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A New Modified Weibull Distribution and Its Applications

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Abstract. In this paper we introduce a four-parameter generalized Weibull distribution. This new distribution has a more general form of failure rate function. It is more general for modeling than six ageing classes of life distributions with appropriate choices of parameter values, so it can display decreasing, increasing, bathtub shaped, unimodal, increasing-decreasing increasing and decreasing-increasing-decreasing failure rates. The new distribution has also a bimodal density function. The moments are obtained and the method of maximum likelihood is used to estimate the model parameters. Also, the observed information matrix is obtained. Two applications are presented to illustrate the advantage of the proposed distribution.

Keywords. Beta distribution; exponentiated Weibull distribution; generalized exponentiated Weibull distribution; modified Weibull distribution; maximum likelihood estimation; observed information matrix.

MSC 2010: 62N05.

1 Introduction

In analyzing lifetime data one often uses the Weibull distribution. The Weibull distribution, having exponential and Rayleigh distribution as special cases, is a very popular distribution for modeling lifetime data as well as modeling phenomenon with monotone failure rates. Therefore, when modeling monotone hazard rates, the Weibull distribution may be an initial choice

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because of its negatively and positively skewed density function. However, the Weibull distribution does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub-shaped. The models that present bathtub-shaped failure rate are very useful in survival analysis. For example, Haupt and Schabe (1992) considered a life-time model with bathtub failure rates. These models, however, are not useful in practice. In recent years, new classes of distributions were proposed based on modifications of the Weibull distribution to cope with bathtub-shaped failure rate. Among these, the exponentiated Weibull (EW) distribution introduced by Mudholkar et al. (1995, 1996), the additive Weibull distribution by Xie and Lai (1996), a new lifetime distribution defined by Chen (2000), extended Weibull distribution presented by Xie et al. (2002), modified Weibull (MW) distribution proposed by Lai et al. (2003), the extended flexible Weibull distribution defined by Bebbington et al. (2007), generalized modified Weibull (GMW) distribution introduced by Carrasco et al. (2008), beta modified Weibull distribution defined by Silva et al. (2010), generalized inverse Weibull distribution introduced by Felipe et al. (2011) and the Modified Beta distribution by Nadarajah et al. (2014). A good review of these models is presented by Pham and Lai (2007).
In this paper, we introduce a new four-parameter distribution called new modified Weibull (NMW) distribution. The NMW distribution has decreasing, unimodal and bimodal probability density function (*pdf*) and can have decreasing, increasing, bathtub shaped, unimodal, increasing-decreasing-increasing and decreasing-increasing- decreasing hazard rate functions.
This paper is organized as follows. We introduce, in Section 2, the NMW distribution. The moments are obtained in Section 3. In Section 4 we discusses order statistics from NMW distribution. The MLEs are provided in Section 5. In Section 6, we discuss simulation study on a variety of sample sizes. Two lifetime data sets are used in Section 7 to illustrate the usefulness of the distribution. Finally, Section 8 deals with some concluding remarks.

2 A New Modified Weibull Distribution

The cumulative distribution function (*cdf*) and *pdf* of the new modified Weibull distribution with four parameters are given respectively by

$$F(t;\alpha,\beta,\gamma,\lambda)=1-e^{-e^{\alpha t^{\gamma}}+e^{-\beta t^{\lambda}}}, \qquad \alpha>0,\beta>0,\gamma\geqslant 0,\lambda\geqslant 0 \tag{1}$$

and

$$f(t) = (\alpha\gamma t^{\gamma-1} e^{\alpha t^{\gamma}} + \lambda\beta t^{\lambda-1} e^{-\beta t^{\lambda}}) e^{-e^{\alpha t^{\gamma}} + e^{-\beta t^{\lambda}}}, \qquad t > 0 \tag{2}$$

where α and β are scale and γ and λ are shape parameters.
We would like to mention that the new distribution has decreasing, unimodal and bimodal *pdf*. Figure 1 shows the plots of NMW density curves for different choices of the parameters.

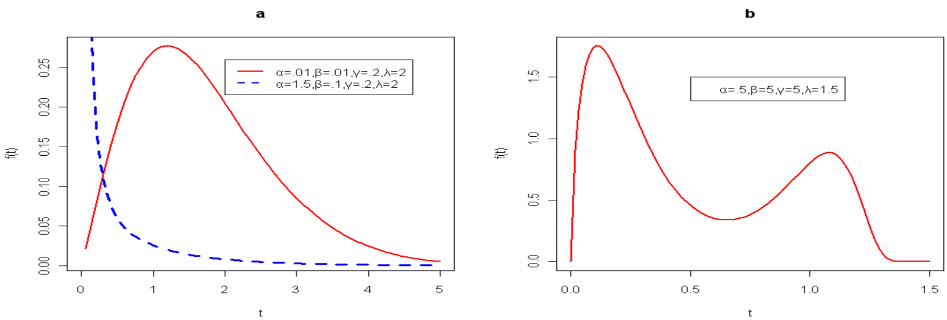

Figure 1. NMW density function curves for different choices of the parameters. (a) the decreasing and unimodal density function and, (b) the bimodal density function.

The corresponding survival and hazard rate functions are, respectively,

$$S(t) = 1 - F(t) = e^{-e^{\alpha t^{\gamma}} + e^{-\beta t^{\lambda}}}, \tag{3}$$

$$h(t) = \alpha\gamma t^{\gamma-1} e^{\alpha t^{\gamma}} + \lambda\beta t^{\lambda-1} e^{-\beta t^{\lambda}}. \tag{4}$$

A new distribution displays with decreasing, increasing, bathtub shaped, unimodal, increasing-decreasing-increasing and decreasing-increasing-decreasing failure rates. Figure 2 shows the plots of hazard rate function curves for different choices of the parameters.

3 Moments

Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis). The rth ordinary moment, $\mu_r^{'} = E(T^r)$, of the NMW is:

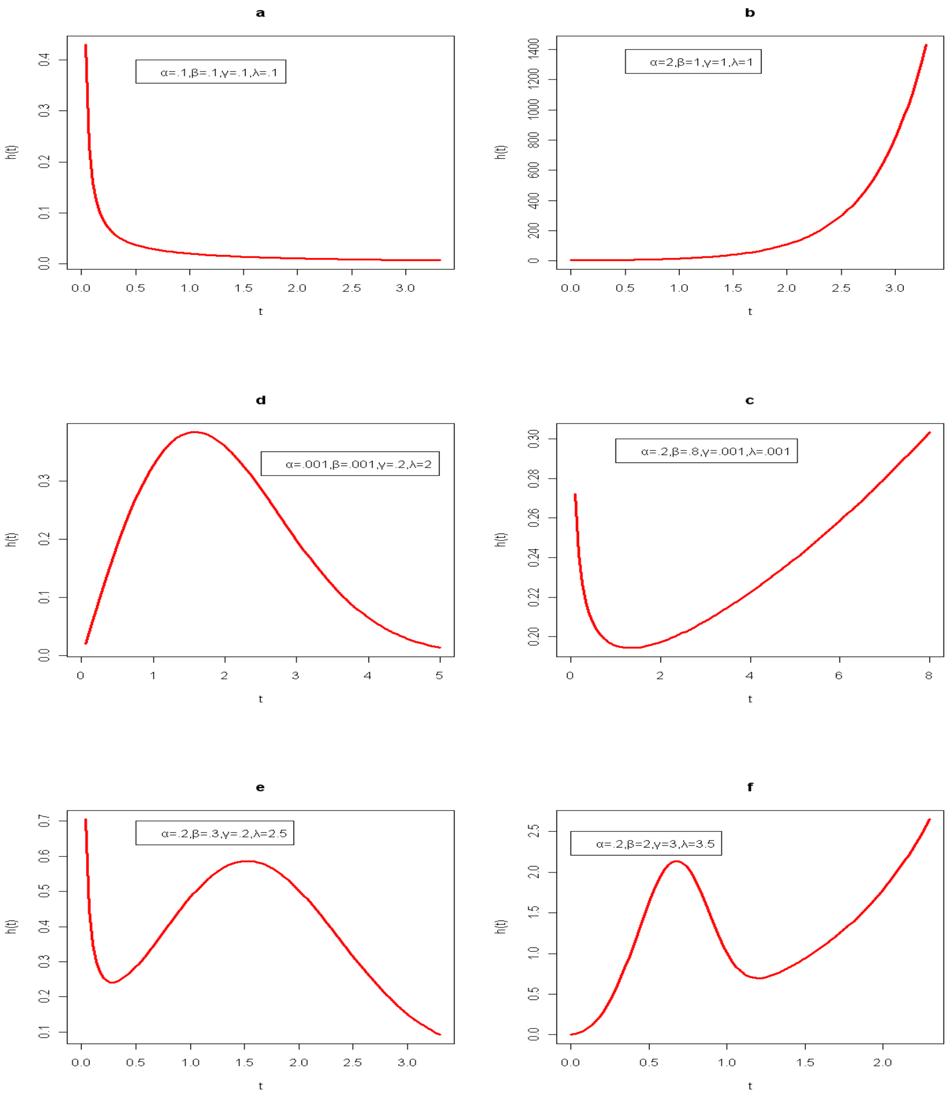

Figure 2. NMW hazard rate functions. (a) The decreasing hazard rate function (b) The increasing hazard rate function. (c) The bathtub hazard rate function. (d) The unimodal hazard rate function. (e) The increasing–decreasing–increasing hazard rate function. (f) The decreasing–increasing– decreasing hazard rate function.

$$\mu_r^{'} = \int_0^\infty (\alpha\gamma t^{r+\gamma-1} e^{\alpha t^\gamma} + \lambda\beta t^{r+\lambda-1} e^{-\beta t^\lambda}) e^{-e^{\alpha t^\gamma} + e^{-\beta t^\lambda}} dt \tag{5}$$

Consider the series expansins:

$$e^{-e^{\alpha t^\gamma}} = \sum_{i=0}^{\infty} \frac{(-1)^i e^{i\alpha t^\gamma}}{i!} \tag{6}$$

$$e^{-e^{\beta t^\lambda}} = \sum_{j=0}^{\infty} \frac{e^{-j\beta t^\lambda}}{j!} \tag{7}$$

$$e^{\alpha t^\gamma (1+i)} = \sum_{k=0}^{\infty} \frac{\alpha^k t^{k\gamma} (1+i)^k}{k!} \tag{8}$$

$$e^{i\alpha t^\gamma} = \sum_{k=0}^{\infty} \frac{i^k \alpha^k t^{k\gamma}}{k!}. \tag{9}$$

Substituting the expansions (6) and (7) into (5), we would obtain

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i}{i!j!} \int_0^\infty \{\alpha\gamma t^{r+\gamma-1} e^{\alpha t^\gamma (1+i) - j\beta t^\lambda} + \lambda\beta t^{r+\lambda-1} e^{-(1+j)\beta t^\lambda + i\alpha t^\gamma}\} dt \tag{10}$$

Now substituting the expansions (8) and (9) into (10), we would obtain

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^i \alpha^{k+1} \gamma (1+i)^k}{i!j!k!} \int_0^\infty t^{k\gamma+r+\gamma-1} e^{-j\beta t^\lambda} dt + \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^i i^k \alpha^k}{i!j!k!} \int_0^\infty t^{k\gamma+r+\lambda-1} e^{-(1+j)\beta t^\lambda} dt \tag{11}$$

Letting $t^\lambda = x$, we have

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^i \alpha^{k+1} \gamma (1+i)^k}{i!j!k!} \int_0^\infty x^{\frac{k\gamma+r+\gamma}{\lambda}-1} e^{-j\beta x} dx + \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^i i^k \alpha^k}{i!j!k!} \int_0^\infty t^{\frac{k\gamma+r+\lambda}{\lambda}-1} e^{-(1+j)\beta x} dx$$

From which the rth moment will be given by

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^i \alpha^{k+1}\gamma(1+i)^k}{i!j!k!} \cdot \frac{\Gamma\left(\frac{k\gamma+r+\gamma}{\lambda}\right)}{(j\beta)^{\frac{k\gamma+r+\gamma}{\lambda}}}$$
$$+ \lambda\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^i i^k \alpha^k}{i!j!k!} \cdot \frac{\Gamma\left(\frac{k\gamma+r+\lambda}{\lambda}\right)}{\{(1+j)\beta\}^{\frac{k\gamma+r+\lambda}{\lambda}}}$$

4 Order Statistics

Let $T_1, T_2, \ldots, T_n$ be a random sample of size n from NMW with *cdf* and *pdf* as defined in (1) and (2), respectively. We now give the density of the ith order statistic $T_{i:n}$, $f_{i:n}(t)$, from the NMW distribution. It is well known that (for $i = 1, 2, \ldots, n$)

$$f_{i:n}(t) = \frac{1}{B(i, n-i+1)} f(t)F(t)^{i-1}\{1-F(t)\}^{n-i}$$

where $B(i, n-i+1)$ is the beta function. From (1) and (2) we can express $f_{i:n}(t)$ as

$$f_{i:n}(t) = \frac{(\alpha\gamma t^{\gamma-1}e^{\alpha t^{\gamma}} + \lambda\beta t^{\lambda-1}e^{-\beta t^{\lambda}})e^{-(n-i+1)(e^{\alpha t^{\gamma}}-e^{-\beta t^{\lambda}})}}{B(i, n-i+1)(1-e^{-e^{\alpha t^{\gamma}}+e^{-\beta t^{\lambda}}})^{1-i}}$$

Using the binomial expansion we can rewrite the density function of the ith order statistic as a finite weighted sum of densities of the NMW distributions

$$f_{i:n}(t) = \frac{1}{B(i, n-i+1)}\alpha\gamma t^{\gamma-1}e^{\alpha t^{\gamma}} + \lambda\beta t^{\lambda-1}e^{-\beta t^{\lambda}}$$
$$\times \sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j e^{-(j+n-i+1)(e^{\alpha t^{\gamma}}-e^{-\beta t^{\lambda}})}$$
$$= \sum_{j=0}^{i-1} w_{i,j}\, g(t)$$

where $g(t)$ is a NMW density function and the weights which are simply given by

$$w_{i,j} = \frac{(-1)^j\binom{i-1}{j}}{(j+n-i+1)B(i, n-i+1)}$$

5 Maximum Likelihood Estimation

Let T_i be a random variable distributed as (1) with the vector of parameters, $\theta = (\alpha, \beta, \gamma, \lambda)^T$. We now determine the maximum likelihood estimates (MLEs) of the parameters of the NMW distribution. Let $t_1, \ldots, t_n$ be observed values of a random sample of size n from the $NMW(\alpha, \beta, \gamma, \lambda)$ distribution. The log-likelihood function for the vector of parameters can be written as:

$$l(\theta) = \sum_{i=1}^{n} \ln(\alpha\gamma {t_i}^{\gamma-1} e^{\alpha {t_i}^{\gamma}} + \lambda\beta {t_i}^{\lambda-1} e^{-\beta {t_i}^{\lambda}}) + \sum_{i=1}^{n} (-e^{\alpha {t_i}^{\gamma}} + e^{-\beta {t_i}^{\lambda}}).$$

The log-likelihood can be maximized directly by using the package R ($nlminb$). The components of the score vector $U(\boldsymbol{\theta})$ are given by

$$U_\alpha(\theta) = \sum_{i=1}^{n} \frac{1 + \alpha t_i^{\gamma}}{\alpha} - \sum_{i=1}^{n} t_i^{\gamma} (e^{\alpha t_i t^{\gamma}} + e^{-\beta t_i^{\lambda}})$$

$$U_\beta(\theta) = \sum_{i=1}^{n} \frac{1 - \beta t_i^{\lambda}}{\beta} + \sum_{i=1}^{n} t_i^{\lambda} (e^{\alpha t_i^{\gamma}} + e^{-\beta t_i^{\lambda}})$$

$$U_\gamma(\theta) = \frac{n}{\gamma} + \sum_{i=1}^{n} \log(t_i)(\alpha t_i^{\gamma} + 1) - \alpha \sum_{i=1}^{n} t_i^{\gamma} e^{\alpha t_i^{\gamma}} \log(t_i)$$

$$U_\lambda(\theta) = \frac{1}{\lambda} - \sum_{i=1}^{n} \log(t_i)(\beta t_i^{\lambda} - 1) - \beta \sum_{i=1}^{n} \frac{t_i^{\lambda} \log(t_i)}{e^{-\beta t_i^{\lambda}}}.$$

The maximum likelihood estimate (MLE) $\hat{\theta}$ of θ is obtained by solving the non-linear likelihood equations $U_\alpha = 0,\ U_\beta = 0,\ U_\gamma = 0$ and $U_\lambda = 0$.
For interval estimation of $\theta = (\alpha, \beta, \gamma, \lambda)$ and tests of hypotheses on these parameters, we obtain the observed information matrix since the information matrix is very complicated and will require numerical integration. The 4×4 observed information matrix $J(\hat{\theta})$ is given by

$$J^{-1}(\hat{\theta}) = \begin{pmatrix} \widehat{L_{\alpha\alpha}} & \widehat{L_{\alpha\beta}} & \widehat{L_{\alpha\gamma}} & \widehat{L_{\alpha\lambda}} \\ . & \widehat{L_{\beta\beta}} & \widehat{L_{\beta\gamma}} & \widehat{L_{\beta\lambda}} \\ . & . & \widehat{L_{\gamma\gamma}} & \widehat{L_{\gamma\lambda}} \\ . & . & . & \widehat{L_{\lambda\lambda}} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha},\hat{\beta}) & \text{cov}(\hat{\alpha},\hat{\gamma}) & \text{cov}(\hat{\alpha},\hat{\lambda}) \\ . & \text{var}(\hat{\beta}) & \text{cov}(\hat{\beta},\hat{\gamma}) & \text{cov}(\hat{\beta},\hat{\lambda}) \\ . & . & \text{var}(\hat{\gamma}) & \text{cov}(\hat{\gamma},\hat{\lambda}) \\ . & . & . & \text{var}(\hat{\lambda}) \end{pmatrix} \quad (12)$$

where, the elements of this matrix are given in the Appendix. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, we have the asymptotic distribution

$$\sqrt{n}(\hat{\theta} - \theta) \sim N_4(0, I^{-1}(\theta))$$

where, $I(\theta)$ is the information matrix. This asymptotic behavior is valid if $I(\theta)$ is replaced by $J(\hat{\theta})$, the observed information matrix evaluated at $\hat{\theta}$. The asymptotic multivariate normal $N_4(0, J^{-1}(\hat{\theta}))$ distribution can be used to construct approximate confidence intervals and confidence regions for the individual parameters and for the hazard rate and survival functions.
We can use the above approach to derive the $100(1-\alpha)\%$ confidence intervals for the parameters $\alpha, \beta, \gamma, \lambda$ in the following forms

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\beta})},$$

$$\hat{\gamma} \pm Z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\gamma})}, \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\lambda})}$$

Here, $Z_{\frac{\alpha}{2}}$ is the upper $100(\frac{\alpha}{2})$ percentile of the standard normal distribution.

6 Simulation Study

For validating the theoretical results reported in Section 5, we carry out simulations by generating n observations from NMW distribution with parametric values θ. The estimated values of the parameters are found using the nlminb method in R package. The considered sample sizes are: $n = 100, 200$ and 500 as 1000 is the number of repetitions. In Table 1, the results from simulated data sets are reported, where the averages of the 1000 MLEs, $av(\hat{\theta})$, standard errors, $se(\hat{\theta})$, and $bias(\hat{\theta})$ are given. We see that convergence has been achieved in all cases, which emphasizes that the numerical stability of the R package, and standard errors of the MLEs decrease as sample size increases.

Table 1. The mean, bias and standard errors of the R package estimator with initial values $\theta^0 = (\alpha^0, \beta^0, \gamma^0, \lambda^0)$ from 1000 samples.

n	θ θ^0	$av(\hat{\theta})$ $bias(\hat{\hat{\theta}})$	$se(\hat{\theta})$
100	(0.4,1,1,0.5) (0.5,0.5,0.5,0.5)	(0.3976,1.0842,1.1013,0.5299) (-0.0023,0.0842,0.1013,0.0299)	(0.1421,0.2794,0.5054,0.3664)
200		(0.4008 ,1.0398,1.0269,0.5027) (0.0008,0.0398,0.0269,0.0027)	(0.0973,0.1872 ,0.3367 ,0.0788)
500		(0.4014,1.0129,1.0159,0.5020) (0.0014,0.0129,0.0159,0.0020)	(0.0639,0.1206,0.2143 ,0.0481)
100	(0.5,0.5,0.5,1.5) (0.1,0.9,0.7,1)	(0.4946,0.5330,0.5621,1.6423) (-0.0053,0.0330,0.0621,0.1423)	(0.0770,0.0807,0.3238,0.8700)
200		(0.4997,0.5143,0.5149,1.5490) (-0.0002,0.0143,0.0149,0.0490)	(0.0528, 0.0455,0.1786,0.4235)
500		(0.5002,0.5037,0.5107,1.5226) (0.0002,0.0037,0.0107,0.0226)	(0.0297 ,0.0245,0.1043,0.2091)
100	(0.2,2,2,2) (1,1,1,1)	(0.1997,2.1312,2.1615,2.0890) (-0.0002,0.1312,0.1615,0.0890)	(0.0835,0.4654,0.7195,0.3251)
200		(0.2018,2.0507,2.0748,2.0407) (0.0018,0.0507,0.0748,0.0407)	(0.0579,0.3076,0.4542,0.2013)
500		(0.2006,2.0244,2.0393,2.0199) (0.0006,0.0244,0.0393,0.01999)	(0.0362,0.1900,0.2599,0.1276)

7 Applications

In this Section we fit NMW distribution to two real data sets and then compare our result with the results of some distributions whose densities are given by

- Modified Beta (MB) distribution

$$f(t) = \frac{\beta^a g(t)\{G(t)\}^{a-1}\{1-G(t)\}^{b-1}}{B(a,b)\{1-(1-\beta)G(t)\}^{a+b}}, \qquad a>0, b>0, \beta>0$$

  where

$$G(t) = 1 - e^{-\alpha t}, \quad g(t) = \alpha e^{-\alpha t}, \qquad t>0,\ \alpha>0$$

  which is the density of the distribution generated by using Beta kernel introduced Nadarajah et al. (2014).

- Generalized Modified Weibull (GMW) distribution

$$f(t) = \frac{\alpha\beta t^{\gamma-1}(\gamma+\lambda t)\exp\{\lambda t - \alpha t^{\gamma}\exp(\lambda t)\}}{[1-\exp\{-\alpha t^{\gamma}\exp(\lambda t)\}]^{1-\beta}},$$
$$t>0, \alpha>0, \gamma>0, \beta>0, \lambda>0$$

  which is the density of the distribution introduced by Carrasco et al. (2008).

- Modified Weibull (MW) distribution

$$f(t) = \alpha t^{\gamma-1}(\gamma+\lambda t)\exp\{\lambda t - \alpha t^{\gamma}\exp(\lambda t)\}, \quad t>0, \alpha>0, \gamma>0, \lambda>0$$

  which is the density of the distribution introduced by Lai et al. (2003).

- Exponentiated Weibull (EW) distribution

$$f(t) = \alpha\beta\gamma t^{\gamma-1}\exp(-\alpha t^{\gamma})\{1-\exp(-\alpha t^{\gamma})\}^{\beta-1},$$
$$t>0, \alpha>0, \gamma>0, \beta>0$$

  which is the density of the distribution introduced by Mudholkar et al. (1995, 1996).

- The Beta Modified Weibull (BMW) distribution

$$f(t) = \frac{1}{B(a,b)} G(t)^{a-1} \{1 - G(t)\}^{b-1} g(t),$$

where

$$G(t) = 1 - \exp\{-\alpha t^{\gamma} \exp(\lambda t)\}$$
$$g(t) = \alpha t^{\gamma-1} (\gamma + \lambda t) \exp\{\lambda t - \alpha t^{\gamma} \exp(\lambda t)\},$$
$$t > 0, \alpha > 0, \gamma > 0, \lambda > 0$$

which is the density of the distribution introduced by Silva et al. (2010).

- The Beta Weibull (BW) distribution

$$f(t) = \frac{1}{B(a,b)} G(t)^{a-1} \{1 - G(t)\}^{b-1} g(t),$$

where,

$$G(t) = 1 - \exp(-\alpha t^{\gamma})$$
$$g(t) = \alpha \gamma t^{\gamma-1} \exp(-\alpha t^{\gamma}), \quad t > 0, \alpha > 0, \gamma > 0$$

which is the density of the distribution introduced by Lee et al. (2007).

- Generalized Rayleigh (GR) distribution

$$f(t) = 2\alpha\beta t \exp(-\alpha t^2) \{1 - \exp(-\alpha t^2)\}^{\beta-1}, \quad t > 0, \alpha > 0, \beta > 0$$

which is the density of the distribution introduced by Kundu and Rakab (2005).

7.1 Application 1

The first data set is given by Aarset (1987), and also reported in Mudholkar and Srivastava (1993); Mudholkar et al. (1996) and Wang (2000), on lifetimes of 50 components, which possess a bathtub-shaped failure rate property. The data are: 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86. In many applications, there is a qualitative information about the failure rate function shape which can help in selecting

a particular model. In this context, a device called the total time on test (TTT) plot (Aarset, 1987) is useful. The TTT plot is obtained by plotting $G(\frac{i}{n}) = \frac{\sum_{i=1}^{i} T_{i:n}}{\sum_{i=1}^{n} T_{i:n}}$ for $i = 1, 2, \ldots, n$ against $\frac{i}{n}$ (Mudholkar et al., 1996), where $T_{i:n}$, for $i = 1, 2, \ldots, n$, are the order statistics of the sample. The TTT plot for the Aarset data in Figure 3a shows a bathtub-shaped hazard rate function. Hence, the NMW distribution could be an appropriate model for the fitting of such data.
Table 2 gives the MLEs of the parameters and the values of the AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) statistics for some models.

Table 2. Estimates of the parameters for some models fitted to the Aarset data (the standard errors are given in parentheses) and the values of the AIC and BIC statistics.

Model	$MLEs$						AIC	BIC
	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$		
NMW	2.7e-20	0.0667	0.6970	10.1423	-	-	434.1	441.8
	(2.0e-19)	(0.0300)	(0.1201)	(1.6294)				
MW	0.0624	0.3548	0.0233	-	-	-	460.3	466.0
	(0.0267)	(0.1127)	(0.0048)					
MB	0.0797	-	-	2.91e-5	0.2748	34.7787	453.4	461.0
	(0.04012)			(1.37e-6)	(0.0726)	(2.5698)		
GMW	0.0002	0.9942	0.0529	0.2975	-	-	455.8	463.4
	(0.0001)	(0.2396)	(0.0183)	(0.0613)				
BMW	0.0002	1.3771	0.0541	-	0.1975	0.1647	451.6	461.2
	(6.69e-5)	(0.3387)	(0.0157)		(0.0462)	(0.0830)		
BW	0.0007	2.3615	-	-	0.18356	0.0748	463.9	471.6
	(0.0004)	(0.1715)			(0.0509)	(0.0353)		
EW	0.0011	1.5936	-	0.4668	-	-	480.5	486.2
	(0.0010)	(0.1858)		(0.0889)				
GR	0.0002	-	-	0.3643	-	-	475.9	479.7
	(4.87e-5)			(0.0624)				

As we can see from these numerical results in Table 2, the AIC and BIC of the NMW model are the smallest among those of the six fitted models, and hence our new model can be chosen as the best model. In order to assess if the model is appropriate, we plot in Figure 3b the empirical survival function and the estimated survival function of the NMW distribution which provides a good fit for the data under analysis. The estimated hazard rate function in Figure 3c is a bathtub-shaped curve.

Further, the plots of the estimated densities and the histogram of the Aarset data given in Figure 3d show that the NMW distribution produces a better fit than the other models.

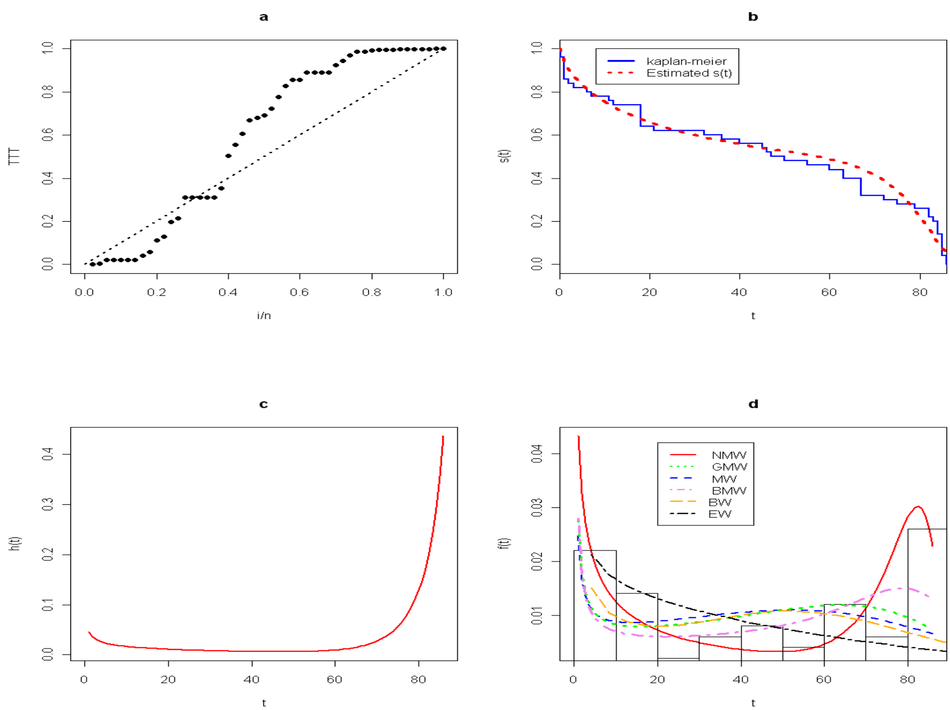

Figure 3. (a) TTT-plot on the Aarset data. (b) Estimated survival function from the fitting of the NMW distribution and the empirical survival function for the Aarset data. (c) Estimated hazard rate function for the Aarset data. (d) Estimated densities of six models fitted to Aarset data.

Substituting the MLE of the unknown parameters in (12), we obtain estimation of the variance covariance matrix as

$$I^{-1}(\hat{\theta}) = \begin{pmatrix} 4.03e-38 & 3.27e-19 & -7.66e-22 & 3.36e-21 \\ 3.27e-19 & 2.6549 & 0.0062 & -0.0274 \\ -7.66e-22 & 0.0062 & 0.0009 & -0.0028 \\ 3.36e-21 & -0.0274 & -0.0028 & 0.0144 \end{pmatrix}.$$

The approximate 95% two sided confidence intervals of the parameters α, β, γ and λ are [0,4.2141e-19], [6.9487,13.3360], [0.0077,0.1257] and [0.4616, 0.9324] respectively.

7.2 Application 2

The second data set is given by Sylwia (2007) on the lifetime of a certain device. The data are: 0.0094, 0.05, 0.4064, 4.6307, 5.1741, 5.8808, 6.3348, 7.1645, 7.2316, 8.2604, 9.2662, 9.3812, 9.5223, 9.8783, 9.9346, 10.0192, 10.4077, 10.4791, 11.076, 11.325, 11.5284, 11.9226, 12.0294, 12.074, 12.1835, 12.3549, 12.5381, 12.8049, 13.4615, 13.853.

The TTT plot for the Sylwia's data in Figure 4a shows a bathtub-shaped hazard rate function and, therefore, indicates the appropriateness of the NMW distribution to fit these data.

The MLEs of the parameters (the standard errors are given in parentheses) and the values of AIC and BIC for the six fitted models are calculated in Table 3. It can be seen from these numerical results, that AIC and BIC of the NMW model are the smallest among those of the six fitted models. Hence our new model is a more suitable model.

Additionally, Kaplan–Meier product limit estimate are plotted in Figure 4b. It can be seen that the NMW distribution is a very competitive model for describing the bathtub- shaped failure rate of the Sylwia data. Figure 4c shows that the estimated hazard rate function has a bathtub-shaped. Further, the plots of the estimated densities and the histogram of the Sylwia data given in Figure 4d shows that the NMW distribution produces a better fit than the other models.

Table 3. Estimates of the parameters for some models fitted to the Sylwia (2007) data (the standard errors are given in parentheses) and the values of the AIC and BIC statistics.

Model	$MLEs$						AIC	BIC
	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$		
NMW	3.54e-5	0.0855	0.2786	4.0734	-	-	146.3	151.9
	(6.98e-5)	(0.0529)	(0.1551)	0.7679)				
MW	0.01813	0.20611	0.33156	-	-	-	152.3	156.5
	(0.0126)	(0.1395)	(0.0590)					
MB	0.3911	-	-	6.27e-3	0.6005	136.8678	155.8	161.4
	(0.1459)			(7.61e-3)	(0.2104)	(3.7582)		
GMW	9.77e-10	1.5623	1.270	0.1510	-	-	148.7	154.3
	(1.62e-9)	(1.0029)	(0.273)	(0.0405)				
BMW	2.89e-3	1.2133	0.4893	-	0.2143	0.0623	149.9	156.9
	(4.89e-3)	(0.7160)	(0.1393)		(0.0513)	(0.0024)		
BW	1.74e-4	5.5351	-	-	0.0587	9.23e-3	149.5	155.1
	(2.83e-4)	(0.6425)			(0.0021)	(0.0004)		
EW	4.69e-6	4.7603	-	0.2442	-	-	174.0	178.2
	(9.85e-6)	(0.7839)		(0.0623)				
GR	7.30e-3	-	-	0.5768	-	-	183.4	186.2
	(2.02e-3)			(0.1236)				

Substituting the MLE of the unknown parameters in (12), we obtain estimation of the variance covariance matrix as

$$I^{-1}(\theta) = \begin{pmatrix} 4.87e-9 & -5.36e-5 & -4.73e-7 & -2.48e-6 \\ -5.36e-5 & 0.5928 & 0.0051 & 0.0267 \\ -4.73e-7 & 0.0051 & 0.0027 & -0.0016 \\ -2.48e-6 & 0.0267 & -0.0016 & 0.0240 \end{pmatrix}.$$

The approximate 95% two sided confidence intervals of the parameters α, β, γ and λ are: [0,1.723e-4], [2.5643,5.5826], [0,0.1892] and [0,0.5827], respectively.

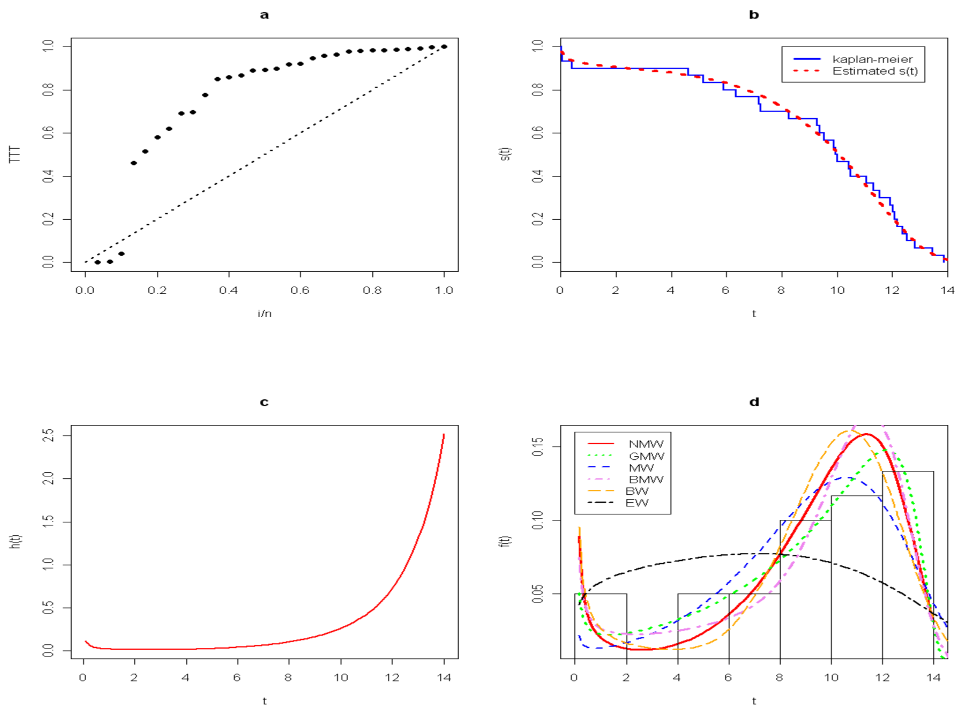

Figure 4. TTT-plot on the Sylwia data. (b) Estimated survival function from the fitting of the NMW distribution and the empirical survival function for the Sylwia data. (c) Estimated hazard rate function for the Sylwia data. (d) Estimated densities of six models fitted to Sylwia data.

8 Conclusions

We have introduce a four parameter lifetime distribution called a new modified Weibull (NMW) distribution. The new model is much more flexible than the Modified Beta distribution (MB), exponentiated Weibull (EW), modified Weibull (MW), beta Weibull (BW), beta modified Weibull (BMW) and generalized modified Weibull (GMW). The proposed distribution could have increasing, decreasing, bathtub, unimodal, increasing-decreasing-increasing and decreasing-increasing-decreasing hazard rate functions also the distribution has decreasing, unimodal and bimodal *pdf*. It is then useful to model lifetime with a bathtub-shaped hazard rate function. We had provide a mathematical treatment for this distribution including the moment and the density of the order statistics. The asymptotic properties of the MLEs of the unknown parameters are obtained. Two application of the NMW distribution to two real data sets is provided, which show that this distribution provides a better fit than other models.

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Appendix

The elements of the observed information matrix $J(\theta)$ for the parameters $(\alpha, \gamma, \lambda, \beta)$ are:

$$L_{\alpha\alpha} = -\sum_{i=1}^{n} \left[\frac{t_i^{2\beta} \exp(\alpha t_i^{\beta})}{\{\gamma\lambda t_i^{\lambda} + \alpha\beta t_i^{\beta} \exp(\alpha t_i^{\beta}) \exp(\gamma t_i^{\lambda})\}^2} \right.$$
$$\left. \times \{\gamma^2\lambda^2 t_i^{2\lambda} + \beta^2 \exp(\alpha t_i^{\beta}) \exp(2\gamma t_i^{\lambda}) + \alpha^2\beta^2 t_i^{2\beta} \exp(2\alpha t_i^{\beta}) \exp(2\gamma t_i^{\lambda})\} \right]$$
$$+ \sum_{i=1}^{n} \frac{t_i^{2\beta} t_i^{\lambda} \exp(\alpha t_i^{\beta}) \exp(\gamma t_i^{\lambda}) \{2\beta\gamma\lambda + \alpha\beta\gamma\lambda t_i^{\beta} - 2\alpha\beta\gamma\lambda t_i^{\beta} \exp(\alpha t_i^{\beta})\}}{\{\gamma\lambda t_i^{\lambda} + \alpha\beta t_i^{\beta} \exp(\alpha t_i^{\beta}) \exp(\gamma t_i^{\lambda})\}^2}$$

$$L_{\alpha\beta} = L_{\beta\alpha} = \sum_{i=1}^{n}\Bigg[\frac{\gamma\lambda t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$

$$\times\{-2\alpha^2\beta t_i^{2\beta}\exp(\alpha t_i^{\beta})\log(t_i)-2\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta})\log(t_i)+1\}\Bigg]$$

$$+\sum_{i=1}^{n}\Bigg[\frac{\gamma\lambda t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$

$$\times\{\beta\log(t_i)+\alpha t_i^{\beta}+3\alpha\beta t_i^{\beta}\log(t_i)+\alpha^2\beta t_i^{2\beta}\log(t_i)\}\Bigg]$$

$$-\sum_{i=1}^{n}\frac{t_i^{\beta}\exp(\alpha t_i^{\beta})\log(t_i)\{\alpha^3\beta^2 t_i^{3\beta}\exp(2\alpha t_i^{\beta}+2\gamma t_i^{\lambda})+\alpha\gamma^2\lambda^2 t_i^{\beta}t_i^{2\lambda}\}}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$

$$-\sum_{i=1}^{n}\Bigg[\frac{t_i^{\beta}\exp(\alpha t_i^{\beta})\log(t_i)}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}2}$$

$$\times\{\gamma^2\lambda^2 t_i^{2\lambda}-\alpha^2\beta^2 t_i^{2\beta}\exp(\alpha t_i^{\beta}+2\gamma t_i^{\lambda})+\alpha^2\beta^2 t_i^{2\beta}\exp(2\alpha t_i^{\beta}+2\gamma t_i^{\lambda})\}\Bigg]$$

$$L_{\alpha\gamma} = L_{\gamma\alpha} = \sum_{i=1}^{n}\frac{\beta\lambda t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})(\alpha t_i^{\beta}+1)(\gamma t_i^{\lambda}-1)}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$

$$L_{\alpha\lambda} = L_{\lambda\alpha} = -\sum_{i=1}^{n}\Bigg[\frac{\beta\gamma t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})(\alpha t_i^{\beta}+1)}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$

$$\times\{\lambda\log(t_i)-\gamma\lambda t_i^{\lambda}\log(t_i)+1\}\Bigg]$$

$$L_{\beta\beta} = \sum_{i=1}^{n}\Bigg[\frac{\alpha\gamma\lambda t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\log(t_i)}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$

$$\times\{-2\alpha^2\beta t_i^{2\beta}\exp(\alpha t_i^{\beta})\log(t_i)-2\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta})\log(t_i)+2\}\Bigg]$$

$$+\sum_{i=1}^{n}\Bigg[\frac{\alpha\gamma\lambda t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$

$$\times\{\beta\log(t_i)+2\alpha t_i^{\beta}+3\alpha\beta t_i^{\beta}\log(t_i)+\alpha^2\beta t_i^{2\beta}\log(t_i)\}\Bigg]$$

$$-\sum_{i=1}^{n}\Bigg[\frac{\alpha t_i^{\beta}\exp(\alpha t_i^{\beta})}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$
$$\times\{\alpha^2\beta^2 t_i^{2\beta}\exp(2\alpha t_i^{\beta}+2\gamma t_i^{\lambda})\log(t_i)^2+\alpha^3\beta^2 t_i^{3\beta}\exp(2\alpha t_i^{\beta}+2\gamma t_i^{\lambda})\log(t_i)^2\}\Bigg]$$
$$-\sum_{i=1}^{n}\Bigg[\frac{\alpha t_i^{\beta}\exp(\alpha t_i^{\beta})}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$
$$\times\{\alpha t_i^{\beta}\exp(\alpha t_i^{\beta}+2\gamma t_i^{\lambda})+\gamma^2\lambda^2 t_i^{2\lambda}\log(t_i)^2-\alpha^2\beta^2 t_i^{2\beta}\exp(\alpha t_i^{\beta}+2\gamma t_i^{\lambda})\log(t_i)^2\}\Bigg]$$
$$-\sum_{i=1}^{n}\frac{\alpha t_i^{\beta}\exp(\alpha t_i^{\beta})\{\alpha\gamma^2\lambda^2 t_i^{\beta}t_i^{2\lambda}\log(t_i)^2)}{(\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$

$$L_{\beta\gamma}=L_{\gamma\beta}=\sum_{i=1}^{n}\Bigg[\frac{\alpha\lambda t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})(\gamma t_i^{\lambda}-1)}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$
$$\times\{\beta\log(t_i)+\alpha\beta t_i^{\beta}\log(t_i)+1\}\Bigg]$$

$$L_{\beta\lambda}=L_{\lambda\beta}=-\sum_{i=1}^{n}\Bigg[\frac{\alpha\gamma t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$
$$\times\{\beta\log(t_i)+\alpha\beta t_i^{\beta}\log(t_i)+1)(\lambda\log(t_i)-\gamma\lambda t_i^{\lambda}\log(t_i)+1\}\Bigg]$$

$$L_{\gamma\gamma}=\sum_{i=1}^{n}\frac{t_i^{2\lambda}}{\exp(\gamma t_i^{\lambda})}$$
$$-\sum_{i=1}^{n}\frac{\lambda t_i^{2\lambda}(\lambda+2\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})-\alpha\beta\gamma t_i^{\beta}t_i^{\lambda}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda}))}{(\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda}))^2}$$

$$L_{\gamma\lambda}=L_{\lambda\gamma}=\sum_{i=1}^{n}\Bigg[\frac{\alpha\beta t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta})}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}$$
$$\times\{-2\gamma\lambda t_i^{\lambda}\log(t_i)-3\gamma\lambda t_i^{\lambda}+2\gamma^2\lambda t_i^{2\lambda}\log(t_i)+\gamma^2\lambda t_i^{2\lambda}\exp(\gamma t_i^{\lambda})\log(t_i)\}\Bigg]$$

$$
+\sum_{i=1}^{n}\Bigg[\frac{\alpha\beta t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta})}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}
$$

$$
\times\{-3\gamma\lambda t_i^{\lambda}\exp(\gamma t_i^{\lambda})\log(t_i)+\exp(\gamma t_i^{\lambda})+\lambda\exp(\gamma t_i^{\lambda})\log(t_i)-\gamma t_i^{\lambda}\exp(\gamma t_i^{\lambda})\}\Bigg]
$$

$$
-\sum_{i=1}^{n}\Bigg[\frac{t_i^{\lambda}\log(t_i)\exp(-\gamma t_i^{\lambda})}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}
$$

$$
\times\{\gamma^2\lambda^2 t_i^{2\lambda}-\gamma^3\lambda^2 t_i^{3\lambda}+\alpha^2\beta^2 t_i^{2\beta}\exp(2\alpha t_i^{\beta}+2\gamma t_i^{\lambda})+\gamma^2\lambda^2 t_i^{2\lambda}\exp(\gamma t_i^{\lambda})\}\Bigg]
$$

$$
+\sum_{i=1}^{n}\Bigg[\frac{t_i^{\lambda}\log(t_i)\exp(-\gamma t_i^{\lambda})}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}
$$

$$
\times\{\alpha^2\beta^2\gamma t_i^{2\beta}t_i^{\lambda}\exp(2\alpha t_i^{\beta}+2\gamma t_i^{\lambda})\}\Bigg]
$$

$$
L_{\lambda\lambda}=\sum_{i=1}^{n}\frac{\gamma t_i^{\lambda}\log(t_i)^2(\gamma t_i^{\lambda}-1)}{\exp(\gamma t_i^{\lambda})}
$$

$$
+\sum_{i=1}^{n}\frac{\beta\{-2\alpha\gamma^2\lambda t_i^{\beta}t_i^{2\lambda}\exp(\alpha t_i^{\beta})\log(t_i)^2(\gamma t_i^{\lambda}-1)\}-\gamma^2 t_i^{2\lambda}\{\gamma\lambda^2 t_i^{\lambda}\log(t_i)^2+1\}}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}
$$

$$
+\sum_{i=1}^{n}\Bigg[\frac{\alpha\beta\gamma t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta})\log(t_i)}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}
$$

$$
\times\{-2\gamma\lambda t_i^{\lambda}\log(t_i)+2\gamma^2\lambda t_i^{2\lambda}\log(t_i)+\gamma^2\lambda t_i^{2\lambda}\exp(\gamma t_i^{\lambda})\log(t_i)\}\Bigg]
$$

$$
-\sum_{i=1}^{n}\frac{\alpha\beta\gamma t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta})\log(t_i)\{3\gamma\lambda t_i^{\lambda}\exp(\gamma t_i^{\lambda})\log(t_i)\}}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}
$$

$$
+\sum_{i=1}^{n}\Bigg[\frac{\alpha\beta\gamma t_i^{\beta+\lambda}\exp(\alpha t_i^{\beta})\log(t_i)}{\{\gamma\lambda t_i^{\lambda}+\alpha\beta t_i^{\beta}\exp(\alpha t_i^{\beta}+\gamma t_i^{\lambda})\}^2}
$$

$$
\times\{2\exp(\gamma t_i^{\lambda})+\lambda\exp(\gamma t_i^{\lambda})\log(t_i)-2\gamma t_i^{\lambda}\exp(\gamma t_i^{\lambda})\}\Bigg]
$$

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