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Estimation for the Type-II Extreme Value Distribution Based on Progressive Type-II Censoring

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Abstract. In this paper, we discuss the statistical inference on the unknown parameters and reliability function of type-II extreme value (EVII) distribution when the observed data are progressively type-II censored. By applying EM algorithm, we obtain maximum likelihood estimates (MLEs). We also suggest approximate maximum likelihood estimators (AMLEs), which have explicit expressions. We provide Bayes estimates using both the symmetric and asymmetric loss functions via squared error loss, LINEX loss, and general entropy loss functions. Bayes estimates are obtained using the idea of Lindley and Markov chain Monte Carlo techniques. Finally, Monte Carlo simulations are presented to illustrate the methods discussed in this paper. Analysis is also carried out for a real data set.

Keywords. Approximate maximum likelihood estimators; Bayes estimates; EM algorithm; Lindley's approximation; Monte Carlo simulation; progressive type-II censoring.

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1 Introduction

Data arising from life-testing and reliability experiments are often censored based on cost and time considerations. Among the different censoring schemes, the progressive censoring scheme has received a specific attention in the last

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few years, particularly in reliability analysis. The statistical inference on the parameters of failure time distributions under progressive type-II censoring schemes has been studied by sevral authors, such as Cohen (1963), Mann (1971), Viveros and Balakrishnan (1994), Balakrishnan and Aggarwala (2000) and Balakrishnan (2007). These schemes of censoring are the most popular censoring schemes which are used in practice. A type-II progressively right censored scheme defined as follows. Suppose that n units are placed on a life test and the experimenter decides beforehand the quantity m, the number of units to be failed. Suppose a censoring scheme $(R_1, \ldots, R_m)$ is prefixed such that following the first failure, denoted with $X_{1:m:n}$, R_1 surviving units are randomly chosen and removed from the experiment. Similarly after the second failure $(X_{2:m:n})$, R_2 surviving items are removed at random from the $n - R_1 - 2$ remaining items, and so on. The procedure is continued until all R_m remaining units are removed after the m-th failure $(X_{m:m:n})$; $n = m + \sum_{i=1}^{m} R_i$. We will denote the m order observed failure times by $X_{1:m:n}, \ldots, X_{m:m:n}$.

In the recent years, the extreme value distribution has received considerable attention in engineering statistics as an appropriate model to represent phenomena with usually large maximum observations. Often, in engineering circles this distribution is called the Frechet model. It is one of the pioneers of extreme value statistics. The type-II extreme value (EVII) is one of the probability distributions used to model extreme events. The generalization of the standard EVII has been introduced by Nadarajah and Kotz (2003) and Abd-Elfattah and Omima (2009). There are over fifty applications ranging from accelerated life testing through to earthquakes, floods, rain fall, queues in supermarkets, sea currents, wind speeds and track race records, see Kotz and Nadarajah (2000). Recently, several articles have been published on estimating of the unknown parameters for different distribution functions. See, for example Balakrishnan et al. (2004), Kim et al. (2011), Mubarak (2012), Rastogi and Tripathi (2012), etc. In the present article, we consider the type-II progressively censored lifetime data, when the lifetime follows type-II extreme value distribution. First, we try to earn the maximum likelihood estimates (MLEs) of the unknown parameters. It is observed that the MLEs can be obtained implicitly by solving two nonlinear equations, but they cannot be obtained in closed form. The EM algorithm is suggested to determine the MLEs which involves solving two one-dimensional optimization problems. Since the MLEs do not have explicit forms, the approximate maximum likelihood estimators (AMLEs) are proposed which have explicit expressions (see Balakrishnan and Varadan, 1991). Furthermore, we consider the Bayes estimates under the assumptions of independent gamma priors on the scale and shape parameters. Based on these priors, the Bayes estimates can not be obtained explicitly, so we obtain the Bayes estimates using Lindley's approximation and Markov chain Monte Carlo (MCMC) techniques. In order to compare the performances of these methods we use Monte Carlo simulations and for illustrative purposes we have analyzed one real data set. This paper is organized as follows. In Section 2 we describe the model and provide the MLEs of the unknown parameters. The AMLEs are presented in Section 3. The Bayes estimates for the shape and scale parameters α and β, respectively, are derived in Section 4. Simulation results and discussions are provided in Section 5 and finally the article concludes in Section 6.

2 Maximum Likelihood Estimators

The form of the probability density function (pdf) and cumulative distribution function (cdf) of EVII distribution, with shape parameter α and scale parameter β are given, respectively, by

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left\{-\left(\frac{\beta}{x}\right)^{\alpha}\right\}, \qquad \alpha,\beta,x>0, \tag{1}$$

$$F(x;\alpha,\beta)=\exp\left\{-\left(\frac{\beta}{x}\right)^{\alpha}\right\}, \qquad x>0. \tag{2}$$

We denote a type-II extreme value distribution with the pdf (1) by EVII(α,β). Also, the reliability function of the EVII(α,β) distribution, for $t>0$ becomes:

$$R(t)=1-\exp\left\{-\left(\frac{\beta}{t}\right)^{\alpha}\right\}.$$

Let $X_{1:m:n},\ldots,X_{m:m:n}$ be a progressively type-II censored sample from EVII(α,β) distribution with censoring scheme $(R_1,\ldots,R_m)$. By considering $\theta=\beta^{\alpha}$, the likelihood function based on the progressive type-II censored sample is given by

$$L(\alpha,\theta|\mathbf{x})=C\alpha^m\theta^m\exp\left\{-\theta\sum_{i=1}^{m}x_{i:m:n}^{-\alpha}\right\}\prod_{i=1}^{m}x_{i:m:n}^{-(\alpha+1)}\prod_{i=1}^{m}\left[1-\exp\{-\theta x_{i:m:n}^{-\alpha}\}\right]^{R_i} \tag{3}$$

where $C = \prod_{i=1}^{m} \gamma_i$ is a normalizing constant and

$$\gamma_1 = n, \quad \gamma_i = n - \sum_{k=1}^{i-1} R_k - i + 1, \qquad i = 2, \ldots, m.$$

Then the log-likelihood function is proportion to

$$l(\alpha, \theta|\mathbf{x}) = \log L(\alpha, \theta|\mathbf{x}) \propto m \ln \alpha + m \ln \theta - (\alpha + 1) \sum_{i=1}^{m} \ln x_{i:m:n} - \theta \sum_{i=1}^{m} x_{i:m:n}^{-\alpha} + \sum_{i=1}^{m} R_i \ln \left[1 - \exp\{-\theta x_{i:m:n}^{-\alpha}\}\right]. \tag{4}$$

To obtain the MLEs of α and θ, say $\hat{\alpha}$ and $\hat{\theta}$, we derivative from (4) with respect to α and θ and we set these derivatives equal to zero.

$$\frac{\partial l(\alpha, \theta|\mathbf{x})}{\partial \alpha} = \frac{m}{\alpha} + \theta \sum_{i=1}^{m} x_{i:m:n}^{-\alpha} \ln x_{i:m:n} - \theta \sum_{i=1}^{m} R_i \ln x_{i:m:n} \frac{x_{i:m:n}^{-\alpha} \exp\{-\theta x_{i:m:n}^{-\alpha}\}}{1 - \exp\{-\theta x_{i:m:n}^{-\alpha}\}} - \sum_{i=1}^{m} \ln x_{i:m:n} = 0, \tag{5}$$

$$\frac{\partial l(\alpha, \theta|\mathbf{x})}{\partial \theta} = \frac{m}{\theta} - \sum_{i=1}^{m} x_{i:m:n}^{-\alpha} + \sum_{i=1}^{m} R_i \frac{x_{i:m:n}^{-\alpha} \exp\{-\theta x_{i:m:n}^{-\alpha}\}}{1 - \exp\{-\theta x_{i:m:n}^{-\alpha}\}} = 0. \tag{6}$$

MLEs can be secured by solving nonlinear equations of (5) and (6), but they cannot be obtained in closed forms. So MLEs of parameters are derived numerically. We use the EM algorithm to compute the MLEs of α and θ which involves solving two one-dimensional optimization problems rather than one two-dimensional problem. Also, we can get MLEs of β and $R(t)$, say $\hat{\beta}$ and $\hat{R}(t)$, by using the invariance property of MLEs.

2.1 EM Algorithm

In the previous section, it was observed that MLEs of α and β can be obtained by solving two nonlinear normal equations, whose explicit solutions cannot be obtained. In this subsection, we propose to use the EM algorithm to compute the MLEs of α and β as suggested by Dempster et al. (1977). This way is a very powerful tool for handling the incomplete data problem. In the EM algorithm method, we use the likelihood function of complete data. First, let us symbolize the observed and the censored data by $\mathbf{X} = (X_{1:m:n}, \ldots, X_{m:m:n})$ and $\mathbf{Z} = (Z_1, \ldots, Z_m)$, respectively, where each Z_i is a $1 \times R_i$ vector with $Z_i = (Z_{i1}, \ldots, Z_{iR_i})$ for $i = 1, \ldots, m$ and they are not observable. The combination of $\mathbf{W} = (\mathbf{X}, \mathbf{Z})$ forms the whole data set. For the E-step of the EM algorithm, one needs to compute the pseudo log-likelihood function. It can be obtained from the likelihood function of complete data by replacing any function of Z_{ik}, say $g(Z_{ik})$, with $E\left[g(Z_{ik})|Z_{ik} > x_{i:m:n}\right]$. Therefore, the pseudo log-likelihood function becomes

$$\begin{aligned} l_c^*(\alpha, \theta|\mathbf{w}) = {} & n \ln \alpha + n \ln \theta - \theta \sum_{i=1}^{m} x_{i:m:n}^{-\alpha} \\ & - \theta \sum_{i=1}^{m} \sum_{k=1}^{R_i} E\left(Z_{ik}^{-\alpha}|Z_{ik} > x_{i:m:n}, \alpha_{(h)}, \theta_{(h)}\right) \\ & - (\alpha+1) \sum_{i=1}^{m} \ln x_{i:m:n} \\ & - (\alpha+1) \sum_{i=1}^{m} \sum_{k=1}^{R_i} E\left(\ln Z_{ik}|Z_{ik} > x_{i:m:n}, \alpha_{(h)}, \theta_{(h)}\right), \end{aligned} \tag{7}$$

where $\alpha_{(h)}$ and $\theta_{(h)}$ are the hth iteration values of the parameters α and θ, respectively. In the M-step of the $(h+1)$th iteration, $(\alpha_{(h+1)}, \theta_{(h+1)})$ can be obtained by maximizing (7). The conditional expectations in (7) are obtained using the result that given $X_{i:m:n} = x_{i:m:n}$, $Z_i^{\prime}s$ have a left-truncated distribution F, truncated at $x_{i:m:n}$. The conditional probability density of $\mathbf{Z}$, given $\mathbf{x}$, is as follows (see Ng et al., 2002):

$$f_{\mathbf{Z}|\mathbf{X}}\left(\mathbf{z}|\mathbf{x}; \alpha, \theta\right) = \prod_{i=1}^{m} \prod_{k=1}^{R_i} f_{Z_{ik}|X_{i:m:n}}\left(z_{ik}|x_{i:m:n}; \alpha, \theta\right),$$

where

$$f_{Z_{ik}|X_{i:m:n}}(z_{ik}|x_{i:m:n}; \alpha, \theta) = \frac{\alpha\theta z_{ik}^{-(\alpha+1)} \exp\{-\theta z_{ik}^{-\alpha}\}}{1 - \exp\{-\theta x_{i:m:n}^{-\alpha}\}}, \qquad z_{ik} > x_{i:m:n}.$$

It can be seen that

$$E\left(Z_{ik}^{-\alpha}|Z_{ik}>x_{i:m:n},\alpha_{(h)},\theta_{(h)}\right)=\frac{\theta_{(h)}}{1-\exp\{-\theta_{(h)}x_{i:m:n}^{-\alpha_{(h)}}\}} \times\sum_{j=0}^{\infty}\frac{(-1)^j\theta_{(h)}^j x_{i:m:n}^{-\alpha-(j+1)\alpha_{(h)}}}{(j+\frac{\alpha}{\alpha_{(h)}}+1)\Gamma(j+1)}, \quad (8)$$

$$E\left(\ln Z_{ik}|Z_{ik}>x_{i:m:n},\alpha_{(h)},\theta_{(h)}\right)=\ln x_{i:m:n}-\frac{\theta_{(h)}}{\alpha_{(h)}\left[1-\exp\{-\theta_{(h)}x_{i:m:n}^{-\alpha_{(h)}}\}\right]} \times\sum_{j=0}^{\infty}\frac{(-1)^{j+1}\theta_{(h)}^j x_{i:m:n}^{-(j+1)\alpha_{(h)}}}{(j+1)\Gamma(j+2)}, \quad (9)$$

$$E\left(Z_{ik}^{-\alpha}\ln Z_{ik}|Z_{ik}>x_{i:m:n},\alpha_{(h)},\theta_{(h)}\right)= \frac{\theta_{(h)}x_{i:m:n}^{-(\alpha+\alpha_{(h)})}}{\alpha_{(h)}\left[1-\exp\{-\theta_{(h)}x_{i:m:n}^{-\alpha_{(h)}}\}\right]}\sum_{j=0}^{\infty}\left[\frac{(-1)^j\theta_{(h)}^j}{\Gamma(j+1)}\times x_{i:m:n}^{-j\alpha_{(h)}}\left\{\frac{1}{(j+\frac{\alpha}{\alpha_{(h)}}+1)^2} +\frac{\alpha_{(h)}\ln x_{i:m:n}}{(j+\frac{\alpha}{\alpha_{(h)}}+1)}\right\}\right], \quad (10)$$

where $\Gamma(\cdot)$ is gamma function. Note that the maximization of (7) can be earned easily as follows.
First, $\alpha_{(h+1)}$ can be obtain by solving the equation $h(\alpha)=\alpha$, where $h(\alpha)$ is defined as

$$h(\alpha)=n\left\{\sum_{i=1}^{m}\ln x_{i:m:n}-\hat{\theta}(\alpha)\left(\sum_{i=1}^{m}x_{i:m:n}^{-\alpha}\ln x_{i:m:n}+\tilde{C}\right)+\tilde{A}\right\}^{-1}$$

and

$$\tilde{A}=\sum_{i=1}^{m}R_iE\left(\ln Z_{ik}|Z_{ik}>x_{i:m:n},\alpha_{(h)},\theta_{(h)}\right),$$
$$\tilde{C}=\sum_{i=1}^{m}R_iE\left(Z_{ik}^{-\alpha}\ln Z_{ik}|Z_{ik}>x_{i:m:n},\alpha_{(h)},\theta_{(h)}\right).$$

Also, MLE of θ can be obtained by solving the equation $\dfrac{\partial l_c^*(\alpha,\theta|\mathbf{w})}{\partial\theta}=0$, in the following form

$$\hat{\theta}(\alpha)=\frac{n}{\sum_{i=1}^m x_{i:m:n}^{-\alpha}+\tilde{B}},$$
$$\tilde{B}=\sum_{i=1}^m R_i E\left(Z_{ik}^{-\alpha}|Z_{ik}>x_{i:m:n},\alpha_{(h)},\theta_{(h)}\right),$$

Once $\alpha_{(h+1)}$ is found, $\theta_{(h+1)}$ is obtained from $\theta_{(h+1)}=\hat{\theta}(\alpha_{(h+1)})$. Therefore, we can use the following algorithm to proceed from the hth iterate to $(h+1)$th iterate.
The resultant estimates of α, β and $R(t)$ via EM algorithm are thereafter referred as $\hat{\alpha}_{EM}$, $\hat{\beta}_{EM}$ and $\hat{R}_{EM}(t)$ respectively in this paper.

3 Approximate Maximum Likelihood Estimators

We use the following notations; $\mu=\ln\beta$, $\sigma=1/\alpha$, $Y_{i:m:n}=\ln X_{i:m:n}$ and $Z_{i:m:n}=(Y_{i:m:n}-\mu)/\sigma$. Then, the likelihood function based on progressive Type-II right censored sample $Z_{1:m:n},Z_{2:m:n},\ldots,Z_{m:m:n}$ can be written as

$$L(\mu,\sigma|\mathbf{z})\propto\sigma^{-m}\prod_{i=1}^m g(z_{i:m:n})\left[1-G(z_{i:m:n})\right]^{R_i},$$

where $G(z)=\exp\{-e^{-z}\}$, $g(z)=exp\{-z-e^{-z}\}$. Upon differentiation of the logarithm of the likelihood function with respect to μ and σ, the score equations to be solved for μ and σ in this case are given by

$$\frac{\partial l(\mu,\sigma|\mathbf{z})}{\partial\mu}=\frac{1}{\sigma}\sum_{i=1}^m R_i\frac{g(z_{i:m:n})}{1-G(z_{i:m:n})}-\frac{1}{\sigma}\sum_{i=1}^m\frac{g'(z_{i:m:n})}{g(z_{i:m:n})}=0 \qquad (11)$$

and

$$\frac{\partial l(\mu,\sigma|\mathbf{z})}{\partial\sigma}=\frac{m}{\sigma}+\frac{1}{\sigma}\sum_{i=1}^m z_{i:m:n}\frac{g'(z_{i:m:n})}{g(z_{i:m:n})}-\frac{1}{\sigma}\sum_{i=1}^m R_i z_{i:m:n}\frac{g(z_{i:m:n})}{1-G(z_{i:m:n})}=0. \qquad (12)$$

The likelihood equations (11) and (12) do not admit explicit solutions. So, we expand the functions $h_1(z_{i:m:n})=\dfrac{g'(z_{i:m:n})}{g(z_{i:m:n})}$ and $h_2(z_{i:m:n})=\dfrac{g(z_{i:m:n})}{1-G(z_{i:m:n})}$

in a Taylor series around the point $v_{i:m:n} = E(Z_{i:m:n})$, where

$$v_{i:m:n} \approx G^{-1}(\eta_i) = -\ln(-\ln \eta_i),$$

and $\eta_i = 1 - \prod_{j=m-i+1}^{m} \frac{j+R_{m-j+1}+\cdots+R_m}{j+1+R_{m-j+1}+\cdots+R_m}$, see Balakrishnan and Aggarwala (2000, pp. 81-83).

We may then consider the following approximations:

$$h_1(z_{i:m:n}) \approx h_1(v_{i:m:n}) + h_1'(v_{i:m:n})(z_{i:m:n} - v_{i:m:n}) = \alpha_i - \beta_i z_{i:m:n}, \quad (13)$$
$$h_2(z_{i:m:n}) \approx h_2(v_{i:m:n}) + h_2'(v_{i:m:n})(z_{i:m:n} - v_{i:m:n}) = \delta_i + \gamma_i z_{i:m:n}, \quad (14)$$

where, for $i = 1, 2, \ldots, m$,

$$\alpha_i = h_1(v_{i:m:n}) - v_{i:m:n} h_1'(v_{i:m:n}), \quad (15)$$
$$\beta_i = -h_1'(v_{i:m:n}), \quad (16)$$
$$\delta_i = h_2(v_{i:m:n}) - v_{i:m:n} h_2'(v_{i:m:n}), \quad (17)$$
$$\gamma_i = h_2'(v_{i:m:n}). \quad (18)$$

By using (13)-(18), the approximate likelihood equations for μ and σ can be written as

$$\frac{\partial l(\mu,\sigma|\mathbf{z})}{\partial \mu} \approx \sum_{i=1}^{m} R_i \left(\delta_i + \gamma_i z_{i:m:n}\right) - \sum_{i=1}^{m} \left(\alpha_i - \beta_i z_{i:m:n}\right) = 0, \quad (19)$$
$$\frac{\partial l(\mu,\sigma|\mathbf{z})}{\partial \sigma} \approx m - \sum_{i=1}^{m} R_i z_{i:m:n} \left(\delta_i + \gamma_i z_{i:m:n}\right) + \sum_{i=1}^{m} z_{i:m:n} \left(\alpha_i - \beta_i z_{i:m:n}\right) = 0. \quad (20)$$

Replacing $(y_{i:m:n} - \mu)/\sigma$ by $z_{i:m:n}$ in (19) and (20), we have

$$\tilde{\mu}(\sigma) = A - B(\sigma), \quad (21)$$

where

$$A = \frac{\sum_{i=1}^{m}(\beta_i + R_i\gamma_i) y_{i:m:n}}{\sum_{i=1}^{m}(\beta_i + R_i\gamma_i)}, \qquad B = \frac{\sum_{i=1}^{m}(\alpha_i - R_i\delta_i)}{\sum_{i=1}^{m}(\beta_i + R_i\gamma_i)}.$$

By considering

$$C = \sum_{i=1}^{m} (\alpha_i - R_i \delta_i)(y_{i:m:n} - A),$$
$$D = \sum_{i=1}^{m} (\beta_i + R_i \gamma_i)(y_{i:m:n} - A)^2,$$

We obtain the AMLE of σ from (20) as

$$\tilde{\sigma} = \frac{-C + \sqrt{(C^2 + 4mD)}}{2m},$$

which is the only positive root. Also, we can get AMLEs of α, β and reliability function of $R(t)$, say $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{R}(t)$ respectively, by using the invariance property of MLEs.

4 Bayes Estimation

In this section we are concerned on the Bayes estimates of the parameters α and β and the reliability function of the EVII distribution. For computing the Bayes estimates, we use different symmetric and asymmetric loss functions. The loss function plays a critical role in Bayesian perspective. Most authors use the simple symmetric loss function and obtain the posterior mean as the Bayes estimate. However, in practice, the real loss function is often not symmetric. For example, the consequences of overestimates, in loss of human life and reliability estimation are much more serious than the consequences of underestimates. Thus, an asymmetric loss function might be more appropriate. A very well known symmetric loss function is the squared error which is given by

$$L_1\left(f(\mu), \hat{f}(\mu)\right) = \left(\hat{f}(\mu) - f(\mu)\right)^2,$$

with $\hat{f}(\mu)$ begin an estimate of $f(\mu)$. Here $f(\mu)$ denotes some parametric function of μ. Bayes estimate, say $\hat{f}_{SB}(\mu)$ is evaluated by the posterior mean of $f(\mu)$. The most popular asymmetric loss function, called the LINEX (Linear-Exponential) loss function has been suggested by Varian (1975), and is the simple generalization of squared error loss function. It is defined as follows

$$L_2\left(f(\mu), \hat{f}(\mu)\right) = e^{c\Delta} - c\Delta - 1, \qquad c \neq 0,$$

where $\Delta = \hat{f}(\mu) - f(\mu)$. Using the loss function L_2 , the Bayes estimator of $f(\mu)$, can be expressed as $\hat{f}_{LB}(\mu) = -\frac{1}{c}\ln\left\{E_\mu\left(e^{-cf(\mu)}|\mathbf{x}\right)\right\}$, provided $E_\mu(\cdot)$ exists and is finite. The general entropy loss function, which is a generalization of the entropy loss function, is another useful asymmetric loss function and is given by

$$L_3\left(f(\mu), \hat{f}(\mu)\right) \propto \left(\frac{\hat{f}(\mu)}{f(\mu)}\right)^q - q\ln\left(\frac{\hat{f}(\mu)}{f(\mu)}\right) - 1, \qquad q \neq 0.$$

The Bayes estimate of $f(\mu)$ under general entropy loss function is obtained as

$$\hat{f}_{EB}(\mu) = \left[E\left\{(f(\mu))^{-q}|\mathbf{x}\right\}\right]^{-\frac{1}{q}},$$

provided $E_\mu(\cdot)$ exists and is finite. It is assumed that α and β have the following independent gamma priors:

$$\pi_1(\alpha|c_1, d_1) \propto \alpha^{c_1-1}e^{-\alpha d_1}, \qquad \pi_2(\beta|c_2, d_2) \propto \beta^{c_2-1}e^{-\beta d_2}, \qquad c_1, d_1, c_2, d_2 > 0. \tag{22}$$

So, the bivariate prior density function of α and β is of the form

$$\pi(\alpha, \beta) \propto \alpha^{c_1-1}e^{-\alpha d_1}\beta^{c_2-1}e^{-\beta d_2}. \tag{23}$$

Then by applying (3), the joint posterior density function of α and β given $\mathbf{x}$ is as

$$\pi(\alpha, \beta|\mathbf{x}) = \frac{1}{k}\alpha^{m+c_1-1}\beta^{m\alpha+c_2-1}\left\{\prod_{i=1}^{m} x_{i:m:n}^{-(\alpha+1)}\left(1 - e^{-s_i}\right)^{R_i}\right\} \times e^{-\left(\sum_{i=1}^{m} s_i + \alpha d_1 + \beta d_2\right)}, \tag{24}$$

where $s_i = \left(\frac{\beta}{x_{i:m:n}}\right)^\alpha$ for $i = 1, \ldots, m$ and

$$k = \int_0^\infty \int_0^\infty \alpha^{m+c_1-1}\beta^{m\alpha+c_2-1}\left\{\prod_{i=1}^{m} x_{i:m:n}^{-(\alpha+1)}\left(1 - e^{-s_i}\right)^{R_i}\right\} \times e^{-\left(\sum_{i=1}^{m} s_i + \alpha d_1 + \beta d_2\right)}\, d\alpha\, d\beta.$$

Therefore, under the loss function L_1, the Bayes estimate of any function of α and β, say $g = g(\alpha, \beta)$ is as

$$\hat{g}_{SB} = \int_0^\infty \int_0^\infty g(\alpha, \beta)\pi(\alpha, \beta|\mathbf{x})\, d\alpha\, d\beta. \tag{25}$$

Similarly, for the loss function L_2, we have

$$\hat{g}_{LB} = -\frac{1}{c} \ln \left\{ E\left(e^{-cg(\alpha,\beta)} |\mathbf{x} \right) \right\}, \tag{26}$$

where $E\left(e^{-cg(\alpha,\beta)}|\mathbf{x}\right) = \int_0^\infty \int_0^\infty e^{-cg(\alpha,\beta)}\pi(\alpha, \beta|\mathbf{x})\, d\alpha\, d\beta$, and under the loss function L_3, the Bayes estimate of $g(\alpha, \beta)$ is obtained as

$$\hat{g}_{EB} = \left[E\left\{ (g(\alpha, \beta))^{-q} |\mathbf{x} \right\} \right]^{-\frac{1}{q}}, \tag{27}$$

where $E\left[(g(\alpha, \beta))^{-q} |\mathbf{x}\right] = \int_0^\infty \int_0^\infty \{g(\alpha, \beta)\}^{-q} \pi(\alpha, \beta|\mathbf{x})\, d\alpha\, d\beta$.
As these estimators can not be evaluated explicitly, so we adopt Lindley's approximation and MCMC methods to approximate them.

4.1 Lindley's Approximation Method

It is easily observed that the Bayes estimators have not explicit closed forms. Therefore, in such a situation, we resort to the use of a numeric integration technique such as Lindley's approximation. An approximate procedure has been developed for the evaluation of the ratio of two integrals by Lindley (1980). This approach has been used by several authors to obtain the approximate Bayes estimates (for details, see Lindley, 1980). Suppose

$$I(\mathbf{x}) = \frac{\int_0^\infty \int_0^\infty h(\alpha, \beta) e^{l(\alpha,\beta|\mathbf{x})}\pi(\alpha, \beta)\, d\alpha\, d\beta}{\int_0^\infty \int_0^\infty e^{l(\alpha,\beta|\mathbf{x})}\pi(\alpha, \beta)\, d\alpha\, d\beta}, \tag{28}$$

where $\pi(\alpha, \beta)$ is the joint prior density function and $h(\alpha, \beta)$ is any function of α and β. Consider $(\lambda_1, \lambda_2) = (\alpha, \beta)$, then utilizing the Lindley's method $I(x)$ can be approximated as

$$\hat{I} = h(\lambda_1, \lambda_2) + \frac{1}{2}(A + l^*_{30}B_{12} + l^*_{03}B_{21} + l^*_{21}C_{12} + l^*_{12}C_{21}) + p_1 A_{12} + p_2 A_{21}, \tag{29}$$

where $p = \ln \pi(\lambda_1, \lambda_2)$,

$$A = \sum_{i=1}^{2}\sum_{j=1}^{2} h_{ij}\tau_{ij}, \quad l^*_{\eta\xi} = \frac{\partial^{\eta+\xi} l(\lambda_1, \lambda_2|\mathbf{x})}{\partial\lambda_1^{\eta}\partial\lambda_2^{\xi}}, \qquad \eta, \xi = 0, 1, 2, 3, \quad \eta + \xi = 3,$$

and for $i, j = 1, 2$, $p_i = \dfrac{\partial p}{\partial \lambda_i}$, $h_i = \dfrac{\partial h(\lambda_1, \lambda_2)}{\partial \lambda_i}$, $h_{ij} = \dfrac{\partial^2 h(\lambda_1, \lambda_2)}{\partial\lambda_i \partial\lambda_j}$, $\tau_{ij} =$ (i, j)th elements of the inverse matrix $[l_{ij}]$ such that $l_{ij} = -\dfrac{\partial^2 l(\lambda_1, \lambda_2|\mathbf{x})}{\partial\lambda_i\partial\lambda_j}$, and for $i \neq j$,

$$A_{ij} = h_i\tau_{ii} + h_j\tau_{ji}, \qquad B_{ij} = (h_i\tau_{ii} + h_j\tau_{ij})\tau_{ii},$$

$$C_{ij} = 3h_i\tau_{ii}\tau_{ij} + h_j(\tau_{ii}\tau_{jj} + 2\tau_{ij}^2),$$

where $l(\lambda_1, \lambda_2|\mathbf{x})$ is the log-likelihood function of the observed data. Expression (29) is to be evaluated at the maximum likelihood estimates of λ_1 and λ_2. So, we get

$$l_{11} = \frac{1}{\alpha^2}\left[m + \sum_{i=1}^{m} s_i \ln^2 s_i - \sum_{i=1}^{m} R_i e^{-s_i} s_i \ln^2 s_i \left\{\frac{1 - s_i}{1 - e^{-s_i}} - \frac{s_i e^{-s_i}}{(1 - e^{-s_i})^2}\right\}\right],$$

$$l_{12} = l_{21} = -\frac{1}{\beta}\left[m - \sum_{i=1}^{m} s_i(1 + \ln s_i) + \sum_{i=1}^{m} R_i s_i e^{-s_i}\right.$$
$$\left.\times\left\{\frac{1 + (1 - s_i)\ln s_i}{1 - e^{-s_i}} - \frac{s_i \ln s_i e^{-s_i}}{(1 - e^{-s_i})^2}\right\}\right],$$

$$l_{22} = \frac{\alpha}{\beta^2}\left[m + (\alpha - 1)\sum_{i=1}^{m} s_i - \sum_{i=1}^{m} R_i s_i e^{-s_i}\left\{\frac{(\alpha - 1) - \alpha s_i}{1 - e^{-s_i}} - \frac{\alpha s_i e^{-s_i}}{(1 - e^{-s_i})^2}\right\}\right],$$

$$l^*_{03} = \frac{\alpha}{\beta^3}\left[2m - (\alpha - 1)(\alpha - 2)\sum_{i=1}^{m} s_i + \sum_{i=1}^{m} R_i s_i e^{-s_i}\left\{\frac{2\alpha^2 s_i^2 e^{-2s_i}}{(1 - e^{-s_i})^3}\right.\right.$$
$$\left.\left.- \frac{3\alpha s_i e^{-s_i}}{(1 - e^{-s_i})^2}\cdot\{\alpha(1 - s_i) - 1\} + \frac{\alpha^2(s_i^2 - 3s_i + 1) - 3\alpha(1 - s_i) + 2}{1 - e^{-s_i}}\right\}\right],$$

$$l^*_{12} = -\frac{1}{\beta^2}\left[m + \sum_{i=1}^{m} s_i\left\{2\alpha - 1 + (\alpha - 1)\ln s_i\right\} - \sum_{i=1}^{m} R_i s_i e^{-s_i}\right.$$

$$\left\{\frac{2\alpha s_i^2 e^{-2s_i}\ln s_i}{(1-e^{-s_i})^3} - \frac{s_i e^{-s_i}}{(1-e^{-s_i})^2}\left[2\alpha + \ln s_i\left\{3\alpha(1-s_i)-1\right\}\right] + \frac{1}{1-e^{-s_i}}\right.$$

$$\left.\left.\times\left[(1-s_i)(2\alpha-\ln s_i) + \alpha\ln s_i\left\{(1-s_i)^2 - s_i\right\} - 1\right]\right\}\right],$$

$$l_{21}^* = -\frac{1}{\beta\alpha}\left[\sum_{i=1}^{m}(1+\ln s_i)s_i\ln s_i - \sum_{i=1}^{m}R_i s_i e^{-s_i}\ln s_i\left\{\frac{2s_i^2e^{-2s_i}\ln s_i}{(1-e^{-s_i})^3}\right.\right.$$

$$-\frac{s_ie^{-s_i}}{(1-e^{-s_i})^2}\cdot\left\{3(1-s_i)\ln s_i + 2\right\} + \frac{1}{1-e^{-s_i}}[2(1-s_i)$$

$$\left.\left.+\left\{(1-s_i)^2 - s_i\right\}\ln s_i]\right\}\right],$$

$$l_{30}^* = \frac{1}{\alpha^3}\left[2m - \sum_{i=1}^{m}s_i\ln^3 s_i + \sum_{i=1}^{m}R_i s_i e^{-s_i}\ln^2 s_i\left\{\frac{2s_i^2e^{-2s_i}\ln s_i}{(1-e^{-s_i})^3}\right.\right.$$

$$\left.\left.-\frac{3s_ie^{-s_i}}{(1-e^{-s_i})^2}\cdot(1-s_i)\ln s_i + \frac{(1-s_i)^2\ln s_i - s_i\ln s_i}{1-e^{-s_i}}\right\}\right],$$

using prior density (23), we obtain

$$p_1 = \frac{c_1-1}{\alpha} - d_1, \qquad p_2 = \frac{c_2-1}{\beta} - d_2.$$

With the above defined expressions, we now get the approximate Bayesian estimates. The Bayes estimates of α and β under the squared error loss function L_1 can be obtained respectively, as

$$\hat{\alpha}_{SB} = \hat{\alpha} + \left[\frac{1}{2}\left\{l_{30}^*\tau_{11}^2 + l_{03}^*\tau_{21}\tau_{22} + 3l_{21}^*\tau_{11}\tau_{12} + l_{12}^*(\tau_{11}\tau_{22} + 2\tau_{21}^2)\right\} + p_1\tau_{11}\right.$$

$$\left. + p_2\tau_{12}\right]_{\alpha=\hat{\alpha},\beta=\hat{\beta}}, \tag{30}$$

$$\hat{\beta}_{SB} = \hat{\beta} + \left[\frac{1}{2}\left\{l_{30}^*\tau_{11}\tau_{12} + l_{03}^*\tau_{22}^2 + l_{21}^*(\tau_{11}\tau_{22} + 2\tau_{12}^2) + 3l_{12}^*\tau_{21}\tau_{22}\right\} + p_1\tau_{21}\right.$$

$$\left. + p_2\tau_{22}\right]_{\alpha=\hat{\alpha},\beta=\hat{\beta}}. \tag{31}$$

Also, under the loss function L_2, the Bayes estimate of α is given by

$$\hat{\alpha}_{LB} = -\frac{1}{c}\ln\left\{e^{-c\alpha}\left[1+\frac{c^2\tau_{11}}{2}-\frac{c}{2}\{l^*_{30}\tau^2_{11}+l^*_{03}\tau_{21}\tau_{22}+l^*_{12}(\tau_{11}\tau_{22}+2\tau^2_{21}) + 3l^*_{21}\tau_{11}\tau_{12}\} - c(p_1\tau_{11}+p_2\tau_{12})\right]\right\}_{\alpha=\hat{\alpha},\beta=\hat{\beta}}, \tag{32}$$

and for β we have

$$\hat{\beta}_{LB} = -\frac{1}{c}\ln\left\{e^{-c\beta}\left[1+\frac{c^2\tau_{22}}{2}-\frac{c}{2}\{l^*_{30}\tau_{11}\tau_{12}+l^*_{03}\tau^2_{22}+l^*_{21}(\tau_{11}\tau_{22}+2\tau^2_{12}) + 3l^*_{12}\tau_{21}\tau_{22}\} - c(p_1\tau_{21}+p_2\tau_{22})\right]\right\}_{\alpha=\hat{\alpha},\beta=\hat{\beta}}. \tag{33}$$

Finally, for the loss function L_3, $\hat{\alpha}_{EB} = \{E(\alpha^{-q}|\mathbf{x})\}^{-\frac{1}{q}}$, where

$$E(\alpha^{-q}|\mathbf{x}) = \hat{\alpha}^{-q} - \left[\frac{q}{2}\alpha^{-(q+1)}\left\{l^*_{30}\tau^2_{11}+l^*_{03}\tau_{21}\tau_{22}+3l^*_{21}\tau_{11}\tau_{12} + l^*_{12}(\tau_{11}\tau_{22}+2\tau^2_{21}) - \frac{1}{\alpha}(1+q)\tau_{11}\right\} - p_1A_{12} - p_2A_{21}\right]_{\alpha=\hat{\alpha},\beta=\hat{\beta}}, \tag{34}$$

and $\hat{\beta}_{EB} = \{E(\beta^{-q}|\mathbf{x})\}^{-\frac{1}{q}}$, where

$$E(\beta^{-q}|\mathbf{x}) = \hat{\beta}^{-q} - \left[\frac{q}{2}\beta^{-(q+1)}\left\{l^*_{30}\tau_{11}\tau_{12}+l^*_{03}\tau^2_{22}+3l^*_{12}\tau_{21}\tau_{22} + l^*_{21}(\tau_{11}\tau_{22}+2\tau^2_{21}) - \frac{1}{\beta}(1+q)\tau_{22}\right\} - p_1A_{12} - p_2A_{21}\right]_{\alpha=\hat{\alpha},\beta=\hat{\beta}}. \tag{35}$$

By considering $h(\alpha,\beta) = 1-e^{-s}$ and

$$h_1 = \frac{1}{\alpha}se^{-s}\ln s, \qquad h_2 = \left(\frac{\alpha}{\beta}\right)se^{-s},$$

$$h_{11} = \frac{1}{\alpha^2}s(1-s)e^{-s}\ln^2 s, \qquad h_{22} = \frac{\alpha s e^{-s}}{\beta^2}\{\alpha(1-s)-1\},$$

$$h_{12} = h_{21} = \frac{s}{\beta}e^{-s}\{1+(1-s)\ln s\},$$

where $s = \left(\frac{\beta}{t}\right)^{\alpha}$.

Table 1. The average biases and MSEs of the non-Bayesian estimates of α, β and $R(t)$ (MSEs reported in second line for any scheme).

									$t=1$			$t=2$		
n	m	*Scheme*	$\hat{\alpha}$	$\tilde{\alpha}$	$\hat{\alpha}_{EM}$	$\hat{\beta}$	$\tilde{\beta}$	$\hat{\beta}_{EM}$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}_{EM}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}_{EM}(t)$
20	10	$(0^{*9}, 10)$	0.2601	0.2777	0.2601	0.0054	0.0134	0.0054	-0.0187	-0.0143	-0.0186	-0.0309	-0.0304	-0.0309
			0.3368	0.3514	0.3367	0.0314	0.0324	0.0314	0.0127	0.0128	0.0127	0.0126	0.0128	0.0126
	10	$(10, 0^{*9})$	0.2042	0.1675	0.1601	0.0157	0.0413	0.0285	-0.0168	-0.0034	-0.0103	-0.0214	-0.0076	0.0013
			0.2585	0.2354	0.2379	0.0453	0.0510	0.0483	0.0148	0.0141	0.0141	0.0136	0.0138	0.0064
	10	(1^{*10})	0.2426	0.2349	0.2268	0.0093	0.0253	0.0130	-0.0181	-0.0094	-0.0161	-0.0283	-0.0217	-0.0248
			0.3001	0.2933	0.2901	0.0357	0.0385	0.0364	0.0131	0.0128	0.0129	0.0131	0.0131	0.0130
	15	$(0^{*14}, 5)$	0.1512	0.1586	0.1512	0.0190	0.0311	0.0190	-0.0035	0.0030	-0.0035	-0.0140	-0.0108	-0.0140
			0.1535	0.1568	0.1535	0.0289	0.0305	0.0289	0.0091	0.0092	0.0091	0.0085	0.0087	0.0085
	15	$(5, 0^{*14})$	0.1467	0.1296	0.1178	0.0240	0.0433	0.0306	-0.0029	0.0073	0.0003	-0.0119	-0.0025	-0.0052
			0.1481	0.1409	0.1399	0.0350	0.0383	0.0363	0.0105	0.0103	0.0102	0.0094	0.0095	0.0095
	15	$(0^{*5}, 1^{*5}, 0^{*5})$	0.1557	0.1420	0.1557	0.0208	0.0398	0.0208	-0.0033	0.0069	-0.0032	-0.0138	-0.0051	-0.0138
			0.1613	0.1543	0.1614	0.0300	0.0328	0.0300	0.0097	0.0096	0.0097	0.0089	0.0091	0.0089
30	10	$(0^{*9}, 20)$	0.2660	0.2831	0.2658	-0.0077	-0.0050	-0.0077	-0.0287	-0.0273	-0.0287	-0.0370	-0.0383	-0.0131
			0.3401	0.3538	0.3400	0.0277	0.0280	0.0277	0.0122	0.0123	0.0122	0.0132	0.0134	0.0132
	10	$(20, 0^{*9})$	0.1729	0.1323	0.0543	0.0033	0.0278	0.0474	-0.0254	-0.0123	-0.0038	-0.0224	-0.0084	0.0050
			0.2145	0.1926	0.1765	0.0442	0.0491	0.0555	0.0149	0.0139	0.0130	0.0135	0.0135	0.0058
	10	(2^{*10})	0.2264	0.2230	0.1749	-0.0046	0.0052	0.0098	-0.0267	-0.0213	-0.0187	-0.0312	-0.0274	-0.0189
			0.2789	0.2770	0.2516	0.0300	0.0312	0.0321	0.0124	0.0122	0.0116	0.0126	0.0127	0.0125
	20	$(0^{*19}, 10)$	0.1123	0.1204	0.1123	0.0144	0.0220	0.0144	-0.0019	0.0021	-0.0019	-0.0105	-0.0090	-0.0105
			0.1099	0.1127	0.1099	0.0191	0.0198	0.0191	0.0061	0.0062	0.0061	0.0063	0.0064	0.0063
	20	$(10, 0^{*19})$	0.1029	0.0878	0.0353	0.0141	0.0281	0.0309	-0.0041	0.0032	0.0036	-0.0098	-0.0025	0.0066
			0.0965	0.0920	0.0926	0.0238	0.0255	0.0265	0.0072	0.0071	0.0068	0.0069	0.0070	0.0032
	20	$(0^{*5}, 1^{*10}, 0^{*5})$	0.1154	0.1062	0.1154	0.0120	0.0255	0.0120	-0.0043	0.0028	-0.0043	-0.0121	-0.0059	-0.0121
			0.1051	0.1026	0.1051	0.0203	0.0218	0.0203	0.0066	0.0065	0.0066	0.0066	0.0067	0.0066

Table 2. The average biases and risk values of the Bayes estimates of α (risk values reported in second line for any scheme).

			Lindley's approximation						MCMC					
			Prior 1			Prior 2			Prior 1			Prior 2		
n	m	*Scheme*	$\hat{\alpha}_{SB}$	$\hat{\alpha}_{LB}$	$\hat{\alpha}_{EB}$	$\hat{\alpha}_{SB}$	$\hat{\alpha}_{LB}$	$\hat{\alpha}_{EB}$	$\hat{\alpha}_{SB}$	$\hat{\alpha}_{LB}$	$\hat{\alpha}_{EB}$	$\hat{\alpha}_{SB}$	$\hat{\alpha}_{LB}$	$\hat{\alpha}_{EB}$
20	10	$(0^{*9}, 10)$	0.1139	0.0457	0.0429	0.0303	-0.0044	-0.0133	0.1931	0.0975	0.0729	0.1663	0.0939	0.0758
			0.2391	0.1407	0.0384	0.0948	0.0713	0.0242	0.2989	0.1571	0.0406	0.1743	0.0795	0.0249
	10	$(10, 0^{*9})$	0.1087	0.0393	0.0336	0.0472	0.0003	-0.0095	0.1763	0.0920	0.0717	0.1544	0.0896	0.0737
			0.2045	0.1105	0.0335	0.0916	0.0574	0.0210	0.2497	0.1295	0.0351	0.1515	0.0707	0.0223
	10	(1^{*10})	0.1141	0.0448	0.0410	0.0386	-0.0019	-0.0107	0.1885	0.0967	0.0731	0.1633	0.0934	0.0759
			0.2198	0.1198	0.0358	0.0919	0.0618	0.0225	0.2730	0.1371	0.0378	0.1629	0.0742	0.0236
	15	$(0^{*14}, 5)$	0.0657	0.0197	0.0136	0.0368	-0.0018	-0.0088	0.1379	0.0835	0.0693	0.1259	0.0796	0.0674
			0.1217	0.0627	0.0221	0.0783	0.0418	0.0166	0.1515	0.0774	0.0237	0.1110	0.0538	0.0177
	15	$(5, 0^{*14})$	0.0741	0.0264	0.0199	0.0463	0.0055	-0.0020	0.1417	0.0873	0.0736	0.1294	0.0833	0.0716
			0.1210	0.0560	0.0221	0.0786	0.0401	0.0164	0.1497	0.0741	0.0238	0.1102	0.0527	0.0178
	15	$(0^{*5}, 1^{*5}, 0^{*5})$	0.0707	0.0221	0.0159	0.0393	-0.0008	-0.0083	0.1422	0.0852	0.0705	0.1295	0.0814	0.0691
			0.1281	0.0680	0.0230	0.0791	0.0435	0.0169	0.1598	0.0838	0.0247	0.1146	0.0558	0.0182
30	10	$(0^{*9}, 20)$	0.1105	0.0422	0.0398	0.0221	-0.0100	-0.0187	0.1792	0.0780	0.0481	0.1558	0.0811	0.0608
			0.2361	0.1357	0.0384	0.0890	0.0677	0.0237	0.2957	0.1492	0.0412	0.1680	0.0746	0.0241
	10	$(20, 0^{*9})$	0.0943	0.0306	0.0238	0.0457	-0.0013	-0.0111	0.1483	0.0720	0.0522	0.1325	0.0734	0.0583
			0.1760	0.0938	0.0297	0.0857	0.0505	0.0189	0.2109	0.1087	0.0313	0.1310	0.0613	0.0200
	10	(2^{*10})	0.0993	0.0347	0.0305	0.0314	-0.0076	-0.0163	0.1612	0.0725	0.0462	0.1424	0.0751	0.0568
			0.2053	0.1178	0.0337	0.0884	0.0614	0.0214	0.2535	0.1330	0.0362	0.1498	0.0689	0.0220
	20	$(0^{*19}, 10)$	0.0481	0.0146	0.0088	0.0326	0.0021	-0.0041	0.1052	0.0670	0.0566	0.0979	0.0638	0.0545
			0.0920	0.0487	0.0173	0.0685	0.0364	0.0140	0.1109	0.0590	0.0185	0.0877	0.0447	0.0149
	20	$(10, 0^{*19})$	0.0531	0.0188	0.0125	0.0392	0.0075	0.0008	0.1047	0.0671	0.0570	0.0982	0.0647	0.0558
			0.0834	0.0428	0.0157	0.0626	0.0320	0.0126	0.1001	0.0516	0.0169	0.0796	0.0397	0.0136
	20	$(0^{*5}, 1^{*10}, 0^{*5})$	0.0530	0.0178	0.0117	0.0369	0.0047	-0.0016	0.1089	0.0690	0.0581	0.1016	0.0663	0.0567
			0.0876	0.0438	0.0167	0.0651	0.0328	0.0134	0.1063	0.0534	0.0181	0.0842	0.0412	0.0144

Table 3. The average biases and risk values of the Bayes estimates of β (risk values reported in second line for any scheme).

			Lindley's approximation						MCMC					
			Prior 1			Prior 2			Prior 1			Prior 2		
n	m	*Scheme*	$\hat{\beta}_{SB}$	$\hat{\beta}_{LB}$	$\hat{\beta}_{EB}$	$\hat{\beta}_{SB}$	$\hat{\beta}_{LB}$	$\hat{\beta}_{EB}$	$\hat{\beta}_{SB}$	$\hat{\beta}_{LB}$	$\hat{\beta}_{EB}$	$\hat{\beta}_{SB}$	$\hat{\beta}_{LB}$	$\hat{\beta}_{EB}$
20	10	$(0^{*9}, 10)$	0.0447	0.0299	0.0168	0.0418	0.0274	0.0149	0.0913	0.0620	0.0448	0.0600	0.0427	0.0291
			0.0385	0.0183	0.0151	0.0286	0.0132	0.0115	0.0525	0.0225	0.0168	0.0315	0.0148	0.0122
	10	$(10, 0^{*9})$	0.0673	0.0430	0.0232	0.0544	0.0318	0.0134	0.1207	0.0808	0.0589	0.0785	0.0551	0.0373
			0.0592	0.0268	0.0206	0.0357	0.0154	0.0135	0.0798	0.0333	0.0230	0.0422	0.0193	0.0153
	10	(1^{*10})	0.0524	0.0353	0.0206	0.0470	0.0304	0.0166	0.0999	0.0679	0.0496	0.0658	0.0468	0.0324
			0.0450	0.0212	0.0168	0.0313	0.0142	0.0121	0.0609	0.0260	0.0188	0.0350	0.0164	0.0132
	15	$(0^{*14}, 5)$	0.0348	0.0214	0.0098	0.0306	0.0175	0.0062	0.0697	0.05180	0.0384	0.0564	0.0423	0.0307
			0.0314	0.0152	0.0129	0.0248	0.0117	0.0105	0.0392	0.0185	0.0142	0.0291	0.0140	0.0114
	15	$(5, 0^{*14})$	0.0460	0.0291	0.0150	0.0391	0.0228	0.0092	0.0855	0.0632	0.0475	0.0675	0.0507	0.0372
			0.0393	0.0188	0.0154	0.0287	0.0134	0.0119	0.0499	0.0233	0.0171	0.0347	0.0166	0.0132
	15	$(0^{*5}, 1^{*5}, 0^{*5})$	0.0411	0.0269	0.0147	0.0367	0.0227	0.0109	0.0763	0.0568	0.0427	0.0608	0.0459	0.0338
			0.0334	0.0160	0.0136	0.0263	0.0124	0.0110	0.0418	0.0195	0.0150	0.0305	0.0145	0.0119
30	10	$(0^{*9}, 20)$	0.0504	0.0381	0.0266	0.0492	0.0371	0.0262	0.1021	0.0665	0.0501	0.0567	0.0395	0.0270
			0.0382	0.0184	0.0144	0.0273	0.0124	0.0103	0.0570	0.0228	0.0165	0.0280	0.0129	0.0107
	10	$(20, 0^{*9})$	0.0678	0.0443	0.0245	0.0548	0.0328	0.0144	0.1216	0.0795	0.0581	0.0738	0.0501	0.0327
			0.0612	0.0279	0.0210	0.0356	0.0151	0.0131	0.0840	0.0344	0.0236	0.0411	0.0187	0.0150
	10	(2^{*10})	0.0519	0.0377	0.0246	0.0488	0.0348	0.0224	0.1012	0.0666	0.0495	0.0593	0.0411	0.0278
			0.0406	0.0193	0.0153	0.0287	0.0129	0.0108	0.0585	0.0238	0.0174	0.0302	0.0139	0.0115
	20	$(0^{*19}, 10)$	0.0286	0.0195	0.0112	0.0267	0.0176	0.0095	0.0518	0.0402	0.0310	0.0444	0.0346	0.0263
			0.0207	0.0100	0.0089	0.0178	0.0085	0.0077	0.0246	0.0118	0.0097	0.0202	0.0097	0.0083
	20	$(10, 0^{*19})$	0.0329	0.0206	0.0098	0.0294	0.0173	0.0066	0.0625	0.0473	0.0357	0.0518	0.0392	0.0287
			0.0264	0.0126	0.0110	0.0212	0.0100	0.0090	0.0325	0.0154	0.0121	0.0248	0.0119	0.0099
	20	$(0^{*5}, 1^{*10}, 0^{*5})$	0.0298	0.0201	0.0112	0.0278	0.0181	0.0094	0.0534	0.0406	0.0308	0.0446	0.0340	0.0252
			0.0224	0.0109	0.0096	0.0189	0.0090	0.0082	0.0269	0.0128	0.0105	0.0213	0.0102	0.0087

Table 4. The average biases and risk values of the Bayes estimates of $R(t)$ for $t = 1$ (risk values reported in second line for any scheme).

			Lindley's approximation						MCMC					
			Prior 1			Prior 2			Prior 1			Prior 2		
n	m	*Scheme*	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$
20	10	$(0^{*9}, 10)$	-0.0128	-0.0173	-0.0289	-0.0044	-0.0085	-0.0187	0.0006	-0.0041	-0.0167	0.0019	-0.0022	-0.0128
			0.0099	0.0050	0.0189	0.0067	0.0034	0.0110	0.0111	0.0055	0.0200	0.0084	0.0042	0.0139
	10	$(10, 0^{*9})$	-0.0121	-0.0186	-0.0347	-0.0050	-0.0109	-0.0264	0.0056	-0.0006	-0.0182	0.0062	0.0008	-0.0136
			0.0119	0.0060	0.0239	0.0077	0.0039	0.0144	0.0131	0.0065	0.0244	0.0097	0.0048	0.0164
	10	(1^{*10})	-0.0125	-0.0176	-0.0304	-0.0043	-0.0088	-0.0207	0.0016	-0.0035	-0.0174	0.0029	-0.0015	-0.0132
			0.0103	0.0052	0.0198	0.0069	0.0035	0.0118	0.0115	0.0058	0.0207	0.0087	0.0044	0.0143
	15	$(0^{*14}, 5)$	-0.0075	-0.0113	-0.0199	-0.0055	-0.0092	-0.0177	0.0077	0.0039	-0.0055	0.0078	0.0043	-0.0041
			0.0078	0.0039	0.0126	0.0062	0.0031	0.0100	0.0087	0.0043	0.0129	0.0073	0.0036	0.0105
	15	$(5, 0^{*14})$	-0.0064	-0.0110	-0.0213	-0.0044	-0.0088	-0.0191	0.0108	0.0064	-0.0051	0.0106	0.0066	-0.0035
			0.0090	0.0045	0.0149	0.0070	0.0035	0.0115	0.0100	0.0050	0.0151	0.0082	0.0041	0.0120
	15	$(0^{*5}, 1^{*5}, 0^{*5})$	-0.0061	-0.0100	-0.0192	-0.0037	-0.0075	-0.0166	0.0089	0.0049	-0.0051	0.0089	0.0053	-0.0037
			0.0083	0.0042	0.0135	0.0065	0.0033	0.0105	0.0093	0.0046	0.0139	0.0077	0.0039	0.0112
30	10	$(0^{*9}, 20)$	-0.0124	-0.0166	-0.0282	-0.0011	-0.0046	-0.0138	-0.0046	-0.0092	-0.0215	-0.0037	-0.0076	-0.0177
			0.0091	0.0045	0.0181	0.0057	0.0028	0.0089	0.0103	0.0051	0.0202	0.0074	0.0037	0.0131
	10	$(20, 0^{*9})$	-0.0144	-0.0209	-0.0377	-0.0069	-0.0128	-0.0287	0.0006	-0.0056	-0.0236	0.0006	-0.0047	-0.0193
			0.01201	0.0060	0.0249	0.0076	0.0038	0.0145	0.0131	0.0065	0.0258	0.0094	0.0047	0.0169
	10	(2^{*10})	-0.0131	-0.0177	-0.0300	-0.0037	-0.0077	-0.0184	-0.0037	-0.0085	-0.0217	-0.0029	-0.0070	-0.0178
			0.0096	0.0048	0.0191	0.0062	0.0031	0.0103	0.0108	0.0054	0.0209	0.0078	0.0039	0.0137
	20	$(0^{*19}, 10)$	-0.0035	-0.0062	-0.0121	-0.0024	-0.0050	-0.0109	0.0067	0.0040	-0.0022	0.0066	0.0041	-0.0017
			0.0054	0.0027	0.0081	0.0047	0.0023	0.0069	0.0060	0.0030	0.0085	0.0053	0.0026	0.0073
	20	$(10, 0^{*19})$	-0.0057	-0.0091	-0.0169	-0.0045	-0.0079	-0.0156	0.0076	0.0042	-0.0041	0.0072	0.0041	-0.0035
			0.0064	0.0032	0.0100	0.0053	0.0027	0.0083	0.0071	0.0035	0.0101	0.0060	0.0030	0.0085
	20	$(0^{*5}, 1^{*10}, 0^{*5})$	-0.0049	-0.0078	-0.0142	-0.0034	-0.0062	-0.0126	0.0051	0.0022	-0.0046	0.0051	0.0024	-0.0039
			0.0058	0.0029	0.0089	0.0049	0.0025	0.0074	0.0064	0.0032	0.0093	0.0055	0.0028	0.0079

Table 5. The average biases and risk values of the Bayes estimates of $R(t)$ for $t = 2$ (risk values reported in second line for any scheme).

			Lindley's approximation						MCMC					
			Prior 1			Prior 2			Prior 1			Prior 2		
n	m	*Scheme*	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$
20	10	$(0^{*9}, 10)$	0.0070	0.0025	-0.0328	0.0133	0.0095	-0.0196	0.0071	0.0035	-0.0338	0.0011	-0.0024	-0.0323
			0.0109	0.0053	0.1231	0.0074	0.0035	0.0603	0.0119	0.0058	0.1419	0.0080	0.0040	0.0891
	10	$(10, 0^{*9})$	0.0121	0.0062	-0.0346	0.0145	0.0092	-0.0296	0.0156	0.0092	-0.0321	0.0077	0.0026	-0.0312
			0.0121	0.0059	0.1278	0.0074	0.0034	0.0684	0.0135	0.0066	0.1448	0.0087	0.0043	0.0897
	10	(1^{*10})	0.0081	0.0033	-0.0339	0.0135	0.0093	-0.0242	0.0090	0.0032	-0.0341	0.0027	-0.0019	-0.0326
			0.0113	0.0055	0.1234	0.0074	0.0035	0.0611	0.0124	0.0061	0.1418	0.0083	0.0041	0.0890
	15	$(0^{*14}, 5)$	0.0067	0.0030	-0.0210	0.0086	0.0051	-0.0185	0.0066	0.0027	-0.0211	0.0039	0.0006	-0.0205
			0.0076	0.0037	0.0653	0.0057	0.0028	0.0465	0.0086	0.0042	0.0739	0.0065	0.0032	0.0555
	15	$(5, 0^{*14})$	0.0086	0.0044	-0.0226	0.0098	0.0058	-0.0209	0.0106	0.0062	-0.0209	0.0070	0.0033	-0.0203
			0.0085	0.0042	0.0718	0.0061	0.0030	0.0510	0.0098	0.0048	0.0802	0.0071	0.0035	0.0591
	15	$(0^{*5}, 1^{*5}, 0^{*5})$	0.0083	0.0044	-0.0214	0.0104	0.0066	-0.0186	0.0082	0.0041	-0.0215	0.0051	0.0015	-0.0208
			0.0080	0.0039	0.0698	0.0059	0.0029	0.0480	0.0090	0.0044	0.0792	0.0067	0.0033	0.0583
30	10	$(0^{*9}, 20)$	0.0087	0.0044	-0.0334	0.0160	0.0124	-0.0182	0.0085	0.0022	-0.0365	0.0002	-0.0017	-0.0362
			0.0114	0.0055	0.1312	0.0076	0.0036	0.0549	0.0124	0.0061	0.1545	0.0079	0.0039	0.0943
	10	$(20, 0^{*9})$	0.0126	0.0068	-0.0344	0.0141	0.0088	-0.0311	0.0169	0.0103	-0.0321	0.0077	0.0025	-0.0321
			0.0122	0.0059	0.1279	0.0075	0.0035	0.0725	0.0137	0.0067	0.1458	0.0086	0.0042	0.0914
	10	(2^{*10})	0.0095	0.0048	-0.0323	0.0147	0.0106	-0.0233	0.0103	0.0042	-0.0338	0.0022	-0.0024	-0.0337
			0.0111	0.0054	0.1235	0.0075	0.0035	0.0591	0.0122	0.0060	0.1442	0.0079	0.0039	0.0905
	20	$(0^{*19}, 10)$	0.0061	0.0034	-0.0141	0.0073	0.0046	-0.0127	0.0046	0.0017	-0.0156	0.0031	0.0005	-0.0152
			0.0058	0.0028	0.0449	0.0047	0.0023	0.0353	0.0064	0.0032	0.0508	0.0052	0.0026	0.0409
	20	$(10, 0^{*19})$	0.0059	0.0026	-0.0177	0.0065	0.0034	-0.0168	0.0073	0.0040	-0.0164	0.0050	0.0020	-0.0163
			0.0064	0.0031	0.0495	0.0050	0.0024	0.0388	0.0072	0.0035	0.0549	0.0057	0.0028	0.0435
	20	$(0^{*5}, 1^{*10}, 0^{*5})$	0.0054	0.0025	-0.0166	0.0068	0.0039	-0.0150	0.0041	0.0010	-0.0180	0.0023	-0.0004	-0.0176
			0.0061	0.0030	0.0479	0.0049	0.0023	0.0373	0.0067	0.0033	0.0541	0.0054	0.0026	0.0431

Now, against the loss function L_1, we can obtain the Bayes estimate of reliability function, $\hat{R}_{SB}(t)$, from (29). Similarly, the Bayes estimates of reliability function under LINEX and general entropy loss functions, say $\hat{R}_{LB}(t)$ and $\hat{R}_{EB}(t)$ respectively can be computed.

4.2 An MCMC Process

Here we study the MCMC process through the application of the Metropolis-Hastings (M-H) algorithm (Hastings, 1970, and Metropolis et al., 1953) to draw samples of α and β, via Gibbs scheme Geman and Geman (1984). Consider $(\lambda_1, \lambda_2) = (\alpha, \beta)$. Hence, the marginal posterior density function of λ_j, $j = 1, 2$, is given as

$$\pi_j(\lambda_j|\mathbf{x}) = \int \pi(\lambda_1, \lambda_2|\mathbf{x})\, d\lambda_k, \qquad k = 1, 2, \quad k \neq j. \tag{36}$$

Given λ_2, the conditional posterior for λ_1 could be written as

$$\pi_1(\lambda|\mathbf{x}, \lambda_2) \propto L(\lambda, \lambda_2|\mathbf{x})\pi(\lambda, \lambda_2),$$

and similarly for given λ_1

$$\pi_2(\lambda|\mathbf{x}, \lambda_1) \propto L(\lambda_1, \lambda|\mathbf{x})\pi(\lambda_1, \lambda),$$

where $L(\cdot, \cdot|\mathbf{x})$ is the likelihood function and $\pi(\cdot, \cdot)$ is equation (23). For $j = 1, 2$, the Markov chain $\lambda_j^{(i)}$, $i = 1, 2, \ldots$, of the jth parameter, λ_j, is constructed by applying the M-H algorithm described as follows.
0. Propose $q_j(\lambda_j^{(*)}|\lambda_j^{(i)})$ as a transition probability from $\lambda_j^{(i)}$ to $\lambda_j^{(*)}$ for $j = 1, 2$. Set $i = 0$ and initial states of $\lambda_j^{(0)}$, $j = 1, 2$, respectively.

1. Let $i = i$ and $j = 1$.

2. Generate $\lambda_j^{(*)}$ from the proposed density $q_j(\lambda_j^{(*)}|\lambda_j^{(i)})$ and u from uniform distribution over $(0, 1)$ interval independently, then for $k = 1, 2$ and $k \neq j$,

$$\lambda_j^{(i+1)} = \begin{cases} \lambda_j^{(*)} & if \quad u \leqslant \min\left\{1, \frac{\pi_j(\lambda_j^{(*)}|\mathbf{x},\lambda_k) q_j(\lambda_j^{(*)}|\lambda_j^{(i)})}{\pi_j(\lambda_j^{(i)}|\mathbf{x},\lambda_k) q_j(\lambda_j^{(i)}|\lambda_j^{(*)})}\right\} \\ \lambda_j^{(i)} & \text{otherwise.} \end{cases} \tag{37}$$

3. Set j=2 and repeat Step 2.

4. Set i=i+1.

5. Repeat Steps 1-4 for a large number, say $i = N + 1$, of periods.

Given $j = 1, 2$, the emprical distribution of λ_j can then be described by the realizations of λ_j from the constructed Markov chain after some burn-in period, N_b. Therefore the Bayes estimates of λ_j and reliability function under the loss function L_1 can be obtained as

$$\hat{\lambda}_{jSB} = \frac{1}{N - N_b} \sum_{i=N_b}^{N} \lambda_j^{(i)}, \tag{38}$$

$$\hat{R}_{SB}(t) = \frac{1}{N - N_b} \sum_{i=N_b}^{N} \exp\left\{ -\left(\frac{\lambda_2^{(i)}}{t}\right)^{\lambda_1^{(i)}} \right\}, \tag{39}$$

respectively. Similarly, the Bayes estimates under LINEX and general entropy loss functions can be computed.

5 Simulations and Data Analysis

It is very difficult to compare the theoretical performances of the different estimates proposed in the previous sections. Therefore, we perform a simulation study to compare the performances of the different methods. In the simulation study performance of all estimates have been compared numerically in terms of their bias values. We also compare the non-Bayesian and Bayes estimates in terms of mean squred error (MSE) and risk values, respectively. The simulation is performed for different choices of n, m, censoring schemes and priors (non-informative and informative). The MLEs of parameters are obtained by solving the nonlinear equations (5) and (6) using the root-solve package, in which the AMLEs are applied as starting values for the iterations. Moreover for the EM algorithm we also employ the AMLEs in role of starting values. We stop iterations in the EM algorithm when absolute difference of estimates in $(h + 1)$th and hth iterate less than 1×10^{-4}. Non-informative prior (prior 1) can be obtained with $c_1 = d_1 = c_2 = d_2 = 0$. Also informative prior is denoted with prior 2. We used the prior 2 by considering $c_1 = 3, d_1 = 2$ and $c_2 = d_2 = 3$. For prior 2 we have chosen the hyper-parameters in such a way that the prior mean became the expected value of the corresponding parameter. To implement the MCMC process a

uniform distribution with center location as the current state value of the parameter is used as the transition probability, $q_j(\lambda_j^{(*)}|\lambda_j^{(i)})$, from the current state value $\lambda_j^{(i)}$ of the parameter λ_j to the next state value $\lambda_j^{(*)}$ of the parameter λ_j where $j = 1, 2$. Then following the iterative process runs up to 50,000 iterations with 30,000 as the burn-in period in the MCMC process.
All calculations are performed on R package. Since β is a scale parameter, we have taken in all cases $\beta = 1$ without loss of generality. For that purpose we report the result only for $\alpha = 1.5$. We replicated the process 5,000 times and report the average biases, MSEs and risks for different censoring schemes. Note that, compact notations have been used to represent different censoring schemes in tables. For example, censoring scheme $(0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$ is denoted as $(0^{*5}, 1^{*5}, 0^{*5})$. Table 1 reports the average bias and MSE values of the non-Bayesian estimates of α, β and $R(t)$, for $t = 1$ and $t = 2$. The average bias and risk values of Bayes estimates of α, β and reliability function $R(t)$ are presented in Tables 2-5. Bayes estimates under the loss functions L_2 and L_3 are obtained for $c = 1$ and $q = 1$, respectively.
From the results of the simulation study, we observe the following.

1. As expected, the performance of the Bayes estimates under prior 2 are better than the Bayes estimates under prior 1 for different n, m and censoring schemes, in terms of risk values. This result is also true for the estimates of α, β and $R(t)$ for $t = 1$, obtianed by Lindley's method as well as β and $R(t)$ for $t = 2$, acquired by MCMC method in terms of bias values.

2. The performance of the Bayes estimates obtained by Lindley's method are better than MCMC method.

3. It is observed that, for prior 1 and 2, the Bayes estimates of α against loss function L_1 perform better than the non-Bayesian estimates. Also, in the case of β, we see that for prior 2, MSEs of the Bayes estimate $\hat{\beta}_{SB}$ obtained by Lindley's method are smaller than the non-Bayesian estimates.

4. For all the censoring schemes, it is observed that performance of the Bayes estimate $\hat{R}_{SB}(t)$ (against squared error loss function L_1) of the reliability function $R(t)$ obtained by Lindley's method is better than the non-Bayesian estimates in terms of MSEs. This result is true for $\hat{R}_{SB}(t)$ obtained by MCMC method for prior 2, when $t = 2$.

5. It is observed that for fixed m, as n increases the performance of most of estimates improves in terms of their bias and mean square error values. Also, for fixed n, as m increases the performance of non-Bayesian and Bayes estimates improves in terms of MSE and risk values, respectively.

6. By increasing t, the bias and risk values of $\hat{R}_{EB}(t)$ increases. But, we cannot opine about the performance of the other Bayes estimates reliability function $R(t)$ in terms of both biase and risk values.

7. When n and m are fixed, choosing the best censoring scheme depends on the parameter of interest. If the shape parameter α is important for us, the censoring scheme $(n-m, 0, \ldots, 0)$ posses the smallest bias, MSE and risk values. Also, for estimates of scale parameter β and reliability function $R(t)$, the censoring scheme $(0, \ldots, 0, n-m)$ is better compare with the other censoring schemes.

8. The best estimates of α, β and reliability function $R(t)$ have been reported in Table 6. Although this table represent the best estimates but there are some other methods of estimation which gives close results in some parts of the table. For example, the performance of the estimate $\hat{\beta}_{EM}$ is also good as $\hat{\beta}$ in terms of MSEs.

Table 6. The best estimates of α, β and $R(t)$ in terms of the bias, MSE and risk values.

		α	β	$R(t)$	
				$t=1$	$t=2$
non − Bayesian					
	bias	$\hat{\alpha}_{EM}$	$\hat{\beta}$	–	–
	MSE	$\hat{\alpha}_{EM}$	$\hat{\beta}$	$\hat{R}_{EM}(t)$	$\hat{R}(t), \hat{R}_{EM}(t)$
Bayes Lindley's approximation (MCMC)					
prior 1	bias	$\hat{\alpha}_{EB}$ $(\hat{\alpha}_{EB})$	$\hat{\beta}_{EB}$ $(\hat{\beta}_{EB})$	$\hat{R}_{SB}(t)$ (-)	$\hat{R}_{LB}(t)$ $(\hat{R}_{LB}(t))$
	risk	$\hat{\alpha}_{EB}$ $(\hat{\alpha}_{EB})$	$\hat{\beta}_{EB}$ $(\hat{\beta}_{EB})$	$\hat{R}_{LB}(t)$ $(\hat{R}_{LB}(t))$	$\hat{R}_{LB}(t)$ $(\hat{R}_{LB}(t))$
prior 2	bias	$\hat{\alpha}_{LB}$ $(\hat{\alpha}_{EB})$	$\hat{\beta}_{EB}$ $(\hat{\beta}_{EB})$	$\hat{R}_{SB}(t)$ (-)	$\hat{R}_{LB}(t)$ (-)
	risk	$\hat{\alpha}_{EB}$ $(\hat{\alpha}_{EB})$	$\hat{\beta}_{EB}$ $(\hat{\beta}_{EB})$	$\hat{R}_{LB}(t)$ $(\hat{R}_{LB}(t))$	$\hat{R}_{LB}(t)$ $(\hat{R}_{LB}(t))$

Example 1. (Real Data).To illustrate the methods of inference proposed in this paper, we perform the following data analysis. We consider the real data set as given by Nelson (1982) concerning the data on time to breakdown of an insulating fluid between electrodes at a voltage of 34 KV (minutes). The 19 time to breakdown are:
0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89.

Before progressing further, we have first fitted the EVII distribution to the complete data set, and it is observed that $\hat{\alpha} = 0.6434$ and $\hat{\beta} = 2.7729$. The Kolmogorov-Smirnov distance is 0.158 and the corresponding p-value is 0.6732. Since the p-value is high, we cannot reject the null hypothesis that the data are coming from the EVII distribution. From these data, a progressively type-II censored sample of size $m = 8$ from $n = 19$ observations recorded at 34 kilovolts has been produced by Viveros and Balakrishnan (1994). These progressively censored data with censoring scheme $(0, 0, 3, 0, 3, 0, 0, 5)$ are as 0.19, 0.78, 0.96, 1.31, 2.78, 4.85, 6.50, 7.35.

The non-Bayesian and Bayes estimates of α, β and $R(t)$ are given in Tables 7 and 8, respectively. Since we do not have any prior information, we take $c_1 = d_1 = c_2 = d_2 = 0$ for Bayes estimates.

6 Conclusion

In this article, we considered the statistical inference of the EVII distribution under progressive type-II censoring. Utilizing different methods of estimation, such as MLE, AMLE and EM algorithm, we obtained non-Bayesian estimates of the unknown parameters. Also we found that, the Bayes estimates cannot be obtained in explicit forms. We used Lindley's approximations and MCMC methods to compute the Bayes estimates under the squared error, LINEX and general entropy loss functions. Furthermore, we compared the performance of the proposed methods by Monte Carlo simulations.

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Table 7. The non-Bayesian estimates of α, β and $R(t)$.

						$t=1$			$t=2$		
$\hat{\alpha}$	$\tilde{\alpha}$	$\hat{\alpha}_{EM}$	$\hat{\beta}$	$\tilde{\beta}$	$\hat{\beta}_{EM}$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}_{EM}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}_{EM}(t)$
0.5115	0.5220	0.5125	3.7075	3.7858	3.6962	0.8584	0.8652	0.8583	0.7462	0.7522	0.7138

Table 8. The Bayes estimates of α, β and $R(t)$.

							$t=1$			$t=2$		
Method	$\hat{\alpha}_{SB}$	$\hat{\alpha}_{LB}$	$\hat{\alpha}_{EB}$	$\hat{\beta}_{SB}$	$\hat{\beta}_{LB}$	$\hat{\beta}_{EB}$	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$	$\hat{R}_{SB}(t)$	$\hat{R}_{LB}(t)$	$\hat{R}_{EB}(t)$
Lindley's approximation	0.4612	0.4543	0.4396	4.7699	2.9634	3.6039	0.8395	0.8375	0.8351	0.7399	0.7362	0.7301
MCMC	0.5042	0.4965	0.4697	4.7113	2.9395	3.2961	0.8466	0.8440	0.8399	0.7397	0.7359	0.7287

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