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پژوهشکدهی آمار

On the Reversed Average Intensity Order

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Abstract. Based on the increasing property of ratio of average reversed hazard rates of two non-negative random variables a new stochastic order for the sake of comparison of two lifetime distributions is proposed. This stochastic order admits some distinguishing properties. In order to illustrate the obtained results, a semi-parametric model which is called reverse hazard power and a mixture model of proportional reversed hazards are taken into account. Some examples are given to explain some facts.

Keywords. Reversed hazard rate; stochastic order; mixture; reversed hazard power.

MSC 2010: 60E15.

1 Introduction and Preliminaries

The concepts of stochastic orders and aging notions are very useful in different branches of sciences such as biology, social sciences, economics and statistics. As two probability distributions are ordered according to a special partial order some information may then be provided. For example for two lifetime devises, a particular stochastic order may determine that which one of two devices ages faster. From other view, to measure various quantities in two distributions several biometric functions have been defined by which the partial orders are constructed. One of these functions is the well known reversed hazard rate (RHR) function. If density function exists then the RHR function is defined as the ratio of the density to the distribution function. We refer to Gupta and Nanda (2001), Finkelstein (2002), Shaked

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and Shanthikumar (2007) and Brito et al. (2011) for some details about ordering properties of the RHR function. Recently some researchers proposed orderings between two models through monotonicity property of ratio of failure rate and mean residual life of two life distributions (cf. Kalashnikov and Rachev, 1986 and Finkelstein, 2006). Recently, Rezaei et al. (2014) proposed a new stochastic order based on the increasing property of ratio of two RHR functions.

In the context of reliability and life testing, Jiang et al. (2003) proposed a quantitative measure, called Aging Intensity (AI) function, defined as the ratio of the hazard rate (HR) $r(t) = f(t)/\bar{F}(t)$ to a baseline hazard rate $H(t)$. One natural choice for $H(t)$ is the average hazard rate $t^{-1}\int_0^t r(x)dx$, for all $t > 0$. Formally, the AI function is

$$L(t) = \frac{r(t)}{\frac{1}{t}\int_0^t r(x)dx}, \qquad \text{for any } t > 0. \tag{1}$$

Recently, Nanda et al. (2007) investigated some useful stochastic and aging properties of the AI function. In their paper, they defined a new stochastic order based on the comparison of the AI functions of two lifetime variables. Denote by r_i the HR of X_i, for $i = 1, 2$. Nanda et al. (2007) said that the random variable X_1 is smaller than the random variable X_2 in average intensity order (denoted by $X_1 \leqslant_{AI} X_2$), if $L_1(t) \geqslant L_2(t)$, for all $t \geqslant 0$, where $L_i(t) = tr_i(t)/\int_0^t r_i(x)dx,\ i = 1, 2$. For two non-negative random variables X_1 and X_2 with respective survival functions $\bar{F}_1$ and $\bar{F}_2$, in Theorem 3.1(iii) of their work it is proved that $X_1 \leqslant_{AI} X_2$ if and only if $\ln \bar{F}_1(x)/\ln \bar{F}_2(x)$ is increasing in $x > 0$. In this paper, we are going to define a dual stochastic order for the AI order in the sense that we use the well known RHR function instead of the HR function. We shall name this order as reversed average intensity (RAI) order. Relationships among the RAI order and other well known stochastic orders along with some aging properties are first investigated. Preservation properties of the new stochastic order under increasing transformations and under the formation of parallel systems are then obtained. To compare two mixture models of proportional reversed hazard rates (PRHR) that is essentially due to Li and Li (2008), we make use of the new stochastic order. Among the results, motivations for the new order are explained by some examples.

Let X_1 and X_2 be two non-negative random variables with respective supports $S_1 = (l_1, u_1)$ and $S_2 = (l_2, u_2)$, having absolutely continuous distribution functions F_1 and F_2 and probability density functions (pdf) f_1 and f_2, respectively. Further, assume that q_1 and q_2 are the RHR's of X_1 and X_2, which are, respectively, given by

$$q_1(x) = \frac{f_1(x)}{F_1(x)} \text{ and } q_2(x) = \frac{f_2(x)}{F_2(x)}, \qquad \forall x > 0.$$

In the following, two stochastic orders are given according to Shaked and Shanthikumar (2007).

Definition 1. The random variable X_1 is said to be smaller than X_2 in usual stochastic order (denoted by $X_1 \leqslant_{ST} X_2$) if $F_1(x) \geqslant F_2(x)$, for all $x \geqslant 0$.

Definition 2. It is said that X_1 is smaller than X_2 in the sense of RHR order (denoted by $X_1 \leqslant_{RHR} X_2$) if, $q_1(x) \leqslant q_2(x)$, for all $x > 0$, or equivalently, if $F_2(x)/F_1(x)$ is increasing in $x \in (0, \infty)$.

The following definition can be found in Nanda et al. (2003).

Definition 3. The random variable X_1 is said to have decreasing reversed hazard rate (DRHR) property, if $q_1(x)$ is decreasing in $x > 0$, or equivalently if F_1 is log-concave.

Definition 4. A non-negative real valued bivariate function h is said to be totally positive of order 2 (abbreviated by TP_2) in $(x, y) \in \chi \times \gamma$, whenever $h(x_1, y_1)h(x_2, y_2) \geqslant h(x_1, y_2)h(x_2, y_1)$, for all $x_1 \leqslant x_2 \in \chi$ and $y_1 \leqslant y_2 \in \gamma$, where χ and γ are two arbitrary subsets of the real line R.

Note that $h_i(t)$ is said to be $\mathrm{TP}_2(i, t)$, in $i = 1, 2$ and $t \in (0, \infty)$ whenever the function $h : (i, t) \mapsto h_i(t)$ is TP_2 in $(i, t) \in \chi \times \gamma$, with $\chi = \{1, 2\}$ and $\gamma = (0, \infty)$.
Suppose that V_1 and V_2 are two non-negative random variables and F_0 is a distribution function on $(0, \infty)$. $G_i(x) = E(F_0^{V_i}(x))$, $i = 1, 2$ is a distribution function over $(0, \infty)$. The random variables V_1 and V_2 are called mixing random variables, F_0 is called baseline population and G_i the overall population. Li and Li (2008) investigated the conditions under which a particular type of a stochastic order between mixing variables implies the same stochastic

order of their corresponding overall populations. We study this topic for our proposed stochastic order. We refer also the reader to Li and Zhao (2011) and Xu and Li (2008) for some analogous results regarding some other mixture models. Throughout the paper, increasing and decreasing mean non-decreasing and non-increasing, respectively. It is implicity assumed that all the integrations, derivatives and expectations exist whenever they are appeared. Recall that a series system with n components works when all of their components work and a parallel system with n components works when at least one of their components works. That is if $X_1, X_2, \ldots, X_n$ denote the life lengths of n components including in a system, then the life length of the system under the series and the parallel structures are $\min\{X_1, X_2, \ldots, X_n\}$ and $\max\{X_1, X_2, \ldots, X_n\}$, respectively.

2 Motivations, Main Definitions and Basic Properties

In this section, we first give some motivations for the study of the RAI order. Then, we obtain some distinguishing properties of this stochastic order including a number of relationships between this order and the other stochastic orders. Some preservation and characterization results regarding monotonic transformations and series or/and parallel systems are given. First of all, we propose the reversed aging intensity function as a dual measure for the AI function given in (1). Likewise the AI function which considers the left tail of distributions, the RAI function serves as a quantitative measure that takes the right tails of distributions into account. Suppose that X is a non-negative random variable with distribution function F, and with support $S_X = (l_X, u_X)$, where $l_X = \inf\{x \ : F(x) > 0\}$ and $U_X = \sup\{x \ : F(x) < 1\}$. By assuming that X is a bounded random variable in the sense that $u_X < \infty$, the RAI function of X is defined by

$$L^*(t) = \frac{q(t)}{\dfrac{1}{u_X - t}\displaystyle\int_t^{u_X} q(x)dx}, \qquad \text{for any } t < u_X, \tag{2}$$

where q is the RHR function of X. Let the non-negative random variable X_i with RHR function q_i, and with support $S_i = (l_i, u_i),\ i = 1, 2$ have the RAI

$$L_i^*(t) = \frac{q_i(t)}{\dfrac{1}{u_i - t}\displaystyle\int_t^{u_i} q_i(x)dx}, \qquad \text{for any } t < u_i.$$

Now, assume that the upper bound u_i of the random variable X_i, for each $i = 1, 2$ is finite. Then, denote by $\Lambda_i(x)$ the average of the RHR function of X_i on the interval (x, u_i) for $i = 1, 2$. It can be obtained, for any $t > 0$, that

$$\Lambda_i(t) = \frac{1}{u_i - t}\int_t^{u_i} q_i(x)dx$$

$$= \frac{\ln F_i(u_i) - \ln F_i(t)}{u_i - t} = \frac{-\ln F_i(t)}{u_i - t}, \qquad i = 1, 2.$$

When the upper bounds of the supports of X_1 and X_2 are not finite but they are equal, i.e. $u_1 = u_2$, we can easily observe that the ratio $\Lambda_2(t)$ / $\Lambda_1(t)$ is increasing for $t > 0$, if and only if $L_1^*(t) \leqslant L_2^*(t)$, for all $t \in (0, u_1)$. It is noticeable here that once this ratio is increasing, it turns out that the average RHR of X_1 decreases faster with respect to the average RHR of X_2. This means that a device with lifetime X_1 ages faster, in some stochastic senses, than another device whose lifetime is X_2. The importance of such an investigation stems from the fact that it provides a concept of relative ageing of two probability distributions (see, e.g., Sengupta and Deshpande, 1994 and Lai and Xie, 2003). For the sake of comparison of two probability distributions according to their RAI functions, we define a new stochastic order as follows. At the continue we always assume that $u_1 = u_2$.

Definition 5. We say that X_1 is smaller than X_2 in the RAI order (denoted by $X_1 \leqslant_{RAI} X_2$) whenever $L_1^*(x) \leqslant L_2^*(x)$, for all $x \in (0, u_1)$, or equivalently if $\frac{\ln F_2(x)}{\ln F_1(x)}$ is increasing in x over $(0, \infty)$.

It can be easily seen that $X_1 \leqslant_{RAI} X_2$ if and only if there exists a non-negative increasing function $\beta(x)$ such that $F_2(x) = [F_1(x)]^{\beta(x)}$, for all $x > 0$, that is the distribution function of the random variable X_2 is an exponentiated function of the distribution function of the random variable X_1. Moreover, one can see that $X_1 \leqslant_{RAI} X_2$ if and only if

$$\frac{q_1(x)}{\int_x^\infty q_1(u)\ du} \geqslant \frac{q_2(x)}{\int_x^\infty q_2(u)\ du},$$

for all $x > 0$. The following definition is due to Rezaei et al. (2014).

Definition 6. The random variable X_1 is said to be smaller than X_2 in relative RHR order (denoted by $X_1 \leqslant_{RRH} X_2$), if $q_2(t)/q_1(t)$ is increasing in $t > 0$, or equivalently if, $q_i(t)$ is $\mathrm{TP}_2(i,t)$ in $(i,t) \in \{1,2\} \times (0,\infty)$.

Example 1. Suppose that X_i has an inverse Weibull distribution with distribution function

$$F_i(x) = \exp\left\{-\left(\frac{\alpha_i}{x}\right)^{\beta_i}\right\}, \qquad x > 0,\ \alpha_i, \beta_i > 0,$$

and we shall denote it by $\mathrm{IW}(\alpha_i, \beta_i)$ for $i = 1, 2$. Then, it can be readily seen that $X_1 \leqslant_{RAI} X_2$ if and only if $\beta_1 \geqslant \beta_2$.

Example 2 (The PRHR model). Suppose that X_i has distribution function F_i and suppose that Y_i has distribution function G_i, for $i = 1, 2$. Let $G_1(x) = \{F_1(x)\}^{v_1}$ and let $G_2(x) = \{F_2(x)\}^{v_2}$, for $v_1 > 0$ and $v_2 > 0$. Then, it is easily verified that $X_i \leqslant_{RAI} Y_i$, for $i = 1, 2$. In parallel, it is easy to see that $X_1 \leqslant_{RAI} X_2$ if and only if $Y_1 \leqslant_{RAI} Y_2$.

Before stating another situation for application of the RAI order, we recall a semi-parametric family of distributions from Marshall and Olkin (2007). This family is called reversed hazard power (RHP) model. Formally, let $F(x|\theta)$ and $F(x)$ be such that $F(x|\theta) = \exp\left[-\{S(x)\}^{\theta}\right];\ \theta > 0$, where $S(x) = -\ln F(x)$. Then the random variables X_θ and X that have distributions $F(x|\theta)$ and $F(x)$, respectively, are said to satisfy in the RHP model.

Proposition 1. $X_{\theta_1} \leqslant_{RAI} X_{\theta_2}$ if and only if $\theta_1 \geqslant \theta_2$.

Proof. We have the identity

$$F(x|\theta_i) = \exp\left[-\{S(x)\}^{\theta_i}\right]$$

with $S(x) = -\ln F(x)$ and $i = 1, 2$. Note that $X_{\theta_1} \leqslant_{RAI} X_{\theta_2}$ if and only if

$$\frac{\ln F(x|\theta_2)}{\ln F(x|\theta_1)} = [S(x)]^{\theta_2 - \theta_1}$$

is increasing in x. Thus, the result follows if and only if $\theta_1 > \theta_2$ since $S(x)$ is decreasing in x. □

Proposition 2. Let X_i and Y_i have distribution functions F_i and G_i, respectively, for $i = 1, 2$. Let $G_i(x) = \exp\left[-\{S_i(x)\}^{\theta_i}\right]$ where $S_i(x) = -\ln F_i(x),\ i = 1, 2$ and let $\theta_1 \geqslant \theta_2$. Then

$$X_1 \leqslant_{RAI} X_2 \ \Rightarrow\ Y_1 \leqslant_{RAI} Y_2.$$

Proof. By assumption $\left\{\frac{S_2(x)}{S_1(x)}\right\}^{\theta_2}$ is non-negative and increasing. Write

$$\frac{\ln G_2(x)}{\ln G_1(x)} = \left\{\frac{S_2(x)}{S_1(x)}\right\}^{\theta_2} \cdot \{S_1(x)\}^{\theta_2-\theta_1}.$$

Note that $S_1(x)$ is decreasing. Since $\theta_1 \geqslant \theta_2$, then $\{S_1(x)\}^{\theta_2-\theta_1}$ is non-negative and increasing. We can now conclude the proof. □

In the following result, we demonstrate some implications among $\leqslant_{RAI}$ order and other well known stochastic orders and also we derive a preservation property under the DRHR aging class. Set $H(x) = \frac{\ln F_2(x)}{\ln F_1(x)},\ x > 0$ and let f_i be the pdf of $X_i,\ i = 1, 2$.
By Definition 5, $X_1 \leqslant_{RAI} X_2$ if and only if the function $S_i(x) = -\ln\{F_i(x)\}$ is $\text{TP}_2(i, x)$ in $i \in \{1, 2\}$ and $x \in (0, \infty)$. As a result, Theorem 2.1 in Rezaei et al. (2014) provides that $X_1 \leqslant_{RRH} X_2$ implies $X_1 \leqslant_{RAI} X_2$. In the literature, it is well known that $X_1 \leqslant_{RHR} X_2$ implies $X_1 \leqslant_{ST} X_2$ (cf. Shaked and Shanthikumar, 2007). One may wonder when does the reversed implication hold. The following theorem presents a condition for this to be hold.

Theorem 1. *If $X_1 \geqslant_{RAI} X_2$ and $X_1 \leqslant_{ST} X_2$, then $X_1 \leqslant_{RHR} X_2$.*

Proof. Note that $X_1 \geqslant_{RAI} X_2$ means that $H(x) = \frac{\ln F_2(x)}{\ln F_1(x)}$ is decreasing in $x > 0$. It holds that

$$H'(x) = \frac{q_2(x)\ln F_1(x) - q_1(x)\ln F_2(x)}{\{\ln F_1(x)\}^2}.$$

Thus, $X_1 \geqslant_{RAI} X_2$ gives $H'(x) \leqslant 0$, for all $x > 0$. Observe that

$$H'(x) \leqslant 0,\ \forall\ x > 0\ \Leftrightarrow \frac{q_2(x)}{q_1(x)} \geqslant \frac{\ln F_2(x)}{\ln F_1(x)},\ \forall\ x > 0.$$

On the other hand, $X_1 \leqslant_{ST} X_2$ provides that $\ln F_2(x)/\ln F_1(x) \geqslant 1$, for all $x > 0$. So by the recent equivalence relation, we get $q_2(x) \geqslant q_1(x)$, for all $x > 0$. That is $X_1 \leqslant_{RHR} X_2$. □

Theorem 2. *The following assertions hold:*

(i). If $X_1 \geqslant_{RAI} X_2$, $H(x)$ *is convex and* X_2 *is DRHR, then* X_1 *is DRHR.*

(ii). If $0 < \lim_{x\to 0^+} \frac{f_2(x)}{f_1(x)} < \infty$ *and* $X_1 \leqslant_{RAI} X_2$, *then* $X_1 \leqslant_{ST} X_2$.

(iii). If $\lim_{x\to\infty} \frac{f_2(x)}{f_1(x)} \leqslant 1$ *and* $X_1 \leqslant_{RAI} X_2$, *then* $X_1 \geqslant_{RHR} X_2$.

Proof. (i). Take $S_i(x) = -\ln F_i(x)$ and notice that $S_i'(x) = -q_i(x)$ for $i = 1, 2$ and $x \geqslant 0$. Also $H(x) = S_2(x) \; / \; S_1(x)$, for all $x > 0$. The assumptions yield $H'(x) \leqslant 0$ and $H''(x) \geqslant 0$ for all $x \geqslant 0$. Note that $X_2 \in$ DRHR means that $S_2''(x) \geqslant 0$, for all $x \geqslant 0$, we then obtain

$$S_1''(x) = \frac{d^2}{dx^2}(S_2(x)H(x))$$

$$= H(x)S_2''(x) + 2H'(x)S_2'(x) + H''(x)S_2(x),$$

which is non-negative, for all $x \geqslant 0$, hence $X_1 \in$ DRHR.

(ii). According to the definition, since $X_1 \leqslant_{RAI} X_2$ thus $H(x)$ is increasing and $F_1(0) = F_2(0) = 0$. Therefore

$$\begin{aligned} H(x) &= \frac{\ln F_2(x)}{\ln F_1(x)} \\ &\geqslant \lim_{x\to 0^+} \left\{ \frac{\ln F_2(x)}{\ln F_1(x)} \right\} \\ &= \lim_{x\to 0^+} \frac{f_2(x)}{f_1(x)} \times \frac{F_1(x)}{F_2(x)} \\ &= \left\{ \lim_{x\to 0^+} \frac{f_2(x)}{f_1(x)} \right\} \left\{ \lim_{x\to 0^+} \frac{F_2(x)}{F_1(x)} \right\}^{-1} = 1 \end{aligned}$$

Hence $H(x) \geqslant 1$ for all $x \geqslant 0$ and so $X_1 \leqslant_{ST} X_2$.

(iii). Note that $F_i(\infty) = 1$ and as a result $\ln F_i(\infty) = 0$, for $i = 1, 2$. We get, for all $x > 0$

$$\begin{aligned} H(x) &= \frac{\ln F_2(x)}{\ln F_1(x)} \\ &\leqslant \lim_{x\to\infty} \frac{\ln F_2(x)}{\ln F_1(x)} \\ &= \lim_{x\to\infty} \frac{f_2(x)}{f_1(x)} \times \frac{F_1(x)}{F_2(x)} \\ &= \lim_{x\to\infty} \frac{f_2(x)}{f_1(x)} \leqslant 1. \end{aligned}$$

Hence $\ln F_1(x) \leqslant \ln F_2(x)$, for all $x > 0$, which is equivalent to $X_1 \geqslant_{ST} X_2$. Now, on using Theorem 1, if $X_1 \leqslant_{RAI} X_2$ and $X_1 \geqslant_{ST} X_2$ hold, then $X_1 \geqslant_{RHR} X_2$ holds.

□

The following example illustrates a situation where Theorem 2(i) is applicable.

Example 3. For a random variable X with distribution F, suppose that X_θ has the RHP distribution $F_\theta(x) = \exp\{-S^\theta(x)\}$, for $x > 0$, and $0 < \theta < 1$, where $S(x) = -\ln F(x)$. Let X be DRHR. Since $H(x) = \ln F(x) \,/\, \ln F_\theta(x) = S^{1-\theta}(x)$ is a decreasing convex function, thus Theorem 2(i) implies that X_θ is DRHR.

We present an example here to show that in Theorem 2(iii) the condition of $\lim_{x\to\infty} \frac{f_2(x)}{f_1(x)} \leqslant 1$ is necessary to conclude the result.

Example 4. Let X_1 and X_2 follow IW(1, 2) and IW(1, 1), respectively. Then, $F_1(x) = \exp[-x^{-2}]$ and $F_2(x) = \exp[-x^{-1}]$. For all $x > 0$, we have

$$\frac{f_2(x)}{f_1(x)} = \frac{x}{2} \exp[x^{-2} - x^{-1}].$$

Thus $\lim_{x\to\infty} \frac{f_2(x)}{f_1(x)} = \infty$. That is the condition of Theorem 2(iii) does not hold. We know by Example 1 that $X_1 \leqslant_{RAI} X_2$. On the other hand, since

$$\frac{F_1(x)}{F_2(x)} = \exp[-x^{-2} + x^{-1}]$$

is not an increasing function in x on $(0,\infty)$, hence $X_1 \geqslant_{RHR} X_2$ fails to hold.

Theorem 3. *If ϕ is strictly increasing, then $X_1 \leqslant_{RAI} X_2$ implies $\phi(X_1) \leqslant_{RAI} \phi(X_2)$.*

Proof. Let $T_i = \phi(X_i)$, for $i = 1, 2$ having distribution function F_{T_i}. Let F_i be the distribution function of $X_i, i = 1, 2$. Then, we have the identity $-\ln\{F_{T_i}(x)\} = S_i(\phi^{-1}(x))$, $i = 1, 2$, where $S_i = -\ln(F_i)$, $i = 1, 2$. The proof now obtains at once from the well known properties of totally positive of order 2 functions. □

The following theorem is related with comparison of parallel systems according to $\leqslant_{RAI}$ order. As we demonstrate in this result, the $\leqslant_{RAI}$ order of components in two parallel systems is equivalent to the $\leqslant_{RAI}$ order among the lifetime of that parallel systems.

Theorem 4. *Let X_i, $i = 1, 2, \ldots, n$ be i.i.d. from F and let Y_i, $i = 1, 2, \ldots, m$ be i.i.d. from G. Then, $X_1 \leqslant_{RAI} Y_1$ if and only if $\max\{X_1, \ldots, X_n\} \leqslant_{RAI} \max\{Y_1, \ldots, Y_m\}$.*

Proof. We know that, for all $x > 0$, $P(\max\{X_1, \ldots, X_n\} \leqslant x) = F^n(x)$, and similarly $P(\max\{Y_1, \ldots, Y_m\} \leqslant x) = G^m(x)$. Hence, for any $x > 0$, we have

$$\frac{\ln P(\max\{Y_1, \ldots, Y_m\} \leqslant x)}{\ln P(\max\{X_1, \ldots, X_n\} \leqslant x)} = \frac{m}{n} \, \frac{\ln G(x)}{\ln F(x)}.$$

By the definition, the above identity proves the equivalence relation in the theorem. □

The following result considers the series systems. This result states that among two series systems, the one that its number of components is more ages faster than the other one in terms of the RAI order. Based on the HR ratio a similar result has been obtained in Theorem 1 of Lai and Xie (2003).

Theorem 5. *Let $X_{1:m}$ and $X_{1:n}$ denote lifetimes of two series systems with m and n i.i.d. components, respectively. If $n > m \geq 1$, then $X_{1:n} \leqslant_{RAI} X_{1:m}$.*

Proof. Take $q_{1:m}(x)$ and $q_{1:n}(x)$ to be the RHR functions of $X_{1:m}$ and $X_{1:n}$, respectively. Let each component of two systems have common distribution function F. To begin with, we take $m = n-1$. Then, we get

$$\frac{q_{1:m}(x)}{q_{1:n}(x)} = \frac{d\ln\left\{1-\overline{F}^m(x)\right\}/dx}{d\ln\left\{1-\overline{F}^n(x)\right\}/dx} = \frac{n-1}{n} g\left\{\overline{F}(x)\right\},$$

where $g : [0,1) \to [0,\infty)$, is defined as $g(x) = (1-x^n)/(x-x^n)$. It obtains that

$$\frac{d}{dx}\left(\frac{x-x^n}{1-x^n}\right) = \frac{(1-x)(1+x+\cdots+x^{n-1}-nx^{n-1})}{(1-x^n)^2},$$

which is non-negative trivially for all $x \in [0,1)$. That is $g(x)$ is decreasing, concluding that $q_{1:m}(x)/q_{1:n}(x)$ is increasing in $x > 0$. Hence, $X_{1:n} \leqslant_{RRH} X_{1:n-1}$. By continuing this procedure, we derive $X_{1:n} \leqslant_{RRH} X_{1:n-1} \leqslant_{RRH} \cdots \leqslant_{RRH} X_{1:m}$. That is to say $X_{1:n} \leqslant_{RRH} X_{1:m}$. This further concludes that $X_{1:n} \leqslant_{RAI} X_{1:m}$. □

3 Comparisons in Some Mixture Models

In this section, we present some results for stochastic comparisons in some mixture models. The mixture PRHR model that was introduced earlier is taken into account. Furthermore, a new mixture model of the reversed hazard power is proposed here which is a possible extension of the RHP model. First we have the following result. It states that the random variable that has the mixture PRHR model is better than the baseline random variable in the model in the sense of the $\leqslant_{RAI}$ order.

***Theorem* 6.** *Let X_1 and X_2 have distribution functions F_1 and F_2, respectively, such that $F_2(x) = E\left\{F_1^V(x)\right\}, x \geqslant 0$. Then $X_1 \leqslant_{RAI} X_2$.*

Proof. We have

$$\frac{d}{dx}\cdot\frac{\ln F_2(x)}{\ln F_1(x)} = \frac{d}{dx}\cdot\frac{\ln E\left\{F_1^V(x)\right\}}{\ln F_1(x)} = \frac{q_1(x)}{\{\ln F_1(x)\}^2 E\{F_1^V(x)\}}\left\{E\left(F_1^V(x)\left[\ln\left\{F_1^V(x)\right\} - \ln E\left\{F_1^V(x)\right\}\right]\right)\right\}$$

Set $T = F^V(x)$ which is a non-negative random variable taking value on $[0,1]$, for all $x \geqslant 0$. Take $g(t) = t[\ln(t) - \ln\{E(T)\}]$, $t \in [0,1]$. Obviously

$g(t)$ is a convex function. Hence Jensen's inequality yields

$$E\left(F_1^V(x)\left[\ln F_1^V(x) - \ln E\left\{F_1^V(x)\right\}\right]\right) = E\{g(T)\} \geqslant g\{E(T)\} = 0.$$

□

The following result is a comparison between two mixture PRHR model.

Theorem 7. *Let* $(V_1 \mid V = v) \leqslant_{ST} (V_2 \mid V = v)$*, for all* $v \geqslant 0$*, where* $V = V_1 + V_2$ *and assume that* V_1 *and* V_2 *are independent such that* $E(V_1) = E(V_2)$*. Then* $X_1 \leqslant_{RAI} X_2$*.*

Proof. For each $i = 1, 2$, it obtains that

$$\frac{d}{dx} \ln E\left\{F_0^{V_i}(x)\right\} = \frac{\frac{d}{dx} E\left\{F_0^{V_i}(x)\right\}}{E\left\{F_0^{V_i}(x)\right\}} = \frac{f_0(x) E\left\{V_i F_0^{V_i - 1}(x)\right\}}{E\left\{F_0^{V_i}(x)\right\}} = q_0(x) \frac{E\left\{V_i F_0^{V_i}(x)\right\}}{E\left\{F_0^{V_i}(x)\right\}}, \qquad (3)$$

where $q_0(x) = f_0(x) \,/\, F_0(x)$, which is the RHR of the baseline distribution F_0. Set $S_i(x) = \ln E\{F_0(x)]\}V_i$, for $i = 1, 2$. Because V_1 and V_2 are, by assumption, independent thus

$$\begin{aligned} E\{F_0^{V_1}(x)\}E\{F_0^{V_2}(x)\} &= E\{F_0^{V_1}(x) F_0^{V_2}(x)\} \\ &= E\{F_0^{V_1+V_2}(x)\} \\ &= E\{F_0^{V}(x)\}, \qquad \text{for any } x \geqslant 0, \end{aligned}$$

and also, for any $x \geqslant 0$, we have the identities

$$E\{V_2 F_0^{V_2}(x)\}E\{F_0^{V_1}(x)\} = E\{V_2 F_0^{V_1+V_2}(x)\},$$

and

$$E\{V_1 F_0^{V_1}(x)\}E\{F_0^{V_2}(x)\} = E\{V_1 F_0^{V_1+V_2}(x)\}.$$

It suffices to show that the ratio $\ln E\{F_0^{V_2}(x)\}/\ln E\{F_0^{V_2}(x)\}$ is increasing in $x > 0$. By using (3) we can get

$$\frac{d}{dx} \cdot \frac{\ln E\{F_0^{V_2}(x)\}}{\ln E(F_0^{V_1}(x))} = q_0(x) \frac{\frac{E\{V_2 F_0^{V_2}(x)\}}{E\{F_0^{V_2}(x)\}} S_1(x) - \frac{E\{V_1 F_0^{V_1}(x)\}}{E\{F_0^{V_1}(x)\}} S_2(x)}{S_1^2(x)}$$
$$= \frac{q_0(x)}{S_1^2(x) E\{F_0^V(x)\}} E[F_0^{V_1+V_2}(x)\{V_2 S_1(x) - V_1 S_2(x)\}].$$

Since $(V_1|V = v) \leqslant_{ST} (V_2|V = v)$, for all $v \geqslant 0$, thus Theorem 1.A.3 (d) in Shaked and Shanthikumar (2007) concludes that $V_1 \leqslant_{ST} V_2$. Because of Theorem 1.A.3 (a) in Shaked and Shanthikumar (2007), it follows that $\phi(V_1) \leqslant_{ST} \phi(V_2)$, for any decreasing function ϕ, and consequently $E\{\phi(V_1)\} \geqslant E\{\phi(V_1)\}$, where the expectations exist. Fix $x > 0$, and take $\phi(v) = F_0^v(x)$, which is decreasing in $v \geqslant 0$. Hence, $S_1(x) = \ln E\{\phi(V_1)\} \geqslant \ln E\{\phi(V_2)\} = S_2(x)$, for all $x > 0$. We know, by assumption, that $h(v) = E(V_2|V = v) - E(V_1|V = v)$ is non-negative and that $E\{h(V)\} = 0$. This implies that $P(h(V) = 0) = 1$, i.e.

$$P\{E(V_1 \mid V) = E(V_2 \mid V)\} = 1.$$

By conditioning on $V = V_1 + V_2$, we, therefore, get

$$E\left[F_0^{V_1+V_2}(x)\{V_2 S_1(x) - V_1 S_2(x)\}\right] = E[E(F_0^{V_1+V_2}(x)\{V_2 S_1(x) - V_1 S_2(x)\}|V)]$$
$$= E[F_0^V(x)\{S_1(x)E(V_2|V) - S_2(x)E(V_1|V)\}]$$
$$= \{S_1(x) - S_2(x)\}E\{F_0^V(x)E(V_1|V)\} \geqslant 0.$$

The proof is complete. □

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