambda_j = \lambda_{j1}] + P[LCL_1 \le \hat{\xi} \le LCL_2|\lambda_j = \lambda_{j1}].$$

So, the probability of process being declared to be out-of-control (P^1_{out}) when the process is shifted is given as follows

$$P_{out}^1 = \frac{P_{out,1}^1}{1 - P_{rep}^1},$$

The out-of-control ARL (ARL_1) for the shifted process is given as follows

$$ARL_1 = \frac{1}{P_{out}^1}.$$

Case 2: Proposed index by Cozzuzoli
The outer and inner control limits for the proposed control chart are given by

$$\begin{aligned}
LCL_1 &= \mu_{\hat{\delta}} - k_1\sigma_{\hat{\delta}} = \sum_{j=1}^{p} d_j P_{j0} - k_1\sqrt{\frac{1}{S_P}\big[\sum_{j=1}^{p} d_j^2 p_{j0} - \big(\sum_{j=1}^{p} d_j p_{j0}\big)^2\big]},\\
UCL_1 &= \mu_{\hat{\delta}} + k_1\sigma_{\hat{\delta}} = \sum_{j=1}^{p} d_j P_{j0} + k_1\sqrt{\frac{1}{S_P}\big[\sum_{j=1}^{p} d_j^2 p_{j0} - \big(\sum_{j=1}^{p} d_j p_{j0}\big)^2\big]},\\
LCL_2 &= \max\Big[0, \sum_{j=1}^{p} d_j P_{j0} - k_2\sqrt{\frac{1}{S_P}\big[\sum_{j=1}^{p} d_j^2 p_{j0} - \big(\sum_{j=1}^{p} d_j p_{j0}\big)^2\big]}\Big],\\
UCL_2 &= \mu_{\hat{\delta}} + k_2\sigma_{\hat{\delta}} = \sum_{j=1}^{p} d_j P_{j0} + k_2\sqrt{\frac{1}{S_P}\big[\sum_{j=1}^{p} d_j^2 p_{j0} - \big(\sum_{j=1}^{p} d_j p_{j0}\big)^2\big]}.
\end{aligned}$$

where k_1 and k_2 are the control limit coefficients and selected in such a way that a specific in-control ARL is attained. The probability of the process being declared to be out-of-control when the process is in control is given as

$$P_{out}^0 = \frac{P_{out,1}^0}{1 - P_{rep}^0} = \frac{2[1 - \Phi(k_1)]}{1 - 2[\Phi(K_1) - \Phi(K_2)]},$$

Where $P_{out.1}^0 = P(\hat{\delta} \leq LCL_1|\lambda_j = \lambda_{j0}) + P(\hat{\delta} \geq UCL_1|\lambda_j = \lambda_{j0}) = 2[1 - \Phi(k_1)]$ and $P_{rep}^0 = P[UCL_2 \leq \hat{\delta} \leq UCL_1|\lambda_j = \lambda_{j0}] + P[LCL_1 \leq \hat{\delta} \leq LCL_2|\lambda_j = \lambda_{j0}] = 1 - 2[\Phi(k_1) - \Phi(k_2)]$. The in-control ARL is defined as

$$ARL_0 = \frac{1}{P_{out}^0}.$$

Suppose now that the process parameter $\lambda_j = \lambda_{j0}$ is shifted to $\lambda_j = \lambda_{j1}$. Then, the probability of the process being declared to be out-of-control based on the signal sample when the process is shifted is

$$P^1_{out,1} = P[\hat{\delta} \leq LCL_1 | \lambda_j = \lambda_{j1}] + P[\hat{\delta} \geq UCL_1 | \lambda_j = \lambda_{j1}].$$

Similarly, the probability of repetition (P^1_{rep}) when the process is shifted is obtained

$$P^1_{rep} = P[UCL_2 \leq \hat{\delta} \leq UCL_1 | \lambda_j = \lambda_{j1}] + P[LCL_1 \leq \hat{\delta} \leq LCL_2 | \lambda_j = \lambda_{j1}].$$

So, the probability of process being declared to be out-of-control (P^1_{out}) when the process is shifted is given as follows

$$P^1_{out} = \frac{P^1_{out,1}}{1 - P^1_{rep}}.$$

The out-of-control ARL (ARL_1) for the shifted process is given as follows

$$ARL_1 = \frac{1}{P^1_{out}}.$$

3 Performance Comparison

This section compares the performance of the proposed control charts with two different indexes using the ARL. The research uses the data in Table 1, which are exactly represented in Table 1 Cozzucoli and Marozzi (2017), while considering more inspection units. It is assumed that the process deteriorates progressively. Table 1 illustrates the values of ARLs for the proposed control chart with two different indexes for the identical values of specified parameters. As depicted in Table 1, the RS scheme with a defined index by Cozzucoli has significantly smaller ARL values in comparison with the RS scheme with an index defined by Jones.

4 Discussion and Conclusions

This paper proposes a control chart for multivariate weighted multivariate Poisson distribution with two different indexes, using repetitive sampling. The efficiency of the proposed chart is compared by out-of-control ARL using

Table 1. Comparison of ARLs proposed control chart with two different indexes

Deterioration	RS scheme with defined index by Cozzucoli $ARL_0 = 200, K_1 = 3.18, K_2 = 0.376$	RS scheme with defined index by Jones $ARL_0 = 200, K_1 = 3.18, K_2 = 0.376$
1	1.015	63.01
2	1.002	37.14
3	1.0005	22.309
4	1.00002	14.88
5	111.17	474.48
6	1.082	162.42
7	1.35	290.31
8	1.012	56.87
9	1.033	71.205
10	1.0018	21.73
11	1.0014	64.28
12	1.03	105.04
13	1.0019	63.02
14	1.003	31.86

different deteriorations when ARL_0 remains unchanged for control charts. The findings of this paper indicate that the proposed chart with a defined index by Cozzacoli yields a small ARL value. So, it can be concluded that this chart performs better. This chart has industrial applications for the evaluation of production processes.

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