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On Moments of the Concomitants of Classic Record Values and Nonparametric Upper Bounds for the Mean under the Farlie-Gumbel-Morgenstern Model

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Extended Abstract. In a sequence of random variables, record values are observations that exceed or fall below the current extreme value. Now consider a sequence of pairwise random variables $\{(X_i, Y_i), i \geqslant 1\}$, when the experimenter is interested in studying just the sequence of records of the first component, the second component associated with a record value of the first one is termed the *concomitant* of that record value. The aim of this paper is to investigate the properties of concomitants of record values in Farlie-Gumbel-Morgenstern (FGM) model. So, the sequence of upper record values and their associated concomitants can be defined as follows

$$(R_0, R_{[0]}) = (X_1, Y_1) \quad \text{with probability one}$$

and for $n \geqslant 1$

$$(R_n, R_{[n]}) = (X_{T_n}, Y_{T_n}),$$

where

$$T_n = \min\{j : X_j > X_{T_{n-1}}\}.$$

Let $f_{Y|X}(y|x)$ denote the conditional pdf of Y given X=x. Then as discussed in Yang (1977), the marginal pdf of nth concomitant $R_{[n]}$ can be obtained as

$$f_{R_{[n]}}(y) = \frac{1}{n!}\int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)\{-\log(S_X(x))\}^n dx,$$

where $S_X(x)$ is $1 - F_X(x)$.

The moments of concomitants of record values can be derived. If we denote $\mathrm{E}(R^k_{[n]})$ by $\alpha^{(k)}_{[n]}$ and $\mathrm{E}(R^t_{[m]}R^s_{[n]})$ by $\alpha^{(t,s)}_{[m,n]}$ then

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$$\alpha_{[n]}^{(k)} = \mathrm{E}\{\mathrm{E}(Y^k|X = R_n)\},$$
$$\alpha_{[m,n]}^{(t,s)} = \mathrm{E}\{\mathrm{E}(Y^t|X = R_m)\mathrm{E}(Y^s|X = R_n)\}.$$

The reader is referred to Khaledi and Kochar (2001) and Raqab and Ahsanullah (2002) for details. In this paper the moment and exact distribution properties of concomitants when the underlying model is FGM are discussed. For the FGM model pdf

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\{1 + \alpha(1 - 2F_X(x))(1 - 2F_Y(y))\},$$

where $\alpha \in (-1, 1)$, the marginal pdf of nth concomitant is obtained, that is

$$f_{R_{[n]}}(y) = f_Y(y)\{1 + \alpha(1 - 2^{-n})(2F_Y(y) - 1)\}.$$

Thus the kth moment of the concomitants of record values under the FGM model is derived as follows

$$\alpha_{[n]}^{(k)} = \alpha^{(k)} + \alpha(1 - 2^{-n})\,(\alpha_{2:2}^{(k)} - \alpha^{(k)}),$$

where $\alpha^{(k)} = \mathrm{E}(Y^k)$ and $\alpha_{2:2}^{(k)}$ denote the kth moment of the largest order statistic in a sample of size two of Y's distribution. Also for $m < n$ the joint pdf of two concomitants is obtained as

$$\begin{aligned} f_{R_{[m]},R_{[n]}}(y_1,y_2) &= f_Y(y_1)f_Y(y_2)\{1 + \alpha c_m H_Y(y_1) + \alpha c_n H_Y(y_2) \\ &\quad + \alpha^2 d_{m,n} H_Y(y_1)H_Y(y_2)\}, \end{aligned}$$

where

$$H_Y(y) = 2F_Y(y) - 1, \quad c_j = 1 - 2^{-j},$$

and

$$d_{m,n} = 3^{-m-1}2^{m+2-n} - 2^{-m} - 2^{-n} + 1.$$

So the product moments of the concomitants are derived as follows

$$\begin{aligned} \alpha_{[m,n]}^{(t,s)} &= \alpha^{(s)}\alpha^{(t)} + \alpha c_n\,\alpha^{(t)}(\alpha_{2:2}^{(s)} - \alpha^{(s)}) + \alpha c_m\,\alpha^{(s)}(\alpha_{2:2}^{(t)} - \alpha^{(t)}) \\ &\quad + \alpha^2\,d_{m,n}(\alpha_{2:2}^{(t)} - \alpha^{(t)})(\alpha_{2:2}^{(s)} - \alpha^{(s)}). \end{aligned}$$

Further, the means, variances, covariances, and correlations of the concomitants are derived from the general moments and discussed. Also several universal upper bounds for the mean of concomitants of record values, under the assumptions $\mathrm{E}(Y) = 0$, $\mathrm{E}(Y^2) = 1$ and $\alpha > 0$, are derived.
The general universal bound for the mean of nth concomitant is obtained as

$$\alpha_{[n]} \leqslant \frac{\alpha}{\sqrt{3}}(1 - 2^{-n}).$$

Although for symmetric populations this bound does not change, but the general extrapolation-type bounds for two known means of concomitants $\alpha_{[m]}$and $\alpha_{[p]}(m < p)$ can be improved for symmetric distributions as follows

$$\alpha_{[n]} \leqslant (L(n,n) - A)^{\frac{1}{2}},$$

where

$$L(m,n) = \frac{\alpha^2}{3}(1-2^{-n})(1-2^{-m}) - 1,$$

and

$$A = \frac{\left(L(m,n) - \frac{\alpha_{[m]}}{\alpha_{[p]}}L(p,n)\right)^2}{L(m,n) - 2\frac{\alpha_{[m]}}{\alpha_{[p]}}L(m,p) + \left(\frac{\alpha_{[m]}}{\alpha_{[p]}}\right)^2 L(p,p)}.$$

Keywords. record statistic; concomitant; Farlie-Gumbel-Morgenstern model; Cauchy-Schwarz inequality; universal bounds.

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